MATH 392 – Geometry Through History
Discussion – Descartes and Coordinate, or "Analytic" Geometry
March 2, 2016

Background

In 1637, the French philosopher and mathematician René Descartes published a pamphlet called *La Géometrie* as one of a series of discussions about particular sciences intended apparently as illustrations of his general ideas expressed in his work *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences* (in English: Discourse on the method of rightly directing one's reason and searching for truth in the sciences). This work introduced other mathematicians to a new way of dealing with geometry. Via the introduction of coordinates, Descartes showed that it was possible to define *geometric objects* by means of *equations* and to apply techniques from *algebra* to deduce geometric properties of those objects.

He illustrated his new methods first by considering a problem first studied by the ancient Greeks after the time of Euclid:

Given three (or four) lines in the plane, find the locus of points P that satisfy the relation that the square of the distance from P to the first line (or the product of the distances from P to the first two of the lines) is equal to the product of the distances from P to the other two lines.

The Greek mathematician Apollonius (ca. 262 BCE - ca. 190 BCE) considered this problem in Book III of his masterwork *Conics* and showed via *extremely elaborate* "synthetic" (i.e. Euclid-style) proofs that *both the three- and four-line problems lead to conic sections – the ellipses, hyperbolas, and parabolas that are obtained as plane sections of a cone.*

For instance, the locus of points satisfying the condition above for the four lines $x + 2 = 0$, $x - 1 = 0$, $x + 2y + 1 = 0$, and $2x - y + 2 = 0$ is shown at the top on the back of this sheet. The point shown in black is $P = (1, 4)$. Note that it satisfies the defining condition *the product of the distances from P to the first two of the lines is equal to the product of the distances from P to the other two lines* since it lies on the line $x - 1 = 0$ so the first product is zero, and it lies on the line $2x - y + 2 = 0$, so the second product is also zero. This case gives an ellipse, the other conics can be obtained by varying the positions of the lines.

Descartes actually learned about this work by reading an account given by the later Greek mathematician Pappus (ca. 290 - 350 CE), whose work *The Collection* preserved much of the earlier work of Greek mathematicians. This was translated into Latin in the 16th century and reintroduced much Greek advanced mathematics to Europe around the time of Descartes. With his coordinate geometry in the plane, Descartes was able to derive Apollonius's results in a much easier way, and he also showed how to solve analogous problems when the locus was described by any number ≥ 3 of lines.

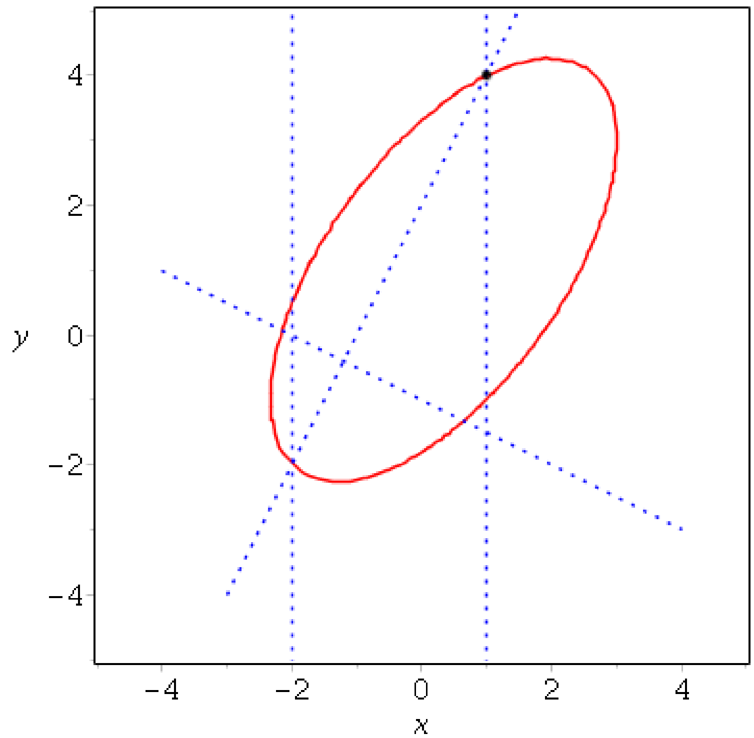

Figure 1: A four-line locus

Today, using our knowledge of coordinate geometry, we want to understand what Descartes did and why it was such an advance of Apollonius's methods.

Questions

(A) Suppose a line L is given by the equation $Ax + By + C = 0$. and $P = (\xi, \eta)$. Show that the (perpendicular) distance from L to P is given by

$$d(L, P) = \frac{|A\xi + B\eta + C|}{\sqrt{A^2 + B^2}}$$

(Hint: $u = \frac{1}{\sqrt{A^2+B^2}}(A, B)$ is a unit vector perpendicular to the line L.)

(B) Use part A to prove Apollonius's result that the three- and four-line loci are both given by curves defined by algebraic equations of total degree 2 in the coordinates of the point P – that is equations of the form:

$$D\xi^2 + E\xi\eta + F\eta^2 + G\xi + H\eta + I = 0$$

for some real numbers D, E, F, G, H, I.

(C) Explain why every equation of the form in part (B) defines a conic section (or perhaps a union of two lines, or a line with multiplicity two, or a single point, or an empty set of solutions).