MATH 392 – Geometry Through History
Saccheri Quadrilaterals and Consequences
Class Friday, February 12

Background

Recall that on Wednesday we were considering the proof of

Theorem 1 *The following are equivalent:*

*(1) Euclid's Postulates I-V (*and *all the additional facts such as Pasch's Axiom and the Axiom of Continuity that Euclid did not state explicitly, but that are needed for complete proofs of the Propositions in Book I of the Elements.)*

(2) Euclid's Postulates I - IV (and all the facts as in (1)), plus: The set of points equidistant from a given line is another line.

We proved (1) $\Rightarrow$ (2) and saw the intuition behind (2) $\Rightarrow$ (1), but we had some work left to establish the implication. We had completed:

Lemma 1 *Let $CDKH$ be a quadrilateral with $CH \perp CD$, $DK \perp CD$ and $CH = DK$. Then $\angle CHK = \angle DKH$.*

Figures as in this Lemma are called "Saccheri quadrilaterals" because Girolamo Saccheri, S.J. considered their properties extensively in his work on geometry.

Lemma 2 (Saccheri-Legendre Theorem) *Assume Euclid's Postulates I - IV (and all the facts as in (1) of the theorem) hold. Then the angle sum in every triangle is $\leq 180°$.*

To complete the proof of (2) $\Rightarrow$ (1), recall that we needed to show that non-parallel lines "diverge" on one side of their intersection point – the distance between points on one line and the other line grows without bound. This will follow from our next statement:

Lemma 3 *Assume Euclid's Postulates I - IV (and all the facts as in (1) of the theorem) hold. Let ΔACM be a right triangle, with $\angle ACM = 90°$. Let B be the midpoint of AM and drop the perpendicular $BD \perp AC$ with D on the segment AC. Then $BD \leq \frac{1}{2}CM$ (or equivalently $CM \geq 2BD$).*

(The statement in parentheses is what guarantees the non-parallel lines $\overleftrightarrow{AM}$ and $\overleftrightarrow{AC}$ diverge as in the discussion above.)

Questions

I. The proof of Lemma 3 is based on this construction: Extend DB to DH with $BH = DB$. Arguing by contradiction, assume $CM < 2BD$. Then we can extend the line segment CM (Postulate II), lay off a segment equal to $2BD$ along that line, and join HK to form a quadrilateral $DCHK$. (Note that the order of the points C, D has been reversed from the above – I'm listing the vertices moving counterclockwise around the quadrilateral.) Follow this outline to provide a proof of Lemma 3:

(A) Show that $\Delta ADB \cong \Delta MBH$.

(B) Show that $\angle DHK > 90°$.

(C) Explain why $DCHK$ is a Saccheri quadrilateral, and deduce that the sum of the angles in that quadrilateral must be $> 360°$.

(D) However, show that (C) gives a contradiction to Lemma 2 (the Saccheri-Legendre Theorem). Hint: What does that result say about the angle sum in a quadrilateral?

II. Let $CDKH$ be a Saccheri quadrilateral and let M be the midpoint of CD and N be the midpoint of HK. Show that $MN \perp CD$ and $MN \perp HK$.

Following Lemma 1 Saccheri basically considered three alternatives in his work. In a Saccheri quadrilateral $CDKH$ as above, recall $\angle CHK = \angle DKH$.

HAA: "hypothesis of the acute angle": the equal angles $\angle CHK = \angle DKH$ are acute.

HRA: "hypothesis of the right angle": the equal angles $\angle CHK = \angle DKH$ are right angles. (This is the case in Euclidean geometry, where a Saccheri quadrilateral is just an ordinary rectangle.)

HOA: "hypothesis of the obtuse angle": the equal angles $\angle CHK = \angle DKH$ are obtuse.

III. Show that HOA contradicts the Saccheri-Legendre Theorem (Lemma 2).

IV. Show that HAA implies $HK > CD$ in the Saccheri quadrilateral. Hint: Argue by contradiction, starting from the assumption $HK \leq CD$. Use the result of II above. If all this is true, you can extend the line segment HK to a point T beyond K with the distance from the midpoint N satisfying $NT = \frac{1}{2}CD$. What is now true about the quadrilateral $MDTN$?

Moreover, it follows that the sum of the angles in any triangle is $< 180°$ under HAA. Saccheri then tried to eliminate HAA to show that Euclidean geometry was the only possibility (showing that Postulate V follows from I - IV). However, even though he eventually found a result that he claimed "was repugnant to the nature of a straight line" and he rejected HAA for that reason, he was actually proving true theorems about *hyperbolic geometry*. We will begin to consider this next week(!)