MATH 392 – Geometry Through History
Saccheri Quadrilaterals and Consequences
Lecture Notes/Solutions From Class Friday, February 12

Background

Recall that on Wednesday we were considering the proof of

Theorem 1 *The following are equivalent:*

*(1) Euclid's Postulates I-V (*and *all the additional facts such as Pasch's Axiom and the Axiom of Continuity that Euclid did not state explicitly, but that are needed for complete proofs of the Propositions in Book I of the Elements.)*

(2) Euclid's Postulates I - IV (and all the facts as in (1)), plus: The set of points equidistant from a given line is another line.

We proved (1) $\Rightarrow$ (2) and saw the intuition behind (2) $\Rightarrow$ (1), but we had some work left to establish the implication. We had completed:

Lemma 1 *Let $CDKH$ be a quadrilateral with $CH \perp CD$, $DK \perp CD$ and $CH = DK$. Then $\angle CHK = \angle DKH$.*

Figures as in this Lemma are called "Saccheri quadrilaterals" because Girolamo Saccheri, S.J. considered their properties extensively in his work on geometry.

Lemma 2 (Saccheri-Legendre Theorem) *Assume Euclid's Postulates I - IV (and all the facts as in (1) of the theorem) hold. Then the angle sum in every triangle is $\leq 180°$.*

To complete the proof of (2) $\Rightarrow$ (1), recall that we needed to show that non-parallel lines "diverge" on one side of their intersection point – the distance between points on one line and the other line grows without bound. This will follow from our next statement:

Lemma 3 *Assume Euclid's Postulates I - IV (and all the facts as in (1) of the theorem) hold. Let ΔACM be a right triangle, with $\angle ACM = 90°$. Let B be the midpoint of AM and drop the perpendicular $BD \perp AC$ with D on the segment AC. Then $BD \leq \frac{1}{2}CM$ (or equivalently $CM \geq 2BD$).*

(The statement in parentheses is what guarantees the non-parallel lines $\overleftrightarrow{AM}$ and $\overleftrightarrow{AC}$ diverge as in the discussion above.)

Questions

I. The proof of Lemma 3 is based on this construction: Extend DB to DH with $BH = DB$. Arguing by contradiction, assume $CM < 2BD$. Then we can extend the line segment CM (Postulate II), lay off a segment equal to $2BD$ along that line, and join HK to form a quadrilateral $DCHK$. (Note that the order of the points C, D has been reversed from the above – I'm listing the vertices moving counterclockwise around the quadrilateral.) Follow this outline to provide a proof of Lemma 3:

(A) Show that $\Delta ADB \cong \Delta MBH$.

Solution: We have $AB = BM$ and $DB = BH$ by construction. Moreover $\angle ABD = \angle MBH$ by equality of vertical angles at intersection of two lines (Proposition I.15). Therefore this congruence is true by Proposition I.4 (SAS).

(B) Show that $\angle DHK > 90°$.

Solution: $\angle DHM = \angle ABD = 90°$ by the congruence in part (A). Then $\angle DHK > 90°$ by Common Notion 5 ("the whole is greater than the part.")

(C) Explain why $DCHK$ is a Saccheri quadrilateral, and deduce that the sum of the angles in that quadrilateral must be $> 360°$.

The first part is true by the definition of a Saccheri quadrilateral: $DH \perp DC$, $KC \perp DC$, and $DH = CK$. Then Lemma 1 implies that the angle sum is equal to two right angles, plus two angles greater than a right angle.

(D) However, show that (C) gives a contradiction to Lemma 2 (the Saccheri-Legendre Theorem). Hint: What does that result say about the angle sum in a quadrilateral?

The contradiction comes from the Saccheri-Legendre theorem. If you join two opposite vertices of a quadrilateral to form two triangles, then the sum of the angles in the two quadrilaterals is $\leq 2 \cdot 180° = 360°$. But (C) says we have a Saccheri quadrilateral where the two summit angles are equal by Lemma 1, and also obtuse. This would give an angle sum $> 360°$. Therefore it must be true that $CM \geq 2BD$, which is what we wanted to show. (The contradiction came from assuming $CM < 2BD$.)

II. Let $CDKH$ be a Saccheri quadrilateral and let M be the midpoint of CD and N be the midpoint of HK. Show that $MN \perp CD$ and $MN \perp HK$.

Solution: Join CN and DN first to form triangles ΔCHN and ΔDKN. We have $CH = DK$ by definition of a Saccheri quadrilateral, $\angle CHN = \angle DKN$ by Lemma 1, and $HN = KN$ since N is the midpoint of HK. Therefore $\Delta CHN \cong \Delta DKN$ by SAS (Proposition I.4). It follows that $CN = DN$ since those are corresponding parts of the congruent triangles. Now, we also have $\Delta CMN \cong \Delta DMN$ by SSS (Proposition I.8). It follows that $\angle CMN = \angle DMN$, so both of them are right angles. The proof that $\angle HNM = \angle KNM$ are right angles is similar, but it proceeds by starting from the diagonals HM and KM, showing $\Delta HMC \cong \Delta KMD$, then deducing $\Delta HMN \cong \Delta KMN$.

Following Lemma 1 Saccheri basically considered three alternatives in his work. In a Saccheri quadrilateral $CDKH$ as above, recall $\angle CHK = \angle DKH$.

HAA: "hypothesis of the acute angle": the equal angles $\angle CHK = \angle DKH$ are acute.

HRA: "hypothesis of the right angle": the equal angles $\angle CHK = \angle DKH$ are right angles. (This is the case in Euclidean geometry, where a Saccheri quadrilateral is just an ordinary rectangle.)

HOA: "hypothesis of the obtuse angle": the equal angles $\angle CHK = \angle DKH$ are obtuse.

III. Show that HOA contradicts the Saccheri-Legendre Theorem (Lemma 2).

Solution: As in I above, HOA would imply that the angle sum in the Saccheri quadrilateral is $> 360°$. But that is not possible because the Saccheri-Legendre theorem implies that the sum is $\leq 360°$ (by splitting the quadrilateral into two triangles with a diagonal).

IV. Show that HAA implies $HK > CD$ in the Saccheri quadrilateral. Hint: Argue by contradiction, starting from the assumption $HK \leq CD$. Use the result of II above. If all this is true, you can extend the line segment HK to a point T beyond K with the distance from the midpoint N satisfying $NT = \frac{1}{2}CD$. What is now true about the quadrilateral $MDTN$?

Solution: Following the Hint: Assume HAA, but $HK \leq CD$. Then $\frac{1}{2}HK \leq \frac{1}{2}CD$ as well and we can extend HK to a point T beyond K with $NT = \frac{1}{2}CD$. Join TD and consider the quadrilateral $MDTN$. By II, the angles $\angle TNM$ and $\angle DMN$ are both right angles. Moreover $NT = MD$ by construction. So $MDTN$ is a Saccheri quadrilateral. But the summit angles are now $\angle MDT > 90°$ (Common Notion 5 again) and $\angle NTD$ which must be equal. This is a contradiction since we were assuming HAA.

Moreover, it follows that the sum of the angles in any triangle is $< 180°$ under HAA. Saccheri then tried to eliminate HAA to show that Euclidean geometry was the only possibility (showing that Postulate V follows from I - IV). However, even though he eventually found a result that he claimed "was repugnant to the nature of a straight line" and he rejected HAA for that reason, he was actually proving true theorems about *hyperbolic geometry*. We will begin to consider this next week(!)