§4 The Gröbner Fan of an Ideal

Gröbner bases for the same ideal but with respect to different monomial orders have different properties and can look very different. For example, the ideal

$$I = \langle z^2 - x + y - 1, x^2 - yz + x, y^3 - xz + 2\rangle \subset \mathbb{Q}[x, y, z]$$

has the following three Gröbner bases:

1. Consider the *grevlex* order with $x > y > z$. Since the leading terms of the generators of I are pairwise relatively prime,
$$\{z^2 - x + y - 1, x^2 - yz + x, y^3 - xz + 2\}$$
is a monic (reduced) Gröbner basis for I with respect to this monomial order. Note that the basis has three elements.
2. Consider the weight order $>_{\mathbf{w},grevlex}$ on $\mathbb{Q}[x, y, z]$ with $\mathbf{w} = (2, 1, 5)$. This order compares monomials first according to the weight vector $\mathbf{w}$ and breaks ties with the *grevlex* order. The monic Gröbner basis for I with respect to this monomial order has the form:
$$\begin{aligned}\{&xy^3 + y^2 - xy - y + 2x + y^3 + 2, yz - x^2 - x,\\ &y^6 + 4y^3 + yx^2 + 4 - y^4 - 2y, x^2y^2 + 2z + xy - x^2 - x + xy^2,\\ &x^3 - y^4 - 2y + x^2, xz - y^3 - 2, z^2 + y - x - 1\}.\end{aligned}$$
This has seven instead of three elements.
3. Consider the *lex* order with $x > y > z$. The monic Gröbner basis for this ideal is:
$$\begin{aligned}\{&z^{12} - 3z^{10} - 2z^8 + 4z^7 + 6z^6 + 14z^5 - 15z^4 - 17z^3 + z^2 + 9z + 6,\\ &y + \tfrac{1}{38977}(1055z^{11} + 515z^{10} + 42z^9 - 3674z^8 - 12955z^7 + 5285z^6\\ &- 1250z^5 + 36881z^4 + 7905z^3 + 42265z^2 - 63841z - 37186),\\ &x + \tfrac{1}{38977}(1055z^{11} + 515z^{10} + 42z^9 - 3674z^8 - 12955z^7 + 5285z^6\\ &- 1250z^5 + 36881z^4 + 7905z^3 + 3288z^2 - 63841z + 1791)\}\end{aligned}$$
This basis of three elements has the triangular form described by the Shape Lemma (Exercise 16 of Chapter 2, §4).

Many of the applications discussed in this book make crucial use of the different properties of different Gröbner bases. At this point, it is natural to ask the following questions about the collection of *all* Gröbner bases of a fixed ideal I:

- Is the collection of possible Gröbner bases of I finite or infinite?
- When do two different monomial orders yield the same monic (reduced) Gröbner basis for I?
- Is there some geometric structure underlying the collection of Gröbner bases of I that can help to elucidate properties of I?

Answers to these questions are given by the construction of the *Gröbner fan* of an ideal I. A fan consists of finitely many closed convex polyhedral cones with vertex at the origin (as defined in §2) with the following properties:

a. Any face of a cone in the fan is also in the fan. (A *face* of a cone σ is $\sigma \cap \{\ell = 0\}$, where $\ell = 0$ is nontrival linear equation such that $\ell \geq 0$ on σ. This is analogous to the definition of a face of a polytope.)
b. The intersection of two cones in the fan is a face of each.

These conditions are similar to the definition of polyhedral complex given in Definition (3.5). The Gröbner fan encodes information about the different Gröbner bases of I and was first introduced in the paper [MR] of Mora and Robbiano. Our presentation is based on theirs.

The first step in this construction is to show that for each fixed ideal I, as $>$ ranges over all possible monomial orders, the collection of monomial ideals $\langle \text{LT}_>(I)\rangle$ is finite. We use the notation

$$\text{Mon}(I) = \{\langle \text{LT}_>(I)\rangle : > \text{ a monomial order}\}.$$

(4.1) Theorem. *For an ideal $I \subset k[x_1, \ldots, x_n]$, the set* $\text{Mon}(I)$ *is finite.*

PROOF. Aiming for a contradiction, suppose that $\text{Mon}(I)$ is an infinite set. For each monomial ideal N in $\text{Mon}(I)$, let $>_N$ be any one particular monomial order such that $N = \langle \text{LT}_{>_N}(I)\rangle$. Let Σ be the collection of monomial orders $\{>_N : N \in \text{Mon}(I)\}$. Our assumption implies that Σ is infinite.

By the Hilbert Basis Theorem we have $I = \langle f_1, \ldots, f_s\rangle$ for polynomials $f_i \in k[x_1, \ldots, x_n]$. Since each f_i contains only a finite number of terms, by a pigeonhole principle argument, there exists an infinite subset $\Sigma_1 \subset \Sigma$ such that the leading terms $\text{LT}_>(f_i)$ agree for all $>$ in Σ_1 and all i, $1 \leq i \leq s$. We write N_1 for the monomial ideal $\langle \text{LT}_>(f_1), \ldots, \text{LT}_>(f_s)\rangle$ (taking any monomial order $>$ in Σ_1).

If $F = \{f_1, \ldots, f_s\}$ were a Gröbner basis for I with respect to some $>_1$ in Σ_1, then we claim that F would be a Gröbner basis for I with respect to *every* $>$ in Σ_1. To see this, let $>$ be any element of Σ_1 other than $>_1$, and let $f \in I$ be arbitrary. Dividing f by F using $>$, we obtain

$$f = a_1 f_1 + \cdots + a_s f_s + r, \tag{4.2}$$

where no term in r is divisible by any of the $\text{LT}_>(f_i)$. However, both $>$ and $>_1$ are in Σ_1, so $\text{LT}_>(f_i) = \text{LT}_{>_1}(f_i)$ for all i. Since $r = f - a_1 f_1 - \cdots - a_s f_s \in I$, and F is assumed to be a Gröbner basis for I with respect to $>_1$, this implies that $r = 0$. Since (4.2) was obtained using the division algorithm, $\text{LT}_>(f) = \text{LT}_>(a_i f_i)$ for some i, so $\text{LT}_>(f)$ is divisible by $\text{LT}_>(f_i)$. This shows that F is also a Gröbner basis for I with respect to $>$.

However, this cannot be the case since the original set of monomial orders $\Sigma \supset \Sigma_1$ was chosen so that the monomial ideals $\langle \text{LT}_>(I)\rangle$ for $>$ in Σ were all distinct. Hence, given any $>_1$ in Σ_1, there must be some $f_{s+1} \in I$ such that $\text{LT}_{>_1}(f_{s+1}) \notin \langle \text{LT}_{>_1}(f_1), \ldots, \text{LT}_{>_1}(f_s)\rangle = N_1$. Replacing f_{s+1} by its remainder on division by $f_1, \ldots, f_s$, we may assume in fact that no term in f_{s+1} is divisible by any of the monomial generators for N_1.

Now we apply the pigeonhole principle again to find an infinite subset $\Sigma_2 \subset \Sigma_1$ such that the leading terms of $f_1, \ldots, f_{s+1}$ are the same for all $>$ in Σ_2. Let $N_2 = \langle \text{LT}_>(f_1), \ldots, \text{LT}_>(f_{s+1})\rangle$ for all $>$ in Σ_2, and note that $N_1 \subset N_2$. The argument given in the preceding paragraph shows that $\{f_1, \ldots, f_{s+1}\}$ cannot be a Gröbner basis with respect to any of the monomial orders in Σ_2, so fixing $>_2 \in \Sigma_2$, we find an $f_{s+2} \in I$ such that no term in f_{s+2} is divisible by any of the monomial generators for $N_2 = \langle \text{LT}_{>_2}(f_1), \ldots, \text{LT}_{>_2}(f_{s+1})\rangle$.

Continuing in the same way, we produce a descending chain of infinite subsets $\Sigma \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \cdots$, and an infinite strictly ascending chain of monomial ideals $N_1 \subset N_2 \subset N_3 \subset \cdots$. This contradicts the ascending chain condition in $k[x_1, \ldots, x_n]$, so the proof is complete. □

We can now answer the first question posed at the start of this section. To obtain a precise result, we introduce some new terminology. It is possible for two monic reduced Gröbner bases of I with respect to different monomial orders to be equal as sets, while the leading terms of some of the basis polynomials are different depending on which order we consider. Examples where I is principal are easy to construct; also see (4.9) below. A *marked Gröbner basis for I* is a set G of polynomials in I, together with an identified leading term in each $g \in G$ such that G is a monic (reduced) Gröbner basis with respect to some monomial order selecting those leading terms. (More formally, we could define a marked Gröbner basis as a set GM of ordered pairs (g, m) where $\{g : (g, m) \in GM\}$ is a monic (reduced) Gröbner basis with respect to some order $>$, and $m = \text{LT}_>(g)$ for each (g, m) in GM.) The idea here is that we do not want to build a specific monomial order into the definition of G. It follows from Theorem (4.1) that each ideal in $k[x_1, \ldots, x_n]$ has only finitely many marked Gröbner bases.

(4.3) Corollary. *The set of marked Gröbner bases of I is in one-to-one correspondence with the set* $\text{Mon}(I)$.

PROOF. The key point is that if the leading terms of two marked Gröbner bases generate the same monomial ideal, then the Gröbner bases must be equal. The details of the proof are left to the reader as Exercise 4. □

Corollary (4.3) also has the following interesting consequence.

Exercise 1. Show that for any ideal $I \subset k[x_1, \ldots, x_n]$, there exists a finite $U \subset I$ such that U is a Gröbner basis simultaneously for all monomial orders on $k[x_1, \ldots, x_n]$.

A set U as in Exercise 1 is called a *universal Gröbner basis* for I. These were first studied by Weispfenning in [Wei], and that article gives an algorithm for constructing universal Gröbner bases. This topic is also discussed in detail in [Stu2].

To answer our other questions we will represent monomial orders using the matrix orders $>_M$ described in Chapter 1, §2. Recall that if M has rows $\mathbf{w}_i$, then $x^\alpha >_M x^\beta$ if there is an ℓ such that $\alpha \cdot \mathbf{w}_i = \beta \cdot \mathbf{w}_i$ for $i = 1 \ldots, \ell - 1$, but $\alpha \cdot \mathbf{w}_\ell > \beta \cdot \mathbf{w}_\ell$.

When $>_M$ is a matrix order, the first row of M plays a special role and will be denoted $\mathbf{w}$ in what follows. We may assume that $\mathbf{w} \neq 0$.

Exercise 2.

a. Let $>_M$ be a matrix order with first row $\mathbf{w}$. Show that

$$\mathbf{w} \in (\mathbb{R}^n)^+ = \{(a_1, \ldots, a_n) : a_i \geq 0, \text{ all } i\}.$$

We call $(\mathbb{R}^n)^+$ the *positive orthant* in $\mathbb{R}^n$. Hint: $x_i >_M 1$ for all i since $>_M$ is a monomial order.

b. Prove that every nonzero $\mathbf{w} \in (\mathbb{R}^n)^+$ is the first row of some matrix M such that $>_M$ is a monomial order.

c. Let M and M' be matrices such that the matrix orders $>_M$ and $>_{M'}$ are equal. Prove that their first rows satisfy $\mathbf{w} = \lambda\mathbf{w}'$ for some $\lambda > 0$.

Exercise 2 implies that each monomial order determines a well-defined ray in the positive orthant $(\mathbb{R}^n)^+$, though different monomial orders may give the same ray. (For example, all graded orders give the ray consisting of positive multiples of $(1, \ldots, 1)$.) Hence it should not be surprising that our questions lead naturally to cones in the positive orthant.

Now we focus on a single ideal I. Let $G = \{g_1, \ldots, g_t\}$ be one of the finitely many marked Gröbner bases of I, with $\text{LT}(g_i) = x^{\alpha(i)}$, and $N = \langle x^{\alpha(1)}, \ldots, x^{\alpha(t)}\rangle$ the corresponding element of $\text{Mon}(I)$. Our next goal is to understand the set of monomial orders for which G is the corresponding marked Gröbner basis of I. This will answer the second question posed at the start of this section. We write

$$g_i = x^{\alpha(i)} + \sum_\beta c_{i,\beta} x^\beta,$$

where $x^{\alpha(i)} > x^\beta$ whenever $c_{i,\beta} \neq 0$. By the above discussion, each such order $>$ comes from a matrix M, so in particular, to find the leading terms we compare monomials first according to the first row $\mathbf{w}$ of the matrix.

If $\alpha(i) \cdot \mathbf{w} > \beta \cdot \mathbf{w}$ for all β with $c_{i,\beta} \neq 0$, the single weight vector $\mathbf{w}$ selects the correct leading term in g_i as the term of highest weight. As we know, however, we may have a tie in the first comparison, in which case we would have to make further comparisons using the other rows of M. This suggests that we should consider the following set of vectors:

$$\begin{aligned} C_G &= \{\mathbf{w} \in (\mathbb{R}^n)^+ : \alpha(i) \cdot \mathbf{w} \geq \beta \cdot \mathbf{w} \text{ whenever } c_{i,\beta} \neq 0\} \\ &= \{\mathbf{w} \in (\mathbb{R}^n)^+ : (\alpha(i) - \beta) \cdot \mathbf{w} \geq 0 \text{ whenever } c_{i,\beta} \neq 0\}. \end{aligned} \tag{4.4}$$

It is easy to see that C_G is an intersection of closed half-spaces in $\mathbb{R}^n$, hence is a closed convex polyhedral cone contained in the positive orthant. There are many close connections between this discussion and other topics we have considered. For example, we can view the process of finding elements of C_G as finding points in the feasible region of a linear programming problem as in §1 of this chapter. Moreover, given a polynomial, the process of finding its term(s) of maximum weight with respect to a given vector $\mathbf{w}$ is equivalent to an integer programming maximization problem on a feasible region given by the Newton polytope $NP(f)$.

The cone C_G has the property that if $>_M$ is a matrix order such that G is the marked Gröbner basis of I with respect to $>_M$, then the first row $\mathbf{w}$ of M lies in C_G. However, you will see below that the converse can fail, so that the relation between C_G and monomial orders for which G is a marked Gröbner basis is more subtle than meets the eye.

In the following example we determine the cone corresponding to a given marked Gröbner basis for an ideal.

(4.5) Example. Consider the ideal

$$I = \langle x^2 - y, xz - y^2 + yz \rangle \subset \mathbb{Q}[x, y, z] \tag{4.6}$$

The marked Gröbner basis with respect to the *grevlex* order with $x > y > z$ is

$$G^{(1)} = \{\underline{x^2} - y, \underline{y^2} - xz - yz\},$$

where the leading terms are underlined. Let $\mathbf{w} = (a, b, c)$ be a vector in the positive orthant of $\mathbb{R}^3$. Then $\mathbf{w}$ is in $C_{G^{(1)}}$ if and only if the following inequalities are satisfied:

$$\begin{aligned} (2,0,0) \cdot (a,b,c) &\geq (0,1,0) \cdot (a,b,c) \quad &\text{or}\quad 2a &\geq b \\ (0,2,0) \cdot (a,b,c) &\geq (1,0,1) \cdot (a,b,c) \quad &\text{or}\quad 2b &\geq a + c \\ (0,2,0) \cdot (a,b,c) &\geq (0,1,1) \cdot (a,b,c) \quad &\text{or}\quad 2b &\geq b + c. \end{aligned}$$

To visualize $C_{G^{(1)}}$, slice the positive orthant by the plane $a + b + c = 1$ (every nonzero weight vector in the positive orthant can be scaled to make this true). The above inequalities are pictured in Figure 4.1, where the a-axis, b-axis and c-axis are indicated by dashed lines and you are looking toward the origin from a point on the ray through $(1, 1, 1)$.

In this figure, the inequality $2a \geq b$ gives the region in the slice to the left (as indicated by the arrow) the line segment connecting $(0, 0, 1)$ at the top the triangle to $(\frac{1}{3}, \frac{2}{3}, 0)$ on the base. The other two inequalities are represented similarly, and their intersection in the first orthant gives the

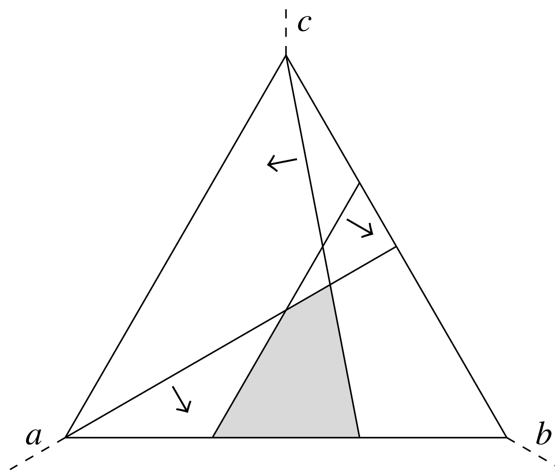

FIGURE 4.1. A Slice of the Cone $C_{G^{(1)}}$

shaded quadrilateral in the slice. Then $C_{G^{(1)}}$ consists of all rays emanating from the origin which go through points of the quadrilateral.

Any weight $\mathbf{w}$ corresponding to a point in the interior of $C_{G^{(1)}}$ (where the inequalities above are strict) will select the leading terms of elements of $G^{(1)}$ exactly; a weight vector on one of the boundary planes in the interior of the positive orthant will yield a "tie" between terms in one or more Gröbner basis elements. For instance $(a, b, c) = (1, 1, 1)$ satisfies $2b = a + c$ and $2b = b + c$, so it is on the boundary of the cone. This weight vector is not sufficient to determine the leading terms of the polynomials.

Now consider a diffferent monomial order, say the *grevlex* order with $z > y > x$. For this order, the monic Gröbner basis for I is

$$G^{(2)} = \{\underline{x^2} - y, \underline{yz} + xz - y^2\},$$

where again the leading terms are underlined. Proceeding as above, the slice of $C_{G^{(2)}}$ in the plane $a + b + c = 1$ is a triangle defined by the inequalities

$$2a \geq b, \;\; b \geq a, \;\; c \geq b.$$

You should draw this triangle carefully and verify that $C_{G^{(1)}} \cap C_{G^{(2)}}$ is a common face of both cones (see also Figure 4.2 below).

Exercise 3. Consider the *grlex* order with $x > y > z$. This order comes from a matrix with $(1, 1, 1)$ as first row. Let I be the ideal from (4.6).

a. Find the marked Gröbner basis G of I with respect to this order.

b. Identify the corresponding cone C_G and its intersections with the two cones $C_{G^{(1)}}$ and $C_{G^{(2)}}$. Hint: The Gröbner basis polynomials contain more terms than in the example above, but some work can be saved by the observation that if $x^{\beta'}$ divides x^β and $\mathbf{w} \in (\mathbb{R}^n)^+$, then $\alpha \cdot \mathbf{w} \geq \beta \cdot \mathbf{w}$ implies $\alpha \cdot \mathbf{w} \geq \beta' \cdot \mathbf{w}$.

Example (4.5) used the *grevlex* order with $z > y > x$, whose matrix has the same first row $(1, 1, 1)$ as the *grlex* order of Exercise 3. Yet they have very different marked Gröbner bases. As we will see in Theorem (4.7) below, this is allowed to happen because the weight vector $(1, 1, 1)$ is on the *boundary* of the cones in question.

Here are some properties of C_G in the general situation.

(4.7) Theorem. *Let I be an ideal in $k[x_1, \ldots, x_n]$, and let G be a marked Gröbner basis of I.*

a. *The interior* $\mathrm{Int}(C_G)$ *of the cone C_G is a nonempty open subset of $\mathbb{R}^n$.*
b. *Let $>_M$ be any matrix order such that the first row of M lies in* $\mathrm{Int}(C_G)$. *Then G is the marked Gröbner basis of I with respect to $>_M$.*
c. *Let G' be a marked Gröbner basis of I different from G. Then the intersection $C_G \cap C_{G'}$ is contained in a boundary hyperplane of C_G, and similarly for $C_{G'}$.*
d. *The union of all the cones C_G, as G ranges over all marked Gröbner bases of I, is the positive orthant* $(\mathbb{R}^n)^+$.

Proof. To prove part a, fix a matrix order $>_M$ such that G is a marked Gröbner basis of I with respect to $>_M$ and let $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be the rows of M. We will show that $\mathrm{Int}(C_G)$ is nonempty by proving that

$$\mathbf{w} = \mathbf{w}_1 + \epsilon \mathbf{w}_2 + \cdots + \epsilon^{m-1} \mathbf{w}_m \in \mathrm{Int}(C_G) \tag{4.8}$$

provided $\epsilon > 0$ is sufficiently small. In Exercise 5, you will show that given exponent vectors α and β, we have

$$x^\alpha >_M x^\beta \Rightarrow \alpha \cdot \mathbf{w} > \beta \cdot \mathbf{w} \text{ provided } \epsilon > 0 \text{ is sufficiently small,}$$

where "sufficiently small" depends on α, β and M. It follows that we can arrange this for any finite set of pairs of exponent vectors. In particular, since $x^{\alpha(i)} = \mathrm{LT}_{>_M}(x^{\alpha(i)} + \sum_{i,\beta} c_{i,\beta} x^\beta)$, we can pick ϵ so that

$$\alpha(i) \cdot \mathbf{w} > \beta \cdot \mathbf{w} \text{ whenever } c_{i,\beta} \neq 0$$

in the notation of (4.4). Furthermore, using $x_i >_M 1$ for all i, we can also pick ϵ so that $\mathbf{e}_i \cdot \mathbf{w} > 0$ (where $\mathbf{e}_i$ is the ith standard basis vector). It follows that w is in the interior of the positive orthant. From here, $\mathbf{w} \in \mathrm{Int}(C_G)$ follows immediately.

For part b, let $>_M$ be a matrix order such that the first row of M lies in $\mathrm{Int}(C_G)$. This easily implies that for every $g \in G$, $\mathrm{LT}_{>_M}(g)$ is the marked term of g. From here, it is straightforward to show that G is the marked Gröbner basis of I with respect to $>_M$. See Exercise 6 for the details.

We now prove part c. In Exercise 7, you will show that if $C_G \cap C_{G'}$ contains interior points of either cone, then by part a it contains interior points of both cones. If $\mathbf{w}$ is such a point, we take any monomial order $>_M$ defined by a matrix with first row $\mathbf{w}$. Then by part b, G and G' are both the marked Gröbner bases of I with respect to $>_M$. The contradicts our assumption that $G \neq G'$.

Part d follows immediately from part b of Exercise 2. □

With more work, one can strengthen part c of Theorem (4.7) to show that $C_G \cap C_{G'}$ is a face of each (see [MR] or [Stu2] for a proof). It follows that as G ranges over all marked Gröbner bases of I, the collection formed by the cones C_G and their faces is a fan, as defined earlier in the section. This is the *Gröbner fan* of the ideal I.

For example, using the start made in Example (4.5) and Exercise 3, we can determine the Gröbner fan of the ideal I from (4.6). In small examples like this one, a reasonable strategy for producing the Gröbner fan is to find the monic (reduced) Gröbner bases for I with respect to "standard" orders (e.g., *grevlex* and *lex* orders with different permutations of the set of variables) first and determine the corresponding cones. Then if the union of the known cones is not all of the positive orthant, select some $\mathbf{w}$ in the complement, compute the monic Gröbner basis for $>_{\mathbf{w},grevlex}$, find the corresponding cone, and repeat this process until the known cones fill the positive orthant.

For the ideal of (4.6), there are seven cones in all, corresponding to the marked Gröbner bases:

(4.9)
$$\begin{aligned}
G^{(1)} &= \{\underline{x^2} - y, \underline{y^2} - xz - yz\}\\
G^{(2)} &= \{\underline{x^2} - y, \underline{yz} + xz - y^2\}\\
G^{(3)} &= \{\underline{x^4} - x^2z - xz, \underline{y} - x^2\}\\
G^{(4)} &= \{\underline{x^2} - y, \underline{xz} - y^2 + yz, \underline{y^2z} + xy^2 - y^3 - yz\}\\
G^{(5)} &= \{\underline{y^4} - 2y^3z + y^2z^2 - yz^2, \underline{xz} - y^2 + yz,\\
&\qquad \underline{xy^2} - y^3 + y^2z - yz, \underline{x^2} - y\}\\
G^{(6)} &= \{\underline{y^2z^2} - 2y^3z + y^4 - yz^2, \underline{xz} - y^2 + yz,\\
&\qquad \underline{xy^2} - y^3 + y^2z - yz, \underline{x^2} - y\}\\
G^{(7)} &= \{\underline{y} - x^2, \underline{x^2z} - x^4 + xz\}.
\end{aligned}$$

(Note that $G^{(5)}$ is the Gröbner basis from Exercise 3.)

Figure 4.2 below shows a picture of the slice of the Gröbner fan in the plane $a+b+c=1$, following the discussion from Example (4.5). The cones are labeled as in (4.9).

For instance, if the Gröbner bases $G^{(1)}, \ldots, G^{(6)}$ in this example are known, the "missing" region of the positive orthant contains (for instance) the vector $\mathbf{w} = (1/10, 2/5, 1/2)$ (see Figure 4.2). Using this weight vector, we find $G^{(7)}$, and the corresponding cone completes the Gröbner fan.

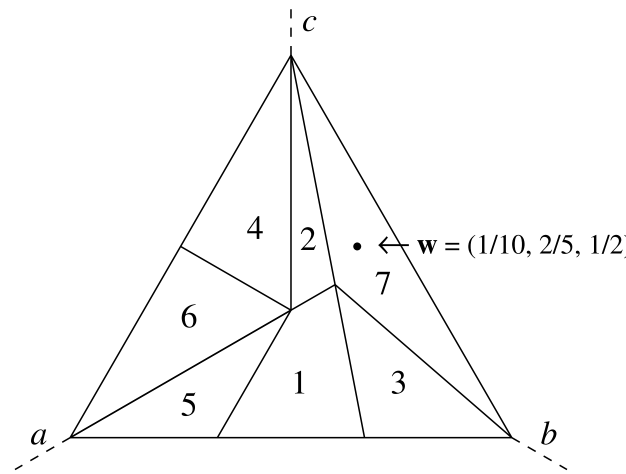

FIGURE 4.2. A Slice of the Gröbner Fan

When the number of variables is larger and/or the ideal generators have more terms, this method becomes much less tractable. Mora and Robbiano propose a "parallel Buchberger algorithm" in [MR] which produces the Gröbner fan by considering all potential identifications of leading terms in the computation and reduction of S-polynomials. But their method is certainly not practical on larger examples either. Gröbner fans can be extremely complicated! Fortunately, Gröbner fans are used primarily as conceptual tools—it is rarely necessary to compute large examples.

If we relax our requirement that $\mathbf{w}$ lie in the first orthant and only ask that $\mathbf{w}$ pick out the correct leading terms of a marked Gröbner basis of I, then we can allow weight vectors with negative entries. This leads to a larger "Gröbner fan" denoted $GF(I)$ in [Stu2]. Then the Gröbner fan of Theorem (4.7) (sometimes called the *restricted Gröbner fan*) is obtained by intersecting the cones of $GF(I)$ with the positive orthant. See [MR] and [Stu2] for more about what happens outside the positive orthant.

We close this section with a comment about a closely related topic. In the article [BM] which appeared at the same time as [MR], Bayer and Morrison introduced the *state polytope* of a homogeneous ideal. In a sense, this is the dual of the Gröbner fan $GF(I)$ (more precisely, the vertices of the state polytope are in one-to-one correspondence with the elements of $\mathrm{Mon}(I)$, and $GF(I)$ is the normal fan of the state polytope). The state polytope may also be seen as a generalization of the Newton polytope of a single homogeneous polynomial. See [BM] and [Stu2] for more details.

In the next section, we will see how the Gröbner fan can be used to develop a general Gröbner basis conversion algorithm that, unlike the FGLM algorithm from Chapter 2, does not depend on zero-dimensionality of I.

Additional Exercises for §4

Exercise 4. Using the proof of Proposition (4.1), prove Corollary (4.3).

Exercise 5. Assume that $x^\alpha >_M x^\beta$, where M is an $m \times n$ matrix giving the matrix order $>_M$. Also define $\mathbf{w}$ as in (4.8). Prove that $\alpha \cdot \mathbf{w} > \beta \cdot \mathbf{w}$ provided that $\epsilon > 0$ is sufficiently small.

Exercise 6. Fix a marked Gröbner basis G of an ideal I and let $>$ be a monomial order such that for each $g \in G$, $\text{LT}_>(g)$ is the marked term of the polynomial g. Prove that G is the marked Gröbner basis of I with respect to $>$. Hint: Divide $f \in I$ by G using the monomial order $>$.

Exercise 7. Show that if the intersection of two closed, n-dimensional convex polyhedral cones C, C' in $\mathbb{R}^n$ contains interior points of C, then the intersection also contains interior points of C'.

Exercise 8. Verify the computation of the Gröbner fan of the ideal from (4.6) by finding monomial orders corresponding to each of the seven Gröbner bases given in (4.9) and determining the cones $C_{G^{(k)}}$.

Exercise 9. Determine the Gröbner fan of the ideal of the affine twisted cubic curve: $I = \langle y - x^2, z - x^3 \rangle$. Explain why all of the cones have a common 1-dimensional edge in this example.

Exercise 10. This exercise will determine which terms in a polynomial $f = \sum_{i=1}^k c_i x^{\alpha(i)}$ can be $\text{LT}(f)$ with respect to some monomial order.
a. Show that $x^{\alpha(1)}$ is $\text{LT}(f)$ for some monomial order if and only if there is some vector $\mathbf{w}$ in the positive orthant such $(\alpha(1) - \alpha(j)) \cdot \mathbf{w} > 0$ for all $j = 2, \ldots, k$.
b. Show that such a $\mathbf{w}$ exists if and only if the origin is *not* in the convex hull of the set of all $(\alpha(1) - \alpha(j))$ for $j = 2, \ldots, k$, together with the standard basis vectors $\mathbf{e}_i$, $i = 1, \ldots, n$ in $\mathbb{R}^n$.
c. Use the result of part b to determine which terms in $f = x^2yz + 2xyw^2 + x^2w - xw + yzw + y^3$ can be $\text{LT}(f)$ for some monomial order. Determine an order that selects each of the possible leading terms.

Exercise 11. Determine the Gröbner fan of the following ideals:
a. $I = \langle x^3yz^2 - 2xy^3 - yz^3 + y^2z^2 + xyz \rangle$.
b. $I = \langle x - t^4, y - t^2 - t \rangle$.

§5 The Gröbner Walk

One interesting application of the Gröbner fan is a general Gröbner basis conversion algorithm known as the *Gröbner Walk*. As we saw in the discussion of the FGLM algorithm in Chapter 2, to find a Gröbner basis with respect to an "expensive" monomial order such as a *lex* order or another elimination order, it is often simpler to find some other Gröbner basis first, then convert it to a basis with respect to the desired order. The algorithm described Chapter 2 does this using linear algebra in the quotient algebra $k[x_1, \ldots, x_n]/I$, so it applies only to zero-dimensional ideals.

In this section, we will present the Gröbner Walk introduced by Collart, Kalkbrener, and Mall in [CKM]. This method converts a Gröbner basis for any ideal $I \subset k[x_1, \ldots, x_n]$ with respect to any one monomial order into a Gröbner basis with respect to any other monomial order. We will also give examples showing how the Walk applies to elimination problems encountered in implicitization.

The basic idea of the Gröbner Walk is pleasingly simple. Namely, we assume that we have a marked Gröbner basis G for I, say the marked Gröbner basis with respect to some monomial order $>_s$. We call $>_s$ the *starting order* for the walk, and we will assume that we have some matrix M_s with first row $\mathbf{w}_s$ representing $>_s$. By the results of the previous section, G corresponds to a cone C_G in the Gröbner fan of I.

The goal is to compute a Gröbner basis for I with respect to some other given *target order* $>_t$. This monomial order can be represented by some a matrix M_t with first row $\mathbf{w}_t$. Consider a "nice" (e.g., piecewise linear) path from $\mathbf{w}_s$ to $\mathbf{w}_t$ lying completely in the positive orthant in $\mathbb{R}^n$. For instance, since the positive orthant is convex, we could use the straight line segment between the two points, $(1-u)\mathbf{w}_s + u\mathbf{w}_t$ for $u \in [0,1]$, though this is not always the best choice. The Gröbner Walk consists of two basic steps:

- Crossing from one cone to the next.
- Computing the Gröbner basis of I corresponding to the new cone.

These steps are done repeatedly until the end of the path is reached, at which point we have the Gröbner basis with respect to the target order. We will discuss each step separately.

Crossing Cones

Assume we have the marked Gröbner basis G_{old} corresponding to the cone C_{old}, and a matrix M_{old} with first row $\mathbf{w}_{old}$ representing $>_{old}$. As we continue along the path from $\mathbf{w}_{old}$, let $\mathbf{w}_{new}$ be the *last point* on the path which lies in the cone C_{old}.

The new weight vector $\mathbf{w}_{new}$ may be computed as follows. Let $G_{old} = \{x^{\alpha(i)} + \sum_{i,\beta} c_{i,\beta} x^\beta : 1 \le i \le t\}$, where $x^{\alpha(i)}$ is the leading term with respect to $>_{M_{old}}$. To simplify notation, let $v_1, \ldots, v_m$ denote the vectors

$\alpha(i) - \beta$ where $1 \le i \le t$ and $c_{i,\beta} \neq 0$. By (4.4), C_{old} consists of those points in the positive orthant $(\mathbb{R}^n)^+$ for which

$$\mathbf{w} \cdot v_j \ge 0, \quad 1 \le j \le m.$$

For simplicity say that the remaining portion of the path to be traversed consists of the straight line segment from $\mathbf{w}_{old}$ to $\mathbf{w}_t$. Parametrizing this line as $(1-u)\mathbf{w}_{old} + u\mathbf{w}_t$ for $u \in [0,1]$, we see that the point for the parameter value u lies in C_{old} if and only if

$$(5.1) \qquad (1-u)(\mathbf{w}_{old} \cdot v_j) + u(\mathbf{w}_t \cdot v_j) \ge 0, \quad 1 \le j \le m.$$

Then $\mathbf{w}_{new} = (1-u_{last})\mathbf{w}_{old} + u_{last}\mathbf{w}_t$, where u_{last} is computed by the following algorithm:

(5.2)

Input: $\mathbf{w}_{old}, \mathbf{w}_t, v_1, \ldots, v_m$

Output: u_{last}

$u_{last} = 1$

FOR $j = 1, \ldots, m$ DO

IF $\mathbf{w}_t \cdot v_j < 0$ THEN $u_j := \dfrac{\mathbf{w}_{old} \cdot v_j}{\mathbf{w}_{old} \cdot v_j - \mathbf{w}_t \cdot v_j}$

IF $u_j < u_{last}$ THEN $u_{last} := u_j$

The idea behind (5.2) is that if $\mathbf{w}_t \cdot v_j \ge 0$, then (5.1) holds for all $u \in [0,1]$ since $\mathbf{w}_{old} \cdot v_j \ge 0$. On the other hand, if $\mathbf{w}_t \cdot v_j < 0$, then the formula for u_j gives the largest value of u such that (5.1) holds for this particular j. Note that $0 \le u_j < 1$ in this case.

Exercise 1. Prove carefully that $\mathbf{w}_{new} = (1-u_{last})\mathbf{w}_{old} + u_{last}\mathbf{w}_t$ is the last point on the path from $\mathbf{w}_{old}$ to $\mathbf{w}_t$ which lies in C_{old}.

Once we have $\mathbf{w}_{new}$, we need to choose the next cone in the Gröbner fan. Let $>_{new}$ be the weight order where we first compare $\mathbf{w}_{new}$-weights and break ties using the *target* order. Since $>_t$ is represented by M_t, it follows that $>_{new}$ is represented by $\binom{\mathbf{w}_{new}}{M_t}$. This gives the new cone C_{new}.

Furthermore, if we are in the situation where M_t is the bottom of the matrix representing $>_{old}$ (which is what happens in the Gröbner Walk), the following lemma shows that whenever $\mathbf{w}_{old} \neq \mathbf{w}_t$, the above process is guaranteed to move us closer to $\mathbf{w}_t$.

(5.3) Lemma. *Let u_{last} be as in (5.2) and assume that $>_{old}$ is represented by $\binom{\mathbf{w}_{old}}{M_t}$. Then $u_{last} > 0$.*

PROOF. By (5.2), $u_{last} = 0$ implies that $\mathbf{w}_{old} \cdot v_j = 0$ and $\mathbf{w}_t \cdot v_j < 0$ for some j. But recall that $v_j = \alpha(i) - \beta$ for some $g = x^{\alpha(i)} + \sum_{i,\beta} c_{i,\beta} x^\beta \in G$, where $x^{\alpha(i)}$ is the leading term for $>_{old}$ and $c_{i,\beta} \neq 0$. It follows that

$$(5.4) \qquad \mathbf{w}_{old} \cdot \alpha(i) = \mathbf{w}_{old} \cdot \beta \quad \text{and} \quad \mathbf{w}_t \cdot \alpha(i) < \mathbf{w}_t \cdot \beta.$$

Since $>_{old}$ is represented by $\binom{\mathbf{w}_{old}}{M_t}$, the equality in (5.4) tells us that $x^{\alpha(i)}$ and x^β have the same $\mathbf{w}_{old}$-weight, so that we break the tie using M_t. But $\mathbf{w}_t$ is the first row of M_t, so that the inequality in (5.4) implies that $x^{\alpha(i)}$ is not the leading term for $>_{old}$. This contradiction proves the lemma. □

Converting Gröbner Bases

Once we have crossed from C_{old} into the C_{new}, we need to convert the marked Gröbner basis G_{old} into a Gröbner basis for I with respect to the monomial order $>_{new}$ represented by $\binom{\mathbf{w}_{new}}{M_t}$. This is done as follows.

The key feature of $\mathbf{w}_{new}$ is that it lies on the boundary of C_{old}, so that some of the inequalities defining C_{old} become equalities. This means that the leading term of some $g \in G_{old}$ has the same $\mathbf{w}_{new}$-weight as some other term in g. In general, given a weight vector $\mathbf{w}$ is the positive orthant $(\mathbb{R}^n)^+$ and a polynomial $f \in k[x_1, \ldots, x_n]$, the *initial form* of f for $\mathbf{w}$, denoted $\mathrm{in}_{\mathbf{w}}(f)$, is the sum of all terms in f of maximum $\mathbf{w}$-weight. Also, given a set S of polynomials, we let $\mathrm{in}_{\mathbf{w}}(S) = \{\mathrm{in}_{\mathbf{w}}(f) : f \in S\}$.

Using this notation, we can form the ideal

$$\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle$$

of $\mathbf{w}_{new}$-initial forms of elements of G_{old}. Note that $\mathbf{w}_{new} \in C_{old}$ guarantees that the marked term of $g \in G_{old}$ appears in $\mathrm{in}_{\mathbf{w}_{new}}(g)$. The important thing to realize here is that in nice cases, $\mathrm{in}_{\mathbf{w}_{new}}(G_{old})$ consists mostly of monomials, together with a small number of polynomials (in the best case, only one binomial together with a collection of monomials).

It follows that finding a monic Gröbner basis

$$H = \{h_1, \ldots, h_s\}$$

of $\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle$ with respect to $>_{new}$ may usually be done very quickly. The surprise is that once we have H, it is relatively easy to convert G_{old} into the desired Gröbner basis.

(5.5) Proposition. *Let G_{old} be the marked Gröbner basis for an ideal I with respect to $>_{old}$. Also let $>_{new}$ be represented by $\binom{\mathbf{w}_{new}}{M_t}$, where $\mathbf{w}_{new}$ is any weight vector in C_{old}, and let H be the monic Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle$ with respect to $>_{new}$ as above. Express each $h_j \in H$ as*

$$h_j = \sum_{g\in G_{old}} p_{j,g}\, \mathrm{in}_{\mathbf{w}_{new}}(g). \tag{5.6}$$

Then replacing the initial forms by the g themselves, the polynomials

$$\overline{h}_j = \sum_{g\in G_{old}} p_{j,g}\, g, \quad 1 \le j \le s, \tag{5.7}$$

form a Gröbner basis of I with respect to $>_{new}$.

Before giving the proof, we need some preliminary observations about weight vectors and monomial orders. A polynomial f is $\mathbf{w}$-*homogeneous* if $f = \mathrm{in}_{\mathbf{w}}(f)$. In other words, all terms of f have the same $\mathbf{w}$-weight. Furthermore, every polynomial can be written uniquely as a sum of $\mathbf{w}$-homogeneous polynomials which are its $\mathbf{w}$-*homogeneous components* (see Exercise 5).

We say that a weight vector $\mathbf{w}$ is *compatible* with a monomial order $>$ if $\mathrm{LT}_>(f)$ appears in $\mathrm{in}_{\mathbf{w}}(f)$ for all nonzero polynomials f. Then we have the following result.

(5.8) Lemma. *Fix* $\mathbf{w} \in (\mathbb{R}^n)^+ \setminus \{0\}$ *and let* G *be the marked Gröbner basis of an ideal* I *for a monomial order* $>$.

a. *If* $\mathbf{w}$ *is compatible with* $>$, *then* $\mathrm{LT}_>(I) = \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{LT}_>(\langle \mathrm{in}_{\mathbf{w}}(I)\rangle)$.

b. *If* $\mathbf{w} \in C_G$, *then* $\mathrm{in}_{\mathbf{w}}(G)$ *is a Gröbner basis of* $\langle \mathrm{in}_{\mathbf{w}}(I)\rangle$ *for* $>$. *In particular,*

$$\langle \mathrm{in}_{\mathbf{w}}(I)\rangle = \langle \mathrm{in}_{\mathbf{w}}(G)\rangle.$$

PROOF. For part a, the first equality $\mathrm{LT}_>(I) = \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(I))$ is obvious since the leading term of any $f \in k[x_1, \ldots, x_n]$ appears in $\mathrm{in}_{\mathbf{w}}(f)$. For the second equality, it suffices to show $\mathrm{LT}_>(f) \in \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(I))$ whenever $f \in \langle \mathrm{in}_{\mathbf{w}}(I)\rangle$. Given such an f, write it as

$$f = \sum_{i=1}^{t} p_i \, \mathrm{in}_{\mathbf{w}}(f_i), \quad p_i \in k[x_1, \ldots, x_n],\ f_i \in I.$$

Each side is a sum of $\mathbf{w}$-homogeneous components. Since $\mathrm{in}_{\mathbf{w}}(f_i)$ is already $\mathbf{w}$-homogeneous, this implies that

$$\mathrm{in}_{\mathbf{w}}(f) = \sum_{i=1}^{t} q_i \, \mathrm{in}_{\mathbf{w}}(f_i),$$

where we can assume that q_i is $\mathbf{w}$-homogeneous and f and $q_i f_i$ have the same $\mathbf{w}$-weight for all i. It follows that $\mathrm{in}_{\mathbf{w}}(f) = \mathrm{in}_{\mathbf{w}}(\sum_{i=1}^{t} q_i\, f_i) \in \mathrm{in}_{\mathbf{w}}(I)$. Then compatibility implies $\mathrm{LT}_>(f) = \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(f)) \in \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(I))$.

Turning to part b, first assume that $\mathbf{w}$ is compatible with $>$. Then

$$\langle \mathrm{LT}_>(I)\rangle = \langle \mathrm{LT}_>(G)\rangle = \langle \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(G))\rangle,$$

where the first equality follows since G is a Gröbner basis for $>$ and the second follows since $\mathbf{w}$ is compatible with $>$. Combining this with part a, we see that $\langle \mathrm{LT}_>(\langle \mathrm{in}_{\mathbf{w}}(I)\rangle)\rangle = \langle \mathrm{LT}_>(\mathrm{in}_{\mathbf{w}}(G))\rangle$. Hence $\mathrm{in}_{\mathbf{w}}(G)$ is a Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}}(I)\rangle$ for $>$, and the final assertion of the lemma follows.

It remains to consider what happens when $\mathbf{w} \in C_G$, which does not necessarily imply that $\mathbf{w}$ is compatible with $>$ (see Exercise 6 for an example). Consider the weight order $>'$ which first compares $\mathbf{w}$-weights and breaks ties using $>$. Note that $\mathbf{w}$ is compatible with $>'$.

The key observation is that since $\mathbf{w} \in C_G$, the leading term of each $g \in G$ with respect to $>'$ is the marked term. By Exercise 6 of §4, it follows that G is the marked Gröbner basis of I for $>'$. Since $\mathbf{w}$ is compatible with $>'$, the earlier part of the argument implies that $\mathrm{in}_{\mathbf{w}}(G)$ is a Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}}(I)\rangle$ for $>'$. However, for each $g \in G$, $\mathrm{in}_{\mathbf{w}}(g)$ has the same leading term with respect to $>$ and $>'$. Using Exercise 6 of §4 again, we conclude that $\mathrm{in}_{\mathbf{w}}(G)$ is a Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}}(I)\rangle$ for $>$. □

We can now prove the proposition.

PROOF OF PROPOSITION (5.5). We will give the proof in three steps. Since $>_{new}$ is represented by $\binom{\mathbf{w}_{new}}{M_t}$, $\mathbf{w}_{new}$ is compatible with $>_{new}$. By part a of Lemma (5.8), we obtain

$$\mathrm{LT}_{>_{new}}(I) = \mathrm{LT}_{>_{new}}(\langle \mathrm{in}_{\mathbf{w}_{new}}(I)\rangle).$$

The second step is to observe that since $\mathbf{w}_{new} \in C_{old}$, the final assertion of part b of Lemma (5.8) implies

$$\langle \mathrm{in}_{\mathbf{w}_{new}}(I)\rangle = \langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle.$$

For the third step, we show that

$$\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle = \langle \mathrm{LT}_{>_{new}}(H)\rangle = \langle \mathrm{LT}_{>_{new}}(\overline{H})\rangle,$$

where $H = \{h_1, \ldots, h_t\}$ is the given Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle$ and $\overline{H} = \{\overline{h}_1, \ldots, \overline{h}_t\}$ is the set of polynomials described in the statement of the proposition. The equality is obvious, and for the second, it suffices to show that for each j, $\mathrm{LT}_{>_{new}}(h_j) = \mathrm{LT}_{>_{new}}(\overline{h}_j)$. Since the $\mathrm{in}_{\mathbf{w}_{new}}(g)$ are $\mathbf{w}_{new}$-homogeneous, Exercise 7 below shows that the same is true of the h_j and the $q_{j,g}$. Hence for each g, all terms in $q_{j,g}(g - \mathrm{in}_{\mathbf{w}_{new}}(g))$ have smaller $\mathbf{w}_{new}$ weight than those in the initial form. Lifting as in (5.7) to get $\overline{h}_j$ adds only terms with smaller $\mathbf{w}_{new}$ weight. Since $>_{new}$ is compatible with $\mathbf{w}_{new}$, the added terms are also smaller in the new order, so the $>_{new}$-leading term of $\overline{h}_j$ is the same as the leading term of h_j.

Combining the three steps, we obtain

$$\langle \mathrm{LT}_{>_{new}}(I)\rangle = \langle \mathrm{LT}_{>_{new}}(\overline{H})\rangle.$$

Since $\overline{h}_j \in I$ for all j, we conclude that $\overline{H}$ is a Gröbner basis for I with respect to $>_{new}$, as claimed. □

The Gröbner basis $\overline{H}$ from Proposition (5.5) is minimal, but not necessarily reduced. Hence a complete interreduction is usually necessary to obtain the marked Gröbner basis G_{new} corresponding to the next cone. In practice, this is a relatively quick process.

In order to use Proposition (5.5), we need to find the polynomials $p_{j,g}$ in (5.6) expressing the Gröbner basis elements h_j in terms of the ideal generators of $\mathrm{in}_{\mathbf{w}_{new}}(G_{old})$. This can be done in two ways:

- First, the $p_{j,g}$ can be computed along with H by an extended Buchberger algorithm (see for instance [BW], Chapter 5, Section 6).
- Second, since $\mathrm{in}_{\mathbf{w}_{new}}(G_{old})$ is a Gröbner basis of $\langle \mathrm{in}_{\mathbf{w}_{new}}(G_{old})\rangle$ with respect to $>_{old}$ by part b of Lemma (5.8), the $p_{j,g}$ can be obtained by dividing h_j by $\mathrm{in}_{\mathbf{w}_{new}}(G_{old})$ using $>_{old}$.

In practice, the second is often more convenient to implement. The process of replacing the $\mathbf{w}_{new}$-initial forms of the g by the g themselves to go from (5.6) to (5.7) is called *lifting* the initial forms to the new Gröbner basis.

The Algorithm

The following algorithm is a basic Gröbner Walk, following the straight line segment from $\mathbf{w}_s$ to $\mathbf{w}_t$.

(5.9) Theorem. *Let*

1. **NextCone** *be a procedure that computes u_{last} from (5.2). Recall that $\mathbf{w}_{new} = (1 - u_{last})\mathbf{w}_{old} + u_{last}\mathbf{w}_t$ is the last weight vector along the path which lies in the cone C_{old} of the previous Gröbner basis G_{old}.*
2. **Lift** *be a procedure that lifts a Gröbner basis for the $\mathbf{w}_{new}$-initial forms of the previous Gröbner basis G_{old} with respect to $>_{new}$ to the Gröbner basis G_{new} following Proposition (5.5).*
3. **Interreduce** *be a procedure that takes a given set of polynomials and interreduces them with respect to a given monomial order.*

Then the following algorithm correctly computes a Gröbner basis for I with respect to $>_t$ and terminates in finitely many steps on all inputs:

Input: M_s and M_t representing start and target orders with first rows $\mathbf{w}_s$ and $\mathbf{w}_t, G_s$ = Gröbner basis with respect to $>_{M_s}$

Output: last value of G_{new} = Gröbner basis with respect to $>_{M_t}$

$M_{old} := M_s$

$G_{old} := G_s$

$\mathbf{w}_{new} := \mathbf{w}_s$

$M_{new} := \begin{pmatrix} \mathbf{w}_{new} \\ M_t \end{pmatrix}$

$done := false$

WHILE $done = false$ DO

  $In := \mathrm{in}_{\mathbf{w}_{new}}(G_{old})$

  $InG := \mathrm{gbasis}(In, >_{M_{new}})$

$G_{new} :=$ Lift$(InG, G_{old}, In, M_{new}, M_{old})$

$G_{new} :=$ Interreduce(G_{new}, M_{new})

$u :=$ NextCone$(G_{new}, \mathbf{w}_{new}, \mathbf{w}_t)$

IF $\mathbf{w}_{new} = \mathbf{w}_t$ THEN

  $done := true$

ELSE

  $M_{old} := M_{new}$

  $G_{old} := G_{new}$

  $\mathbf{w}_{new} := (1-u)\mathbf{w}_{new} + u\mathbf{w}_t$

  $M_{new} := \begin{pmatrix} \mathbf{w}_{new} \\ M_t \end{pmatrix}$

RETURN(G_{new})

PROOF. We traverse the line segment from $\mathbf{w}_s$ to $\mathbf{w}_t$. To prove termination, observe that by Corollary (4.3), the Gröbner fan of $I = \langle G_s \rangle$ has only finitely many cones, each of which has only finitely many bounding hyperplanes as in (4.4). Discarding those hyperplanes which contain line segment from $\mathbf{w}_s$ to $\mathbf{w}_t$, the remaining hyperplanes determine a finite set of distingushed points on our line segment.

Now consider u_{last} = NextCone$(G_{new}, \mathbf{w}_{new}, \mathbf{w}_t)$ as in the alorithm. This uses (5.2) with $\mathbf{w}_{old}$ replaced by the current value of $\mathbf{w}_{new}$. Furthermore, notice tha the monomial order always comes from a matrix of the form $\binom{\mathbf{w}_s}{M_t}$. It follows that the hypothesis of Lemma (5.3) is always satisfied. If $u_{last} = 1$, then the next value of $\mathbf{w}_{new}$ is $\mathbf{w}_t$, so that the algorithm terminates after one more pass through the main loop. On the other hand, if $u_{last} = u_j < 1$, then the next value of $\mathbf{w}_{new}$ lies on the hyperplane $\mathbf{w} \cdot v_j = 0$, which is one of our finitely many hyperplanes. However, (5.2) implies that $\mathbf{w}_t \cdot v_j < 0$ and $\mathbf{w}_{new} \cdot v_j \geq 0$, so that the hyperplane meets the line segment in a single point. Hence the next value of $\mathbf{w}_{new}$ is one of our distingushed points. Furthermore, Lemma (5.3) implies that $u_{last} > 0$, so that if the current $\mathbf{w}_{new}$ differs from $\mathbf{w}_t$, then we must move to a distingushed point further along the line segment. Hence we must eventually reach $\mathbf{w}_t$, at which point the algorithm terminates.

To prove correctness, observe that in each pass through the main loop, the hypotheses of Proposition (5.5) are satisfied. Furthermore, once the value of $\mathbf{w}_{new}$ reaches $\mathbf{w}_t$, the next pass through the loop computes a Gröbner basis of I for the monomial order represented by $\binom{\mathbf{w}_t}{M_t}$. Using Exercise 6 of §4, it follows that the final value of G_{new} is the marked Gröbner basis for $>_t$. □

The complexity of the Gröbner Walk depends most strongly on the number of cones that are visited along the path through the Gröbner fan, and the number of different cones that contain the point $\mathbf{w}_{new}$ at each step. We will say more about this in the examples below.

Examples

We begin with a simple example of the Gröbner Walk in action. Consider the ideal $I = \langle x^2 - y, xz - y^2 + yz \rangle \subset \mathbb{Q}[x, y, z]$ from (4.6). We computed the full Gröbner fan for I in §4 (see Figure 4.2). Say we know

$$G_s = G^{(1)} = \{\underline{x^2} - y, \underline{y^2} - xz - yz\}$$

from (4.9). This is the Gröbner basis of I with respect to $>_{(5,4,1),grevlex}$ (among many others!). Suppose we want to determine the Gröbner basis with respect to $>_{(6,1,3),lex}$ (which is $G^{(6)}$). We could proceed as follows. Let

$$M_s = \begin{pmatrix} 5 & 4 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so $\mathbf{w}_s = (5, 4, 1)$. Following Exercise 6 from Chapter 1, §2, we have used a square matrix defining the same order instead of the 4×3 matrix with first row $(5, 4, 1)$ and the next three rows from a 3×3 matrix defining the *grevlex* order (as in part b of Exercise 6 of Chapter 1, §2). Similarly,

$$M_t = \begin{pmatrix} 6 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\mathbf{w}_t = (6, 1, 3)$. We will choose square matrices defining the appropriate monomial orders in all of the following computations by deleting appropriate linearly dependent rows.

We begin by considering the order defined by

$$M_{new} = \begin{pmatrix} 5 & 4 & 1 \\ 6 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

(using the weight vector $\mathbf{w}_{new} = (5, 4, 1)$ first, then refining by the target order). The $\mathbf{w}_{new}$ initial forms of the Gröbner basis polynomials with respect to this order are the same as those for G_s, so the basis does not change in the first pass through the main loop.

We then call the NextCone procedure (5.2) with $\mathbf{w}_{new}$ in place of $\mathbf{w}_{old}$. The cone of $>_{M_{new}}$ is defined by the three inequalities obtained by comparing x^2 vs. y and y^2 vs. xz and yz. By (5.2), u_{last} is the largest u such that $(1-u)(5,4,1)+u(6,1,3)$ lies in this cone and is computed as follows:

x^2 vs. y :

$v_1 = (2,-1,0), \mathbf{w}_t \cdot v_1 = 6 \geq 0 \Rightarrow u_1 = 1$

y^2 vs. xz :

$v_2 = (-1,2,-1), \mathbf{w}_t \cdot v_2 = -7 < 0 \Rightarrow u_2 = \frac{\mathbf{w}_{new}\cdot v_2}{\mathbf{w}_{new}\cdot v_2 - (-7)} = \frac{2}{9}$

y^2 vs. yz :

$v_2 = (0,-,-1), \mathbf{w}_t \cdot v_3 = -2 < 0 \Rightarrow u_3 = \frac{\mathbf{w}_{new}\cdot v_3}{\mathbf{w}_{new}\cdot v_3 - (-2)} = \frac{3}{5}.$

The smallest u value here is $u_{last} = \frac{2}{9}$. Hence the new weight vector is $\mathbf{w}_{new} = (1-\frac{2}{9})(5,4,1) + \frac{2}{9}(6,1,3) = (47/9, 10/3, 13/9)$, and M_{old} and

$$M_{new} = \begin{pmatrix} 47/9 & 10/3 & 13/9 \\ 6 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

are updated for the next pass through the main loop.

In the second pass, $In = \{y^2 - xz, x^2\}$. We compute the Gröbner basis for $\langle In \rangle$ with respect to $>_{new}$ (with respect to this order, the leading term of the first element is xz), and find

$$H = \{-y^2 + xz, x^2, xy^2, y^4\}.$$

In terms of the generators for $\langle In \rangle$, we have

$$\begin{aligned} -y^2 + xz &= -1 \cdot (y^2 - xz) + 0 \cdot (x^2) \\ x^2 &= 0 \cdot (y^2 - xz) + 1 \cdot (x^2) \\ xy^2 &= x \cdot (y^2 - xz) + z \cdot (x^2) \\ y^4 &= (y^2 + xz) \cdot (y^2 - xz) + z^2 \cdot (x^2). \end{aligned}$$

So by Proposition (5.5), to get the next Gröbner basis, we lift to

$$\begin{aligned} -1 \cdot (y^2 - xz - yz) + 0 \cdot (x^2 - y) &= xz + yz - y^2 \\ 0 \cdot (y^2 - xz - yz) + 1 \cdot (x^2 - y) &= x^2 - y \\ x \cdot (y^2 - xz - yz) + z \cdot (x^2 - y) &= xy^2 - xyz - yz \\ (y^2 + xz) \cdot (y^2 - xz - yz) + z^2 \cdot (x^2 - y) &= y^4 - y^3 z - xyz^2 - yz^2. \end{aligned}$$

Interreducing with respect to $>_{new}$, we obtain the marked Gröbner basis G_{new} given by

$$\{\underline{xz} + yz - y^2, \underline{x^2} - y, \underline{xy^2} - y^3 + y^2 z - yz, \underline{y^4} - 2y^3 z + y^2 z^2 - yz^2\}.$$

(This is $G^{(5)}$ in (4.9).) For the call to NextCone in this pass, we use the parametrization $(1-u)(47/9, 10/3, 13/9) + u(6,1,3)$. Using (5.2) as above, we obtain $u_{last} = 17/35$, for which $\mathbf{w}_{new} = (28/5, 11/5, 11/5)$.

In the third pass through the main loop, the Gröbner basis does not change as a set. However, the leading term of the initial form of the last polynomial $y^4 - 2y^3z + y^2z^2 - yz^2$ with respect to $>_{M_{new}}$ is now y^2z^2 since

$$M_{new} = \begin{pmatrix} 28/5 & 11/5 & 11/5 \\ 6 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}.$$

Using Proposition (5.5) as usal to compute the new Gröbner basis G_{new}, we obtain

$$(5.10)\quad \{\underline{xz} + yz - y^2, \underline{x^2} - y, \underline{xy^2} - y^3 + y^2z - yz, \underline{y^2z^2} - 2y^3z + y^4 - yz^2\},$$

which is $G^{(6)}$ in (4.9). The call to NextCone returns $u_{last} = 1$, since there are no pairs of terms that attain equal weight for any point on line segment parametrized by $(1-u)(75/13, 22/13, 33/13) + u(6, 1, 3)$. Thus $\mathbf{w}_{new} = \mathbf{w}_t$. After one more pass through the main loop, during which G_{new} doesn't change, the algorithm terminates. Hence the final output is (5.10), which is the marked Gröbner basis of I with respect to the target order.

We note that it is possible to modify the algorithm of Theorem (5.9) so that the final pass in the above example doesn't occur. See Exercise 8.

Exercise 2. Verify the computation of u_{last} in the steps of the above example after the first.

Exercise 3. Apply the Gröbner Walk to convert the basis $G^{(3)}$ for the above ideal to the basis $G^{(4)}$ (see (4.9) and Figure (4.2)). Take $>_s = >_{(2,7,1),grevlex}$ and $>_t = >_{(3,1,6),grevlex}$.

Many advantages of the Walk are lost if there are many terms in the $\mathbf{w}_{new}$ initial forms. This tends to happen if a portion of the path lies in a face of some come, or if the path passes through points where many cones intersect. Hence in [AGK], Amrhein, Gloor, and Küchlin make systematic use of perturbations of weight vectors to keep the path in as general a position as possible with respect to the faces of the cones. For example, one possible variant of the basic algorithm above would be to use (4.8) to obtain a perturbed weight vector in the interior of the corresponding cone each time a new marked Gröbner basis is obtained, and resume the walk to the target monomial order from there. Another variant designed for elimination problems is to take a "sudden-death" approach. If we want a Gröbner basis with respect to a monomial order eliminating the variables $x_1, \ldots, x_n$, leaving $y_1, \ldots, y_m$, and we expect a single generator for the elimination ideal, then we could terminate the walk as soon as some polynomial in $k[y_1, \ldots, y_m]$ appears in the current G_{new}. This is only guaranteed to be a multiple of the generator of the elimination ideal, but even a polynomial satisfying that condition can be useful in some circumstances. We refer the interested reader to [AGK] for a discussion of other implementation issues.

In [Tran], degree bounds on elements of Gröbner bases are used to produce weight vectors in the interior of each cone of the Gröbner fan, which gives a deterministic way to find good path perturbations. A theoretical study of the complexity of the Gröbner Walk and other basis conversion algorithms has been made by Kalkbrener in [Kal].

Our next example is an application of the Gröbner Walk algorithm to an implicitization problem inspired by examples studied in robotics and computer aided design. Let C_1 and C_2 be two curves in $\mathbb{R}^3$. The *bisector surface* of C_1 and C_2 is the locus of points P equidistant from C_1 and C_2 (that is, P is on the bisector if the closest point(s) to P on C_1 and C_2 are the same distance from P.) See, for instance, [EK]. Bisectors are used, for instance, in motion planning to find paths avoiding obstacles in an environment. We will consider only the case where C_1 and C_2 are smooth complete intersection algebraic curves $C_1 = \mathbf{V}(f_1, g_1)$ and $C_2 = \mathbf{V}(f_2, g_2)$. (This includes most of the cases of interest in solid modeling, such as lines, circles and other conics, etc.) $P = (x, y, z)$ is on the bisector of C_1 and C_2 if there exist $Q_1 = (x_1, y_1, z_1) \in C_1$ and $Q_2 = (x_2, y_2, z_2) \in C_2$ such that the distance from P to C_i is a minimum at Q_i, $i = 1, 2$ and the distance from P to Q_1 equals the distance from P to Q_2. Rather than insisting on an absolute minimum of the distance function from P to C_i at Q_i, it is simpler to insist that the distance function simply have a critical point there. It is easy to see that this condition is equivalent to saying that the line segment from P to Q_i is *orthogonal* to the tangent line to C_i at Q_i.

Exercise 4. Show that the distance from C_i to P has a critical point at Q_i if and only if the line segment from P to Q_i is *orthogonal* to the tangent line to C_i at Q_i, and show that this is equivalent to saying that

$$(\nabla f_i(Q_i) \times \nabla g_i(Q_i)) \cdot (P - Q_i) = 0,$$

where $\nabla f_i(Q_i)$ denotes the gradient vector of f_i at Q_i, and $\times$ is the cross product in $\mathbb{R}^3$.

By Exercise 4, we can find the bisector as follows. Let (x_i, y_i, z_i) be a general point Q_i on C_i, and $P = (x, y, z)$. Consider the system of equations

$$\begin{aligned}
0 &= f_1(x_1, y_1, z_1)\\
0 &= g_1(x_1, y_1, z_1)\\
0 &= f_2(x_2, y_2, z_2)\\
0 &= g_2(x_2, y_2, z_2)\\
0 &= (\nabla f_1(x_1, y_1, z_1) \times \nabla g_1(x_1, y_1, z_1)) \cdot (x - x_1, y - y_1, z - z_1)\\
0 &= (\nabla f_2(x_2, y_2, z_2) \times \nabla g_2(x_2, y_2, z_2)) \cdot (x - x_2, y - y_2, z - z_2)\\
0 &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\\
&\quad - (x - x_2)^2 - (y - y_2)^2 - (z - z_2)^2
\end{aligned} \tag{5.11}$$

Let $J \subset \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2, x, y, z]$ be the ideal generated by these 7 equations. We claim the bisector will be contained in $\mathbf{V}(I)$, where I is the elimination ideal $I = J \cap \mathbb{R}[x, y, z]$. A proof proceeds as follows: $P = (x, y, z)$ is on the bisector of C_1 and C_2 if and only if there exist $Q_i = (x_i, y_i, z_i)$ such that $Q_i \in C_i$, Q_i is a minimum of the distance function to P, restricted to C_i, and $PQ_1 = PQ_2$. Thus P is in the bisector if and only if the equations in (5.11) are satisfied for some $(x_i, y_i, z_i) \in C_i$. Therefore, P is the projection of some point in $\mathbf{V}(J)$, hence in $\mathbf{V}(I)$. Note that (5.11) contains 7 equations in 9 unknowns, so we expect that $\mathbf{V}(J)$ and its projection $\mathbf{V}(I)$ have dimension 2 in general.

For instance, if C_1 is the twisted cubic $\mathbf{V}(y - x^2, z - x^3)$ and C_2 is the line $\mathbf{V}(x, y - 1)$, then our ideal J is

$$\begin{aligned} J = \langle & y_1 - x_1^2, z_1 - x_1^3, x_2, y_2 - 1, \\ & x - x_1 + 2x_1(y - y_1) + 3x_1^2(z - z_1), z - z_2, \\ & (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \\ & - (x - x_2)^2 - (y - y_2)^2 - (z - z_2)^2 \rangle \end{aligned} \tag{5.12}$$

We apply the Gröbner Walk with $>_s$ the *grevlex* order with $x_1 > y_1 > z_1 > x_2 > y_2 > z_2 > x > y > z$, and $>_t$ the $>_{\mathbf{w},grevlex}$ order, where $\mathbf{w} = (1, 1, 1, 1, 1, 1, 0, 0, 0)$, which has the desired elimination property to compute $J \cap \mathbb{R}[x, y, z]$.

Using our own (somewhat naive) implementation of the Gröbner Walk based on the `Groebner` package in Maple, we computed the $>_{\mathbf{w},grevlex}$ basis for J as in (5.12). As we expect, the elimination ideal is generated by a single polynomial: $J \cap \mathbb{R}[x, y, z] =$

$$\begin{aligned} &\langle 5832z^6y^3 - 729z^8 - 34992x^2y - 14496yxz - 14328x^2z^2 \\ &+ 24500x^4y^2 - 23300x^4y + 3125x^6 - 5464z^2 - 36356z^4y \\ &+ 1640xz^3 + 4408z^4 + 63456y^3xz^3 + 28752y^3x^2z^2 \\ &- 201984y^3 - 16524z^6y^2 - 175072y^2z^2 + 42240y^4xz - 92672y^3zx \\ &+ 99956z^4y^2 + 50016yz^2 + 90368y^2 + 4712x^2 + 3200y^3x^3z \\ &+ 6912y^4xz^3 + 13824y^5zx + 19440z^5xy^2 + 15660z^3x^3y + 972z^4x^2y^2 \\ &+ 6750z^2x^4y - 61696y^2z^3x + 4644yxz^5 - 37260yz^4x^2 \\ &- 85992y^2x^2z^2 + 5552x^4 - 7134xz^5 + 64464yz^2x^2 \\ &- 5384zyx^3 + 2960zy^2x^3 - 151z^6 + 1936 \\ &+ 29696y^6 + 7074z^6y + 18381z^4x^2 - 2175z^2x^4 + 4374xz^7 \\ &+ 1120zx - 7844x^3z^3 - 139264y^5 - 2048y^7 - 1024y^6z^2 \\ &- 512y^5x^2 - 119104y^3x^2 - 210432y^4z^2 + 48896y^5z^2 \\ &- 104224y^3z^4 + 28944y^4z^4 + 54912y^4x^2 - 20768y + 5832z^5x^3 \\ &- 8748z^6x^2 + 97024y^2x^2 + 58560y^2zx + 240128y^4 + 286912y^3z^2 \\ &+ 10840xyz^3 + 1552x^3z - 3750zx^5 \rangle. \end{aligned}$$

The computation of the full Gröbner basis (including the initial computation of the *grevlex* Gröbner basis of J) took 43 seconds on a 866MHz Pentium III using the Gröbner Walk algorithm described in Theorem (5.9). Apparently the cones corresponding to the two monomial orders $>_s, >_t$ are very close together in the Gröbner fan for J, a happy accident. The $\mathbf{w}_{new}$-initial forms in the second step of the Walk contained a large number of distinct terms, though. With the "sudden death" strategy discussed above, the time was reduced to 23 seconds and produced the same polynomial (not a multiple). By way of contrast, a direct computation of the $>_{\mathbf{w},grevlex}$ Gröbner basis using the `gbasis` command of the `Groebner` package was terminated after using 20 minutes of CPU time and over 200Mb of memory. In our experience, in addition to gains in speed, the Gröbner Walk tends also to use much less memory for storing intermediate polynomials than Buchberger's algorithm with an elimination order. This means that even if the Walk takes a long time to complete, it will often execute successfully on complicated examples that are not feasible using the Gröbner basis packages of standard computer algebra systems. Similarly encouraging results have been reported from several experimental implementations of the Gröbner Walk.

As of this writing, the Gröbner Walk has not been included in the Gröbner basis packages distributed with general purpose computer algebra systems such as Maple or *Mathematica*. An implementation is available in Magma, however. The CASA Maple package developed at RISC-Linz (see `http://www.risc.uni-linz.ac.at/software/casa/`) also contains a Gröbner Walk procedure.

ADDITIONAL EXERCISES FOR §5

Exercise 5. Fix a nonzero weight vector $\mathbf{w} \in (\mathbb{R}^n)^+$. Show that every $f \in k[x_1, \ldots, x_n]$ can be written uniquely as a sum of $\mathbf{w}$-homogeneous polynomials.

Exercise 6. Fix a monomial order $>$ and a nonzero weight vector $\mathbf{w} \in (\mathbb{R}^n)^+$. Also, given an ideal $I \subset k[x_1, \ldots, x_n]$, let $C_>$ be the cone in the Gröbner fan of I corresponding to $\langle \mathrm{LT}_>(I) \rangle \in \mathrm{Mon}(I)$.

a. Prove that $\mathbf{w}$ is compatible with $>$ if and only if $\mathbf{w} \cdot \alpha > \mathbf{w} \cdot \beta$ always implies $x^\alpha > x^\beta$.
b. Prove that if $\mathbf{w}$ is compatible with $>$, then $\mathbf{w} \in C_>$.
c. Use the example of $>_{lex}$ for $x > y$, $I = \langle x + y \rangle \subset k[x, y]$ and $\mathbf{w} = (1, 1)$ to show that the naive converse to part b is false. (See part d for the real converse.)
d. Prove that $\mathbf{w} \in C_>$ if and only if there is a monomial order $>'$ such that $C_{>'} = C_>$ and $\mathbf{w}$ is compatible with $>'$. Hint: See the proof of part b of Lemma (5.8).

Exercise 7. Suppose that J is an ideal generated by $\mathbf{w}$-homogeneous polynomials. Show that every reduced Gröbner basis of I consists of $\mathbf{w}$-homogeneous polynomials. Hint: This is generalizes the corresponding fact for homogeneous ideals. See [CLO], Theorem 2 of Chapter 8, §3.

Exercise 8. It is possible to get a slightly more efficient version of the algorithm described in Theorem (5.9). The idea is to modify (5.2) so that u_{last} is allowed to be greater than 1 if the ray from $\mathbf{w}_{old}$ to $\mathbf{w}_t$ leaves the cone at a point beyond $\mathbf{w}_t$.

a. Modify (5.2) so that it behaves as described above and prove that your modification behaves as claimed.
b. Modify the algorithm described in Theorem (5.9) in two ways: first, $\mathbf{w}_{new}$ is defined using $\min\{1, u_{last}\}$ and second, the IF statement tests whether $u_{last} > 1$ or $\mathbf{w}_{new} = \mathbf{w}_t$. Prove that this modified algorithm correctly converts G_s to G_t.
c. Show that the modified algorithm, when applied to the ideal $I = \langle x^2 - y, y^2 - xz - yz\rangle$ discussed in the text, requires one less pass through the main loop than without the modificiation.

Exercise 9. In a typical polynomial implicitization problem, we are given $f_i \in k[t_1, \ldots, t_m]$, $i = 1, \ldots, n$ (the coordinate functions of a parametrization) and we want to eliminate $t_1, \ldots, t_m$ form the equations $x_i = f_1(t_1, \ldots, t_m)$, $i = 1, \ldots, n$. To do this, consider the ideal

$$J = \langle x_1 - f_1(t_1, \ldots, t_m), \ldots, x_n - f_n(t_1, \ldots, t_m)\rangle$$

and compute $I = J \cap k[x_1, \ldots, x_n]$ to find the implicit equations of the image of the parametrization. Explain how the Gröbner Walk could be applied to the generators of J directly to find I without any preliminary Gröbner basis computation. Hint: They are already a Gröbner basis with respect to a suitable monomial order.

Exercise 10. Apply the Gröbner Walk method suggested in Exercise 9 to compute the implicit equation of the parametric curve

$$\begin{cases} x = t^4 \\ y = t^2 + t. \end{cases}$$

(If you do not have access to an implementation of the Walk, you will need to perform the steps "manually" as in the example given in the text.) Also see part b of Exercise 11 in the previous section.