

MATH 357 -- Combinatorics
Some Generating Function Computations
March 17, 2017

These computations show how to determine

1. The number of integer solutions of the equations

$$\begin{aligned}x_1 + x_2 + x_3 + 2x_4 &= 7 \\ 2x_1 + x_2 + 4x_3 + 5x_4 &= 22\end{aligned}$$

2. Then the number of integer solutions of the inequalities:

$$\begin{aligned}x_1 + x_2 + x_3 + 2x_4 &\leq 7 \\ 2x_1 + x_2 + 4x_3 + 5x_4 &\leq 22\end{aligned}$$

We'll start with the equalities. Note first that the equalities imply that $x_1 \leq 7, x_2 \leq 7, x_3 \leq 5, x_4 \leq 3$. This means that we only need to consider the *truncated generating function*:

$$(1 + u \cdot v^2 + \dots + (u \cdot v^2)^7) \cdot (1 + u \cdot v + \dots + (u \cdot v)^7) \cdot (1 + u \cdot v^4 + \dots + (u \cdot v^4)^5) \cdot (1 + u^2 \cdot v^5 + \dots + (u^2 \cdot v^5)^3)$$

Here's one way to compute that. We start by creating a function that will expand one of the truncated geometric series:

restart;

geom := k → sum(xⁱ, i = 0..k);

$$k \rightarrow \sum_{i=0}^k x^i \tag{1}$$

m12 := subs(x = u · v², eval(geom(7)));

$$u^7 v^{14} + u^6 v^{12} + u^5 v^{10} + u^4 v^8 + u^3 v^6 + u^2 v^4 + u v^2 + 1 \tag{2}$$

m11 := subs(x = u · v, eval(geom(7)));

m14 := subs(x = u · v⁴, eval(geom(5)));

m25 := subs(x = u² · v⁵, eval(geom(3)));

$$u^7 v^7 + u^6 v^6 + u^5 v^5 + u^4 v^4 + u^3 v^3 + u^2 v^2 + u v + 1$$

$$u^5 v^{20} + u^4 v^{16} + u^3 v^{12} + u^2 v^8 + u v^4 + 1$$

$$u^6 v^{15} + u^4 v^{10} + u^2 v^5 + 1 \tag{3}$$

gf := expand(m12 · m11 · m14 · m25);

$\text{coeff}(\text{coeff}(gf, u^7), v^{22});$

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So the number of solutions of the equalities is 4. For the inequalities, we basically want to truncate this generating function again, removing all $u^m \cdot v^n$ where $m > 7$ or $n > 22$

$tgf := gf;$

for i **to** $\text{nops}(gf)$ **do**

if $\text{degree}(\text{op}(i, gf), u) > 7$ **or** $\text{degree}(\text{op}(i, gf), v) > 22$ **then**

$tgf := \text{expand}(tgf - \text{op}(i, gf));$

end if;

end do;

$\text{nops}(tgf);$

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$tgf;$

$$\begin{aligned} &4 u^7 v^{22} + 2 u^7 v^{21} + u^6 v^{22} + 4 u^7 v^{20} + 2 u^6 v^{21} + 6 u^7 v^{19} + u^6 v^{20} + 4 u^7 v^{18} \\ &+ 2 u^6 v^{19} + u^5 v^{20} + 6 u^7 v^{17} + 4 u^6 v^{18} + 8 u^7 v^{16} + 2 u^6 v^{17} + u^5 v^{18} + 5 u^7 v^{15} \\ &+ 4 u^6 v^{16} + 2 u^5 v^{17} + 6 u^7 v^{14} + 6 u^6 v^{15} + u^5 v^{16} + 6 u^7 v^{13} + 4 u^6 v^{14} + 2 u^5 v^{15} \\ &+ u^4 v^{16} + 3 u^7 v^{12} + 5 u^6 v^{13} + 4 u^5 v^{14} + 3 u^7 v^{11} + 6 u^6 v^{12} + 2 u^5 v^{13} + u^4 v^{14} \\ &+ 3 u^7 v^{10} + 3 u^6 v^{11} + 4 u^5 v^{12} + 2 u^4 v^{13} + u^7 v^9 + 3 u^6 v^{10} + 5 u^5 v^{11} + u^4 v^{12} \\ &+ u^7 v^8 + 3 u^6 v^9 + 3 u^5 v^{10} + 2 u^4 v^{11} + u^3 v^{12} + u^7 v^7 + u^6 v^8 + 3 u^5 v^9 + 4 u^4 v^{10} \\ &+ u^6 v^7 + 3 u^5 v^8 + 2 u^4 v^9 + u^3 v^{10} + u^6 v^6 + u^5 v^7 + 3 u^4 v^8 + 2 u^3 v^9 + u^5 v^6 \\ &+ 3 u^4 v^7 + u^3 v^8 + u^5 v^5 + u^4 v^6 + 2 u^3 v^7 + u^2 v^8 + u^4 v^5 + 3 u^3 v^6 + u^4 v^4 + u^3 v^5 \\ &+ u^2 v^6 + u^3 v^4 + 2 u^2 v^5 + u^3 v^3 + u^2 v^4 + u^2 v^3 + u v^4 + u^2 v^2 + u v^2 + u v + 1 \end{aligned}$$

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$\text{subs}(u = 1, v = 1, tgf);$

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