MATH 392 – Geometry Through History
Bolyai-Lobachevsky Theorem and Hyperbolic Pythagorean Theorem
February 22, 24, 26

Recap

In the geometry of the hyperbolic plane, the following notions are keys to the general properties:

- Given a line ℓ, the collection of *all lines parallel to* ℓ is called $\mathbf{P}_\ell$.
- Given a pencil $\mathbf{P}$ of lines (of any one of the three types – but we'll be concerned mostly with the parallel pencils), we say two points X and Y *correspond with respect to* $\mathbf{P}$ if the line of the pencil that passes through the midpoint of the line segment $\overline{XY}$ meets that segment at a right angle (or equivalently, the lines of the pencil through X and through Y make equal angles with the segment $\overline{XY}$).
- *In the case* $\mathbf{P} = \mathbf{P}_\ell$ *is the pencil of lines parallel to some line* ℓ, the set of all points corresponding to a given point X is called a *horocycle* through X. In Euclidean geometry, horocycles are lines; in hyperbolic geometry, horocycles are *not* lines, but they can be thought of as "circles with center at infinity.")
- Given a pencil of parallel lines, there is a horocycle passing through every point in the plane. Two horocycles are *concentric* of they come from the same pencil of parallel lines.

Our next goal is to prove a result known as the Bolyai-Lobachevsky theorem:

Theorem 1 (Bolyai-Lobachevsky) *Let* $\Pi(x)$ *denote the angle of parallelism of a segment of length* x*. Then*

$$\tan(\Pi(x)/2) = e^{-x/k}$$

for some constant k*.*

Along the way, we will take a "detour" into *three-dimensional* hyperbolic space and see a result analogous to the Pythagorean theorem that holds for triangles in the hyperbolic plane.

Some preliminary results

Proposition 1 *(1) Let A, B, X, Y be two pairs of points on the same horocycle. If the line segments $\overline{AB}$ and $\overline{XY}$ satisfy $AB = XY$, then the horocycle arcs satisfy $\widehat{AB} = \widehat{XY}$.*

(2) Let A, A' be on one line of a parallel pencil, while X, X' lie on another line of the same parallel pencil. If A, X lie on the same horocycle and similarly for A', X' then $AA' = XX'$.

(3) In the situation of the previous part, if $\overrightarrow{AA'}$ and $\overrightarrow{XX'}$ are in the direction of parallelism, then $\widehat{AX} > \widehat{A'X'}$ (that is, horocycle arcs intercepted by the lines of a pencil decrease *in the direction of parallelism).*

For proofs, see Lemma 4.8 in McCleary.

Proposition 2 *The ratio of the lengths of arcs of two concentric horocycles intercepted by two lines of a parallel pencil is expressible in terms of an exponential function of the distance between the arcs (along any one of the lines – see Proposition 1, part (2)).*

- First suppose A, B, C are on one horocycle and A', B', C' are on a concentric horocycle.
- If $n\widehat{AB} = m\widehat{BC}$ for some $m, n \in \mathbb{Z}$, then

$$\frac{\widehat{AB}}{\widehat{A'B'}} = \frac{\frac{m}{n}\widehat{BC}}{\frac{m}{n}\widehat{B'C'}} = \frac{\widehat{BC}}{\widehat{B'C'}}$$

- So, the ratio depends only on the distance between the horocycles.
- If the ratio is irrational, approximate by rational numbers and take a limit.

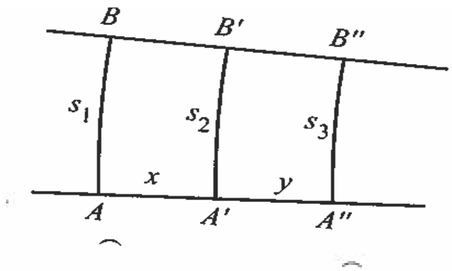

Figure 1: Figure for Proposition 2

- Write $x = AA'$ and $y = A'A''$ along the line.
- Then we have

$$\frac{\widehat{AB}}{\widehat{A'B'}} = f(x), \frac{\widehat{A'B'}}{\widehat{A''B''}} = f(y), \frac{\widehat{AB}}{\widehat{A''B''}} = f(x+y)$$

  and

$$f(x+y) = f(x)f(y)$$

- (Assuming f differentiable), we have, using the previous bullet:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} = f(x) \cdot \frac{1}{k}$$

  for some k.

- Then standard facts from calculus show

$$f(x) = e^{x/k}$$

  so

$$\widehat{AB} = \widehat{A'B'} e^{x/k}$$

  Note, this is increasing in the *opposite of the direction of parallelism.*

Next, following Bolyai and Lobachevsky, we will make a detour into *three-dimensional* space. Assume Postulates I-IV, and Saccheri's HAA hold in all planes.

Definition. Two lines ℓ_1, ℓ_2 in space are *parallel* if they lie in the same plane and are parallel in that plane.

Note that there are also *skew* pairs of lines, and coplanar pairs of lines that do not meet, but are not parallel, since the definition of parallel is hyperbolic one(!)

First there are some results that should look very familiar, since they are the same as in Euclidean three-dimensional space. You will be asked to provide proofs on Problem Set 3, due Monday, February 29.

Lemma 1 *Let ℓ_1, ℓ_2 be parallel lines in space and $\ell_1 \subset T_1$, $\ell_2 \subset T_2$, where T_1, T_2 are distinct planes. If T_1 and T_2 meet, the line $m = T_1 \cap T_2$ is parallel to both ℓ_1 and ℓ_2.*

Definition. We say a line ℓ is *perpendicular to a plane T* if ℓ meets T at a point P and every line through P in T meets ℓ at a right angle.

Lemma 2 *The following are equivalent for a line ℓ intersecting a plane T at a point P.*

(1) $\ell \perp T$

(2) There exist lines $m_1 \neq m_2$ through P in T such that ℓ meets m_1 and m_2 at right angles.

Corollary 1 *Given a line ℓ and a point P, there is a unique plane containing P and perpendicular to ℓ.*

Lemma 3 *If $\ell \perp T$ with $\ell \cap T = P$, let m be any line in T and $A \in m$ such that $\overline{AP} \perp m$. If B is any point on ℓ, then $\overline{AB} \perp m$.*

Corollary 2 *Given a plane T and a point P in T, there is a unique line through P that is perpendicular to T.*

Definition. Two planes T, T' are perpendicular if there is a line $\ell \subset T$ that is perpendicular to T' (or equivalently, there is a line $\ell' \subset T'$ that is perpendicular to T.

Lemma 4 *(1) If T and T' are perpendicular planes, and ℓ is a line on T, then $\ell \perp T'$ is equivalent to $\ell \perp m$ for $m = T \cap T'$.*

(2) If $\ell_1 \perp T$ and $\ell_2 \perp T$, then ℓ_1, ℓ_2 are coplanar and non-intersecting (but not necessarily parallel – this is a difference in the hyperbolic case!)

Lemma 5 *(1) Given a line ℓ and a plane T not containing ℓ, there is a unique plane T' containing ℓ that is perpendicular to T.*

(2) Given any two distinct planes T_1, T_2, there exist planes T that are perpendicular to both.

Definitions.

(a) Let T' be the unique plane containing a given line ℓ and perpendicular to a plane T (note that the assumption this plane is unique rules out the case where $\ell \perp T$). Then $m = T \cap T'$ is called the *perpendicular projection of ℓ onto T*.

(b) $\ell \subset T'$ is *parallel to T* if ℓ and the perpendicular projection m into T are parallel lines in T'

(c) *Two planes are parallel* if the intersections with some common perpendicular plane are parallel lines.

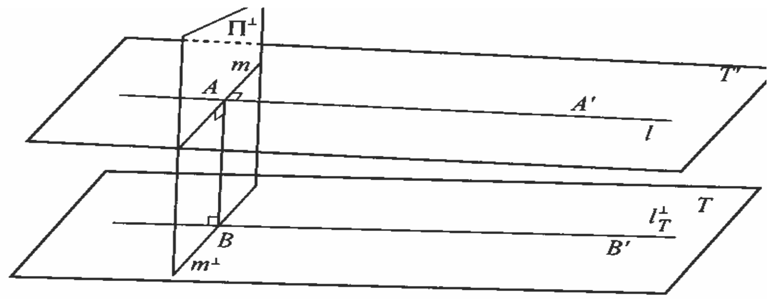

Figure 2: Figure for Theorem 2

Next, we have a *very important* analog of the Playfair Postulate in Euclidean plane geometry:

Theorem 2 *Through a line ℓ parallel to a plane T, there exists a unique plane that contains ℓ and is parallel to T.*

Definition. The set of all points corresponding to a given point P with respect to the family of all lines parallel to a given line is called a *horosphere*, $\mathcal{H}_P$.

To visualize this:

- Let ℓ be any one of the lines in the family
- Let $T \supset \ell$ be a plane
- Then $\mathcal{H}_P \cap T$ is a *horocycle* in the plane T
- That horocycle contains the point P if P is on ℓ, but not otherwise

In fact, a horosphere can be thought of as a sort of "surface of revolution" generated by a horocycle in one of the planes containing ℓ.

One of Bolyai and Lobachevsky's main ideas: *We can "do geometry" on the horosphere* if we take *horocycles as the analogs of lines*. For instance, it's not hard to see that any two points P, P' on the horosphere determine coplanar parallel lines $P \in \ell, P' \in \ell'$, hence a horocycle in the plane containing ℓ, ℓ' passing through P, P'. This and Theorem 2 leads to a most surprising and important result:

Theorem 3 *The geometry of points and horocycles on a horosphere is* Euclidean*.*

(That is, Postulates I-V all hold if we replace "line" by "horocycle!")

Hyperbolic trig functions

Let

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

This gives a family of functions parallel to the usual trig functions according to the following definitions

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)}, \quad \coth(t) = \frac{\cosh(t)}{\sinh(t)}$$

As you'll see on Problem Set 3, these have many properties similar to the usual trig functions, except they correspond to properties of *hyperbolas* rather than *circles*. Most important hyperbolic identity:

$$\cosh^2(t) - \sinh^2(t) = 1$$

for all t (see where the hyperbolas are entering?)

Two basic results:

Lemma 6 *Let $\overrightarrow{PQ}$ and $\overrightarrow{XX'}$ be parallel in the direction of Q and X' with $\overline{XP} \perp \overline{PQ}$ with $PX = u$. If we also draw in the horocycle arc through P (for the pencil of parallels including the two given lines), and call $s = \widehat{PB}$ the distance along that arc from P to where it intersects $\overline{XX'}$ at B. Finally let $v = BX$. Then the following relations hold:*

$$e^{v/k} = \cosh(u/k),$$

and

$$s = \sigma \tanh(u/k)$$

where σ is a constant that will be determined in the course of the proof.

We will assume these for now and come back to their proof later. We will next use them to deduce the Bolyai-Lobachevsky theorem, Theorem 1.

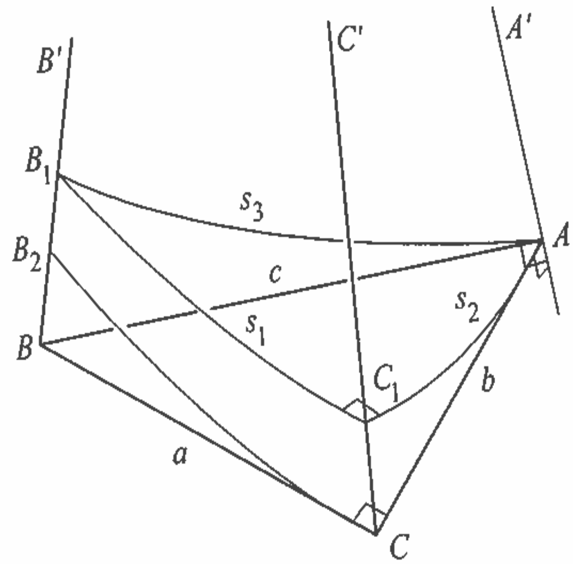

Figure 3: Figure for proof of Theorem 1

- Let ΔABC be a right triangle with right angle at C, and
- In three-dimensional space let $\overleftrightarrow{AA'}$ be the perpendicular line to the plane of the triangle.
- Let $\overleftrightarrow{BB'}$ and $\overline{CC'}$ be parallels to $\overleftrightarrow{AA'}$ in direction of A'.
- These lines intersect the horosphere $\mathcal{H}_P$ at B_1, C_1.
- Hence we get a triangle (with horocycle sides) ΔAB_1C_1 in the (Euclidean) geometry on $\mathcal{H}_P$.

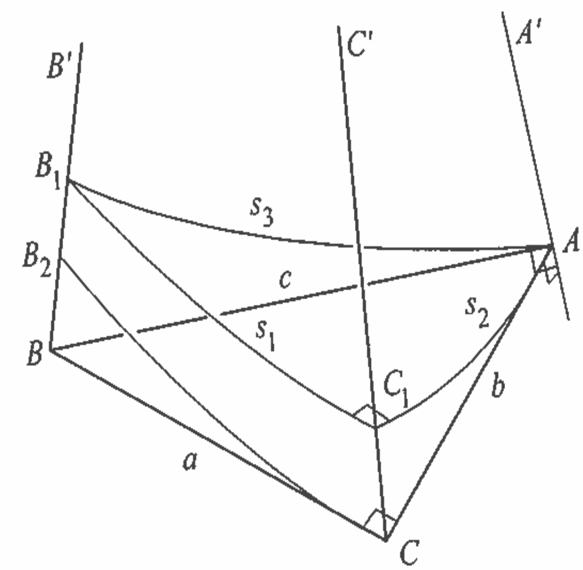

Figure 4: Figure for proof of Theorem 1

- Write $s_1 = \widehat{B_1C_1}$, $s_2 = \widehat{AC_1}$, $s_3 = \widehat{AB_1}$ and $a = BC$, $b = AC$, $c = AB$.
- By Lemma 6, we have

$$s_2 = \sigma \tanh(b/k), \qquad s_3 = \sigma \tanh(c/k)$$

and combining with Lemma 2

$$s_1 = \sigma \tanh(a/k) e^{-CC_1/k}.$$

- But also by Lemma 6,

$$e^{CC_1/k} = \cosh(b/k),$$

so

$$s_1 = \sigma \frac{\tanh(a/k)}{\cosh(b/k)}.$$

- In the Euclidean triangle ΔAB_1C_1 on the horosphere, we have

$$s_2 = s_3 \cos(\angle CAB)$$

so

$$\cos(\angle CAB) = \frac{\tanh(b/k)}{\tanh(c/k)} \tag{1}$$

Similarly,

$$\cos(\angle ABC) = \frac{\tanh(a/k)}{\tanh(c/k)}$$

and then

$$\sin(\angle CAB) = \frac{\tanh(a/k)}{\cosh(b/k)\tanh(c/k)} \tag{2}$$

- Let $a \to \infty$. Then $c \to \infty$ as well and $\tanh(c/k) \to 1$
- At the same time, in the right triangle $\angle ABC$, we have $\angle CAB \to \Pi(b)$ (the angle of parallelism)
- From (1) we get

  $$\cos(\Pi(b)) = \tanh(b/k)$$

  while from (2) we get

  $$\sin(\Pi(b)) = \frac{1}{\cosh(b/k)}$$

- Now, use the trig identity

  $$\tan(\theta/2) = \frac{1-\cos(\theta)}{\sin(\theta)}$$

  and we deduce

  $$\tan(\Pi(b)/2) = \frac{1-\tanh(b/k)}{1/\cosh(b/k)} = e^{-b/k}.$$

  This is what we wanted to prove for the Bolyai-Lobachevsky Theorem(!)

A byproduct of this set-up: The geometry of the horosphere is Euclidean so in fact we have the relation $s_3^2 = s_1^2 + s_2^2$, which yields:

$$\cosh^2(c/k) = \cosh^2(a/k)\cosh^2(b/k)$$

which is the analog of the Pythagorean Theorem for hyperbolic triangles(!!)