MATH 392 – Geometry Through History
Finishing Bolyai-Lobachevsky Theorem
February 29

To finish up our discussion of the Bolyai-Lobachevsky theorem and hyperbolic geometry, we need to prove the following lemma that we deferred last time. (Note: some names of points changed to agree with a diagram I'm copying!)

Reminder about notation: $\overline{AB}$ is a segment of a (hyperbolic) line; AB is the length of the line segment; $\widehat{AB}$ is the *horocycle arc*, or its arclength.

Lemma 6 *Let $\overrightarrow{OX} \perp \overrightarrow{OY}$, let A be along the ray $\overrightarrow{OY}$ with $OA = u$, and $\overrightarrow{AA'}$ be parallel to $\overrightarrow{OX}$ in the direction of X and A'. Let $\widehat{OB}$ be the horocycle arc through O (for the pencil of parallel lines containing $\overleftrightarrow{OX}$ and $\overleftrightarrow{AA'}$) intersecting $\overrightarrow{AA'}$ at B. Let $v = AB$ and $s = \widehat{OB}$. Then the following relations hold:*

$$e^{v/k} = \cosh(u/k),$$

and

$$s = \sigma \tanh(u/k)$$

where σ is a constant that will be determined in the course of the proof.

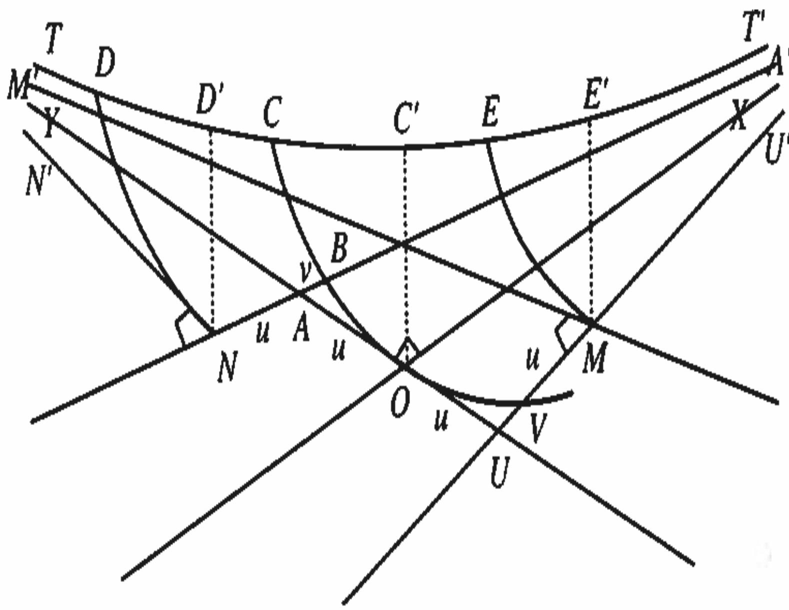

Figure 1: Figure for proof of Lemma 6

- Let $\overleftrightarrow{TT'}$ be parallel with $\overrightarrow{OX}$ in the direction of X and T *and* with $\overrightarrow{OY}$ in the direction of Y and T'. (You showed how to get such a line in Problem Set 3.)

- Construct N as in diagram so $AN = u$ and let $\overrightarrow{NN'}$ be perpendicular to $\overleftrightarrow{AA'}$ and parallel to $\overrightarrow{OY}$ in the direction of Y.

- Construct U as in the diagram so $OU = u$, let $\overrightarrow{UU'}$ be parallel to $\overrightarrow{OX}$ in the direction of X, and extend the horocycle through O, B so that it meets $\overrightarrow{UU'}$ at point V.

- Construct M so $UM = u$ again, and let $\overrightarrow{MM'}$ be perpendicular to $\overleftrightarrow{UU'}$ parallel to $\overrightarrow{OY}$ in the direction of Y.

- Construct the perpendiculars $\overline{ND'}, \overline{OC'}, \overline{ME'}$ to $\overleftrightarrow{TT'}$ as in the diagram.

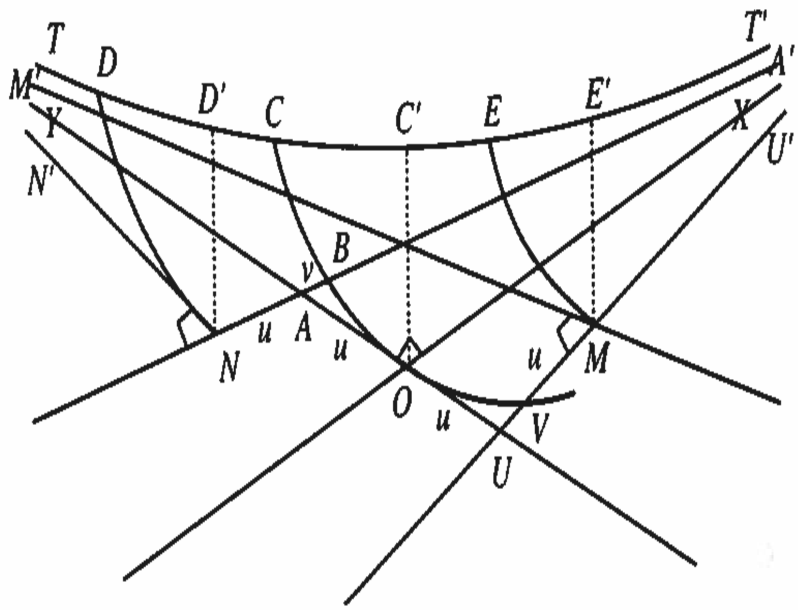

Figure 2: Figure for proof of Lemma 6

- By construction, it follows that $ND' = OC' = ME'$ (since all equal the x such that $\Pi(x) = 45°$)
- Consider the concentric horocycle arcs $\widehat{ND}, \widehat{OC}, \widehat{ME}$. By the previous bullet, these are also all equal in length: $\widehat{ND} = \widehat{OC} = \widehat{ME} = \sigma$. This defines the constant in the second formula we are trying to prove.
- By our previous result about ratios of horocycle arcs, since $BN = u+v$, and $\widehat{BC} = \sigma - s$, we have

$$\frac{\sigma - s}{\sigma} = e^{-(u+v)/k} \tag{1}$$

- Similarly, $VM = u - v$, and $\widehat{CV} = \sigma + s$, so

$$\frac{\sigma}{\sigma + s} = e^{-(u-v)/k} \Leftrightarrow \frac{\sigma + s}{\sigma} = e^{(u-v)/k} \tag{2}$$

Adding (1) and (2) gives

$$2 = \frac{\sigma - s}{\sigma} + \frac{\sigma + s}{\sigma} = e^{-v/k}(e^{u/k} + e^{-u/k}),$$

which shows

$$e^{v/k} = \cosh(u/k)$$

This is the first part of Lemma 6.

The second part also comes from (1) and (2), which we copy for your convenience:

$$\frac{\sigma - s}{\sigma} = e^{-(u+v)/k} \tag{1}$$

and

$$\frac{\sigma}{\sigma + s} = e^{-(u-v)/k} \Leftrightarrow \frac{\sigma + s}{\sigma} = e^{(u-v)/k} \tag{2}$$

If we subtract these instead of adding we get:

$$\frac{2s}{\sigma} = e^{(u-v)/k} - e^{-(u+v)/k} = e^{-v/k}(e^{u/k} - e^{-u/k})$$

so by the first part:

$$s = \sigma e^{-v/k} \sinh(u/k) = \sigma \frac{\sinh(u/k)}{\cosh(u/k)} = \sigma \tanh(u/k)$$

and the proof is complete.