

TORIC CODES AND FINITE GEOMETRIES

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ABSTRACT. We show how the theory of affine geometries over the ring $\mathbb{Z}/\langle q-1 \rangle$ can be used to understand the properties of toric and generalized toric codes over \mathbb{F}_q . The minimum distance of these codes is strongly tied to the collections of lines in the finite geometry that contain subsets of the exponent vectors of the monomials that are evaluated to produce the standard generator matrix for the code. We argue that this connection is, in fact, even more direct than the connection with the lattice geometry of those exponent vectors considered as elements of \mathbb{Z}^2 or \mathbb{R}^2 . This point of view should be useful both as a way to visualize properties of these codes and as a guide to heuristic searches for good codes constructed in this fashion. In particular, we will use these ideas to see a reason why these constructions have been so successful over the field \mathbb{F}_8 , but less successful in other cases.

1. INTRODUCTION

We will consider a particular construction of linear block codes over a finite field \mathbb{F}_q . Mathematically, our codes are simply vector subspaces $C \subset \mathbb{F}_q^n$ whose elements serve as a set of codewords for representing information. This sort of encoding is done to increase the reliability of communication over noisy channels and has a number of engineering applications. Our standard reference for basic notions and notation in coding theory is [7]. As usual, n always denotes the block length and k denotes the vector space dimension of $\dim_{\mathbb{F}_q} C$, so that the set of codewords contains q^k elements. The important parameters of a code are n, k and a third integer d called the minimum Hamming distance. For these linear codes,

$$d = \min_{x \neq 0 \in C} |\{i \mid x_i \neq 0\}|.$$

If we fix n, k , the larger the parameter d is, the larger the error detection and error correction capacity of a code is.

The toric codes studied here are a class of m -dimensional cyclic codes introduced by J. Hansen in [5], [6]. (The term “toric code” is also used in another context that has no direct connection with this one.) Hansen uses the geometry of the projective toric variety corresponding to a polytope P in \mathbb{R}^m to describe toric codes, but these may also be understood in a somewhat more concrete way within the general context of evaluation, or functional, codes.

Definition 1.1. *Let P be the convex hull of a finite set of integer lattice points, contained in $[0, q-2]^m \subset \mathbb{R}^m$ and let $L = \text{Span}\{x^e : e \in P \cap \mathbb{Z}^m\}$ be the \mathbb{F}_q -linear span of the monomials x^e in variables x_1, \dots, x_m corresponding to the lattice points e in P . We get a linear block code, that we will denote by $C_P(\mathbb{F}_q)$, as the image*

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of the evaluation mapping on the \mathbb{F}_q -rational points in the standard m -dimensional torus over \mathbb{F}_q :

$$\begin{aligned} \text{ev} : L &\rightarrow \mathbb{F}_q^{(q-1)^m} \\ g &\mapsto (g(p) : p \in (\mathbb{F}_q^*)^m). \end{aligned}$$

The condition that $P \subset [0, q-2]^m$ implies that the x^e are linearly independent as functions on $(\mathbb{F}_q^*)^m$. In terms of generator matrices, this construction can also be described as follows. Let α be a primitive element for \mathbb{F}_q . If $f \in \mathbb{Z}^m$ is a vector with $0 \leq f_i \leq q-2$ for all i , let p_f denote the point $p_f = (\alpha^{f_1}, \dots, \alpha^{f_m})$ in $(\mathbb{F}_q^*)^m$. If $e = (e_1, \dots, e_m) \in P \cap \mathbb{Z}^m$, write

$$(p_f)^e = (\alpha^{f_1})^{e_1} \dots (\alpha^{f_m})^{e_m} = \alpha^{\langle f, e \rangle}.$$

Then the standard generator matrix for $C_P(\mathbb{F}_q)$ is the $(\dim_{\mathbb{F}_q} L) \times (q-1)^m$ matrix

$$G = ((p_f)^e),$$

whose rows are indexed by $e \in P \cap \mathbb{Z}^m$, and whose columns are indexed by f or $p_f \in (\mathbb{F}_q^*)^m$. We note that if P is the interval $[0, \ell-1] \subset \mathbb{R}$, then $C_P(\mathbb{F}_q)$ is simply the Reed-Solomon code $RS(\ell, q)$. So toric codes are, in a sense, higher-dimensional generalizations of Reed-Solomon codes.

In applying these ideas, it has turned out to be worthwhile to generalize this construction slightly, using arbitrary sets $S \subset [0, q-2]^m \subset \mathbb{R}^m$ instead of the whole set of lattice points in a convex polytope. These codes will be denoted by the analogous notation $C_S(\mathbb{F}_q)$. If $P = \text{conv}(S)$, then the code $C_S(\mathbb{F}_q)$ is a subcode of $C_P(\mathbb{F}_q)$. In the algebraic geometric language used by Hansen, the $C_S(\mathbb{F}_q)$ codes can be defined using incomplete linear systems $V \subset |\mathcal{O}_{X_P}(D_P)|$, where X_P is the toric variety determined by P and D_P is the corresponding divisor class on X_P .

The survey [13] covers most of the work on these codes contained in [8], [11], [14], [15], [16], and [1].

Toric codes or generalized toric codes are not all as good as Reed-Solomon codes from the coding theory perspective, but there are some very good codes first found by this construction. For instance, [2] gives a number of codes over \mathbb{F}_8 found by this method better than any previously known examples.

In many of the works cited above, the main focus has been on identifying conditions on P or on S that imply results about the minimum distance of the corresponding codes using the geometry of $P \cap \mathbb{Z}^m$ or S as subsets of the integer lattice $\mathbb{Z}^m \subset \mathbb{R}^m$. In particular, the role of Minkowski sum decompositions of subpolytopes of P and factorizations of the sections of the corresponding line bundle on the toric surface X_P has been studied rather intensively in [11], [16], and [10].

In this note we will describe and exploit a somewhat different point of view. We relate the properties of the $C_S(\mathbb{F}_q)$ codes to properties of the images of the sets S in the finite m -dimensional affine ring geometry over $\mathbb{Z}/\langle q-1 \rangle$, obtained by simply reading the exponent vectors e above as elements of $(\mathbb{Z}/\langle q-1 \rangle)^m$. The results here are, in a way, complementary to those from [10], where we compared the properties of $C_S(\mathbb{F}_q)$ and the related code $C_P(\mathbb{F}_q)$ for $P = \text{conv}(S)$ and q sufficiently large. Here the focus will be on the special properties of certain S for specific q .

We will concentrate mainly on the case $m = 2$ for simplicity, although the extension to larger m is essentially immediate. By itself, this amounts mostly to a relatively simple translation of known algebraic facts into another sort of geometric language with some unusual properties. However, we will argue that this this

alternative point of view is, if anything, even more natural and direct than studying toric codes via properties of polytopes and integer lattice vectors in \mathbb{Z}^2 . Moreover, this approach should prove useful both for visualizing how d is determined by the properties of S and hence for heuristic searches for codes with good d .

We will recall the known properties of these geometries in §2. We will then apply these properties to the study of generalized toric codes in §3. In particular, we will see a very concrete explanation for why \mathbb{F}_8 appears to be a particularly favorable choice of base field, and for why the construction succeeds so well there, yet performs relatively poorly over other fields of comparable small size. Finally, in §4, we will offer some more speculative comments about the potential of this code construction and an indication of which other finite fields should have properties analogous to those of \mathbb{F}_8 .

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2. FINITE RING GEOMETRIES

The properties of the finite affine and projective geometries over a finite field are very well-known and, of course, form the basis for algebraic geometry over finite fields and many different sorts of applications to coding theory. Perhaps less well-known to many mathematicians not working in the area is that there is also a quite well-developed theory of affine and projective coordinate geometries over rings. We will only need the following relatively simple case discussed in [9] and called affine *Barbilian planes* there. These are geometries with more of the “usual properties” one expects from the geometry of the Euclidean plane than the even more general structures called Hjelmslev planes.

Let R be a ring with multiplicative identity 1 in which $a \cdot b = 1$ implies $b \cdot a = 1$. Examples include commutative rings with identity as well as various noncommutative rings such as matrix rings over a field. Let B be a subset of $R^2 = R \times R$ that satisfies

- (E₁) $(1, 0), (0, 1) \in B$,
- (E₂) If $(u, v) \in B$ and r is a unit in R , then $r(u, v) = (ru, rv) \in B$,
- (E₃) Every $(u, v) \in B$ can be completed to an invertible 2×2 matrix $\begin{pmatrix} u & v \\ s & t \end{pmatrix}$ with $(s, t) \in B$,
- (E₄) If $\begin{pmatrix} u & v \\ s & t \end{pmatrix}$ is an invertible 2×2 matrix with $(u, v), (s, t) \in B$, then $(u, v) + \ell(s, t) \in B$ for all $\ell \in R$.

It is easy to see that if $(u, v) \in B$ there must be $s, t \in R$ such that $su + tv = 1$ and this shows that in the cases we will consider, there is only one choice for B , namely the set of all (u, v) appearing as rows in 2×2 invertible matrices with entries in R .

From now on, B will refer to this set; we will not include any indication of the ring R in the notation, though, since that should always be clear from the context.

Then one can define a geometric structure

$$\mathcal{G} = (\mathcal{P}, \mathcal{L}, \phi, \parallel)$$

associated to R as follows:

- \mathcal{P} , called the set of *points*, is simply R^2 .
- The subsets of \mathcal{P} of the form

$$(a, b) + R(u, v) = \{(a + \ell u, b + \ell v) \mid \ell \in R\}$$

with $(u, v) \in B$ are called *lines* and \mathcal{L} is the collection of all such lines.

- Two points (a, b) and (c, d) are said to be *non-neighbors*, written

$$(a, b) \not\phi (c, d),$$

if $(a - c, b - d) \in B$. If this does not hold, we write $(a, b) \circ (c, d)$ and say that the two points are *neighbors*.

- Two lines $\ell_1 = (a, b) + R(u, v)$ and $\ell_2 = (c, d) + R(s, t)$ are said to be *parallel* if and only if $R(u, v) = R(s, t)$. We write $\ell_1 \parallel \ell_2$ if this is true. Parallelism is an equivalence relation on the set \mathcal{L} .

For simplicity, we will call \mathcal{G} the *affine plane over R* .

The lines in the affine plane \mathcal{G} have a familiar-looking parametric form and the points on a line are in one-to-one correspondence with the elements of R because it is required that $(u, v) \in B$. Two non-neighbor points are contained in a unique line and parallel lines either coincide or are disjoint. But it is also possible for two distinct neighbor points to be contained in *more than one line*, and similarly, it is possible for two non-parallel lines to intersect in *more than one point*.

The exact properties of the geometries obtained by this construction are captured by the list of six axioms from [9] part I, defining the affine Barbilian planes. In addition to the properties already mentioned, there is a nice analog of the Playfair form of the Euclidean Parallel Postulate that holds here. We will not list all of these properties because we will not need to make use of them in the following.

On the other hand, [9] part II also contains a number of results characterizing special properties of these ring geometries corresponding to some standard ring-theoretic properties of R . For instance, we will need the following statements.

Theorem 2.1 ([9], part II). *Let R be a ring with identity with the property that $a \cdot b = 1$ implies $b \cdot a = 1$ and let \mathcal{G} be the corresponding affine plane over R . Then*

- (1) *The geometry satisfies the analog of Pappus's theorem (on triples of points on two distinct lines) if and only if R is commutative*
- (2) *The neighbor relation on the set of points is transitive if and only if R is a local ring and B is the set of pairs (u, v) where at least one of u, v is a unit in R .*
- (3) *There is at most one line containing any pair of distinct points if and only if R has no zero divisors.*
- (4) *The following are equivalent:*
 - (a) *Every pair of distinct points on a line are non-neighbors.*
 - (b) *Every pair of distinct points are non-neighbors.*
 - (c) *R is a field (not necessarily commutative).*
 - (d) *The affine plane satisfies the analog of Desargues' theorem.*

We will want to make use of this construction in the particular case $R = \mathbb{Z}/\langle r \rangle$ for some integer $r > 1$. From Theorem 2.1, we easily derive the following statements.

Corollary 2.2. *Let $R = \mathbb{Z}/\langle r \rangle$ and \mathcal{G} be the affine plane over R . In Theorem 2.1, in this case:*

- (a) *The analog of Pappus's theorem in (1) always holds.*
- (b) *The statements in part (2) hold if and only if r is a prime power.*
- (c) *The statements in parts (3) and (4) hold if and only if r is prime and \mathcal{G} is the affine plane over a field.*

Example 2.3. Consider the affine plane over $R = \mathbb{Z}/\langle 8 \rangle$. We will represent elements of R by the smallest nonnegative elements of the corresponding congruence classes. The set B consists of vectors (u, v) where either u or v is a *unit mod 8*, hence equals 1, 3, 5, or 7. Then for instance $P = (0, 0)$ and $Q = (1, 4)$ satisfy $P \not\in Q$ since $P - Q = (7, 4) \in B$. P and Q are contained in exactly one line:

$$(0, 0) + R(1, 4) = \{(0, 0), (1, 4), (2, 0), (3, 4), (4, 0), (5, 4), (6, 0), (7, 4)\}.$$

From this list of points, we can see already that the affine plane over R has some unusual properties. For example, note that $(2, 0)$ would also be on the line

$$(0, 0) + R(1, 0).$$

So $(0, 0) \circ (2, 0)$ and these are examples of neighbors lying on 2 distinct lines. Similarly the points $(0, 0) \circ (4, 0)$ are neighbors and they actually both lie on *four* distinct lines:

$$(0, 0) + R(1, 0), (0, 0) + R(1, 2), (0, 0) + R(1, 4), (0, 0) + R(1, 6).$$

The set of all neighbors of $(0, 0)$ is

$$N = \{(a, b) \mid a, b \in \{0, 2, 4, 6\}\}.$$

The neighbor relation is transitive in this case since R is a local ring with unique maximal ideal $\langle 2 \rangle R$ (as in part (2) of Theorem 2.1). \diamond

The properties seen in this example generalize immediately.

Proposition 2.4. *Let \mathcal{G} be the affine plane over $\mathbb{Z}/\langle r \rangle$ and let $(a, b) \circ (c, d)$ be distinct neighboring points. The number of distinct lines containing both points is equal to $r/o((a - c, b - d))$, where $o((a - c, b - d))$ is the order of the element $(a - c, b - d)$ in the additive group R^2 .*

Proof. Without loss of generality, we reduce to the case $(c, d) = (0, 0)$. Let (a, b) be any point contained in a line $R(s, t)$ and change coordinates by an invertible 2×2 matrix with entries in R to map (s, t) to $(1, 0)$, hence mapping (a, b) to $\ell(1, 0) = (\ell, 0)$ for some $\ell \in R$. Then for each line containing $(0, 0)$ and $(\ell, 0)$, there is a direction vector $(u, v) \in B$ and $\ell' \in R$ such that

$$(2.1) \quad (\ell, 0) = \ell'(u, v).$$

Note that (2.1) implies v cannot be a unit in R . Hence u must be a unit (since the vector $(u, v) \in B$ by definition) and we can replace (u, v) by another direction vector for the same line having the form $(1, v')$. Then there must be an equation similar to (2.1) with (u, v) replaced by $(1, v')$. Then the scalar multiple $\ell'(1, v')$ giving $(\ell, 0)$ must have $\ell' = \ell$ and $\ell v' = 0$. Moreover, there is a one-to-one correspondence between the lines containing $(0, 0)$ and $(\ell, 0)$ and solutions of the equation $\ell v' = 0$

in R . The number of solutions of this equation is equal to the index of the subgroup $\langle \ell \rangle \subseteq R$, which is equal to $r/o(\ell)$. This establishes the claim. \square

3. RING GEOMETRIES AND GENERALIZED TORIC CODES

We will now consider how the finite ring geometries introduced in the previous section relate to toric codes. We again take $m = 2$ for simplicity although everything extends without difficulty to larger m as well. The first observation is that since we are evaluating the monomials x^e at points p_f in $(\mathbb{F}_q^*)^2$ (as in the introduction), the fact that primitive elements α for \mathbb{F}_q satisfy $\alpha^{q-1} = 1$ implies that

$$e \equiv e' \pmod{q-1} \Rightarrow (p_f)^e = (p_f)^{e'}.$$

Hence, in a sense, it is probably *even more natural* to consider the exponent vectors e used in the evaluation mapping producing a toric surface code or one of the generalized toric codes $C_S(\mathbb{F}_q)$ with $m = 2$ as elements of the affine plane \mathcal{G} over $\mathbb{Z}/\langle q-1 \rangle$ rather than as vectors in \mathbb{Z}^2 or \mathbb{R}^2 . Our first result is a variation on the fact noted in Theorem 3.3 of [12] that lattice equivalent polytopes give monomially equivalent toric codes, giving some additional evidence for this claim. This statement appears in a technical report written by three of my students at the 2009 MSRI-UP undergraduate summer research program. We reproduce the proof here for the convenience of the reader.

Theorem 3.1 ([3], Theorem 1). *Let M be an invertible 2×2 matrix with entries in $\mathbb{Z}/\langle q-1 \rangle$, v be a fixed column vector with entries in $\mathbb{Z}/\langle q-1 \rangle$, and consider the affine mapping*

$$\begin{aligned} T : (\mathbb{Z}/\langle q-1 \rangle)^2 &\rightarrow (\mathbb{Z}/\langle q-1 \rangle)^2 \\ w &\mapsto Mw + v \end{aligned}$$

Let S_1 and S_2 be subsets of $(\mathbb{Z}/\langle q-1 \rangle)^2$ such that $S_2 = T(S_1)$. Then the generalized toric codes $C_{S_1}(\mathbb{F}_q)$ and $C_{S_2}(\mathbb{F}_q)$ are monomially equivalent.

Proof. The proof is essentially the same as that of Theorem 3.3 from [12]. The component of the vector $\text{ev}(x^e)$ corresponding to $e \in S_1$ and $p_f \in (\mathbb{F}_q^*)^2$ is $\alpha^{\langle e, f \rangle}$. Similarly, evaluating x^{Me+v} , where $Me+v \in S_2$, we obtain

$$\alpha^{\langle Me+v, f \rangle} = \alpha^{\langle v, f \rangle} \cdot \alpha^{\langle e, M^t f \rangle}.$$

Because it is assumed invertible, M defines a permutation of $(\mathbb{Z}/\langle q-1 \rangle)^2$, and similarly M^t induces a permutation of $(\mathbb{F}_q^*)^2$. Moreover, the translation vector induces different constant multiples in each component of the evaluation of a monomial. Hence the $C_{S_2}(\mathbb{F}_q)$ code is monomially equivalent to the $C_{S_1}(\mathbb{F}_q)$ code. \square

The transformations T described here form a group under composition, known as the affine general linear group over $\mathbb{Z}/\langle q-1 \rangle$ and denoted by $\text{AGL}(2, \mathbb{Z}/\langle q-1 \rangle)$. If $S_2 = T(S_1)$ for some such T , the sets are said to be $\text{AGL}(2, \mathbb{Z}/\langle q-1 \rangle)$ -equivalent. Because $\det(M)$ can be any unit in $\mathbb{Z}/\langle q-1 \rangle$, not just ± 1 as for invertible integer affine transformations, we tend to obtain somewhat larger equivalence classes here than when we consider lattice equivalence classes of sets S . But the generalized toric codes for all S in one of these equivalence classes are equivalent from the coding theory perspective – they have the same total weight enumerators, for instance.

The following simple algebraic fact will play a key role in relating properties of toric codes to the properties of the affine plane over $\mathbb{Z}/\langle q-1 \rangle$. The ring

$\mathbb{F}_q[x, y]/\langle x^{q-1} - 1, y^{q-1} - 1 \rangle$ is precisely the coordinate ring of the torus $(\mathbb{F}_q^*)^2$ over \mathbb{F}_q . We will show that the geometry of the affine plane over $\mathbb{Z}/\langle q-1 \rangle$ and the algebra of polynomial functions on the torus are closely connected. We will now abandon the multiindex notation and write out monomials in two variables explicitly.

Theorem 3.2. *Let \mathcal{G} be the affine plane over $R = \mathbb{Z}/\langle q-1 \rangle$ and let $(0, 0) \circ (a, b)$ be neighbors. Then the binomial $x^a y^b - 1$ factors in $\mathbb{F}_q[x, y]/\langle x^{q-1} - 1, y^{q-1} - 1 \rangle$ into a product of N distinct factors, where N is the number of distinct lines in \mathcal{G} containing both $(0, 0)$ and (a, b) , or equivalently (by Proposition 2.4), $N = (q-1)/o((a, b))$, where $o((a, b))$ is the order of the element (a, b) in the additive group R^2 .*

Proof. The integer N is a factor of $q-1$. Hence \mathbb{F}_q^* contains N distinct N th roots of unity and $u^N - 1$ factors completely into linear factors in $\mathbb{F}_q[u]$. But then the same will be true for $x^a y^b - 1$ since $(a, b) = N(u, v)$ for some vector $(u, v) \in B$. If α is a primitive element for \mathbb{F}_q , the factorization can be written explicitly as

$$(3.1) \quad x^a y^b - 1 = \prod_{j | (\alpha^j)^N = 1} (x^u y^v - \alpha^j).$$

This establishes the theorem. \square

We are now ready to see some first consequences for toric codes.

Corollary 3.3. *Suppose the set S used to produce the generalized toric code $C_S(\mathbb{F}_q)$ contains $(0, 0)$ and (a, b) as in the statement of Theorem 3.2 (or more generally any two elements of $(\mathbb{Z}/\langle q-1 \rangle)^2$ that differ by an element of order $(q-1)/N$). Then the minimum distance of $C_S(\mathbb{F}_q)$ satisfies*

$$d(C_S(\mathbb{F}_q)) \leq (q-1)^2 - N(q-1).$$

We assume nothing about other points on the lines containing $(0, 0)$ and (a, b) . Note that this does not contradict the statement of Proposition 3.4 of [12] because that article considers only toric codes from polytopes and we are considering generalized toric codes here. The analogous pairs of points here, yielding codes where the bound has $N = 1$, are those pairs differing by an element of order $q-1$ in $(\mathbb{Z}/\langle q-1 \rangle)^2$, or equivalently an element of the set B .

Proof. First, if S contains $(0, 0)$ and (a, b) , then the \mathbb{F}_q -span of the monomials corresponding to S contains all linear combinations of 1 and $x^a y^b$. Therefore, from (3.1), we obtain a codeword containing zero entries at positions corresponding to each of the $(x, y) \in (\mathbb{F}_q^*)^2$ with

$$x^u y^v - \alpha^j = 0$$

as α^j runs through the N th roots of unity in \mathbb{F}_q^* . There are exactly $q-1$ such points for each j . Moreover the sets of zeroes are clearly pairwise disjoint. Hence that codeword has weight $(q-1)^2 - N(q-1)$, and we have an upper bound for $d(C_S(\mathbb{F}_q))$ as claimed. The more general case given in parentheses in the statement of the Corollary follows from this. If $x^a y^b$ and $x^c y^d$ are in S and $(c-a, d-b)$ is an element of order $(q-1)/N$ in $(\mathbb{Z}/\langle q-1 \rangle)^2$, then

$$x^c y^d - x^a y^b = x^a y^b (x^{c-a} y^{d-b} - 1).$$

The monomial $x^a y^b$ is nonzero at all points in $(\mathbb{F}_q^*)^2$ and we proceed as before with the other factor. \square

A direct consequence of this is the following statement about a related configuration of points.

Corollary 3.4. *Suppose the S used to produce the generalized toric code $C_S(\mathbb{F}_q)$ contains the vertices of a “parallelogram” – that is four points of the form*

$$(0, 0), (a, b), (c, d), (a + c, b + d),$$

where the sum is taken in $(\mathbb{Z}/\langle q-1 \rangle)^2$ (or more generally something obtained from this by translating by a fixed vector in $(\mathbb{Z}/\langle q-1 \rangle)^2$). Assume that $(a, b) = N_1(a', b')$ for $(a', b') \in B$ has order $(q-1)/N_1$, and $(c, d) = N_2(c', d')$ for $(c', d') \in B$ has order $(q-1)/N_2$ in $(\mathbb{Z}/\langle q-1 \rangle)^2$. If (a', b') and (c', d') generate the additive group $(\mathbb{Z}/\langle q-1 \rangle)^2$, then

$$d(C_S(\mathbb{F}_q)) \leq (q-1)^2 - (N_1 + N_2)(q-1) + N_1N_2.$$

Proof. Among the linear combinations of the monomials corresponding to the points in S are combinations that factor as

$$(x^a y^b - 1)(x^c y^d - 1).$$

Because of the hypothesis on (a', b') and (c', d') the curves $x^{a'} y^{b'} - \alpha^j = 0$ and $x^{c'} y^{d'} - \alpha^k = 0$ always intersect in a single point in $(\mathbb{F}_q^*)^2$. Applying Theorem 3.2 and the proof of Corollary 3.3, this polynomial has $(N_1 + N_2)(q-1) - N_1N_2$ zeroes in $(\mathbb{F}_q^*)^2$. \square

We leave it to the reader to formulate and prove a result describing the possibilities that can occur when (a', b') and (c', d') fail to generate the additive group $(\mathbb{Z}/\langle q-1 \rangle)^2$. In Example 3.8 below, we will see one example of what can happen.

Example 3.5. Consider generalized toric codes $C_S(\mathbb{F}_9)$. Here $q = 9$ so $q-1 = 8$ and the relevant affine plane is the one whose properties were studied in Example 2.3. From Corollary 3.3, we obtain, for instance that if S contains $(0, 0)$ and $(4, 0)$ (or $(1, 0)$ and $(5, 0)$, etc.) then

$$d(C_S(\mathbb{F}_9)) \leq 64 - 4 \cdot 8 = 32.$$

From [4], the best possible d for a code with $n = 64$ and $k = 2$ over \mathbb{F}_9 has $d = 57$. Hence such $C_S(\mathbb{F}_9)$ can be far from optimal. Similarly if S contains any two points differing by an element of order $N = 2$ in $(\mathbb{Z}/\langle 8 \rangle)^2$, then

$$d(C_S(\mathbb{F}_9)) \leq 64 - 2 \cdot 8 = 48.$$

We can summarize the pattern here by saying that the presence of neighboring points in S tends to reduce $d(C_S(\mathbb{F}_q))$ directly in proportion to the number of distinct lines through the neighbors. \diamond

By part (3) of Theorem 2.1, there will be analogous more or less “bad” configurations of pairs or other small numbers of points that must be avoided in S in order to produce generalized toric codes over \mathbb{F}_q with good minimum distance. Here are several examples illustrating these claims.

Example 3.6. In [10] we discussed several cases where, even though S_0 contains “gaps” (that is, if $P = \text{conv}(S_0)$, then some points of $P \cap \mathbb{Z}^2$ are not contained in S_0) the generalized code $C_{S_0}(\mathbb{F}_q)$ behaves like a toric code $C_P(\mathbb{F}_q)$ where P contains a whole line segment and there are linear combinations of the corresponding monomials that factor completely. The same kind of thing can now

be recognized and predicted in many additional examples. For instance consider the set $S_0 = \{(0, 0), (3, 1), (1, 3)\}$ with $q = 9$. Even though these points are not collinear as elements of \mathbb{Z}^2 , they *are* collinear in the affine plane over $\mathbb{Z}/\langle 8 \rangle$ because $(1, 3) = 3 \cdot (1, 3)$. Moreover, if β_i are any distinct elements of \mathbb{F}_9 with $\beta_1 + \beta_2 + \beta_3 = 0$, then

$$xy^3 + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)x^3y + \beta_1\beta_2\beta_3$$

factors as

$$(x^3y + \beta_1)(x^3y + \beta_2)(x^3y + \beta_3)$$

in $\mathbb{F}_9[x, y]/\langle x^8 - 1, y^8 - 1 \rangle$. This implies that if S contains any set $\text{AGL}(2, \mathbb{Z}/\langle 8 \rangle)$ -equivalent to S_0 , then generalized toric code satisfies

$$d(C_S(\mathbb{F}_9)) \leq 64 - 3 \cdot 8 = 40.$$

The behavior seen in cases like this one, and the similar factorization over \mathbb{F}_8 from Example 5.6 of [11], becomes much less mysterious with the viewpoint provided by the finite geometry. \diamond

In the following examples, we will consider codes over \mathbb{F}_9 and we will use a primitive element α for this field given as a root of $u^2 + u + 2 = 0$.

Example 3.7. Consider $S_0 = \{(1, 0), (0, 1), (6, 3)\}$ in the affine plane over $\mathbb{Z}/\langle 8 \rangle$. These points are not collinear, but replacing x by x^9 and y by y^9 , we obtain a factorization of a linear combination of x^9, y^9, x^6y^3 as follows:

$$x^9 + y^9 + x^6y^3 = (x + \alpha y)^3(x + \alpha^3 y)^3(x + \alpha^4 y)^3$$

So if S contains any configuration $\text{AGL}(2, \mathbb{Z}/\langle 8 \rangle)$ -equivalent to S_0 , then

$$d(C_S(\mathbb{F}_9)) \leq 64 - 3 \cdot 8 = 40.$$

Whenever $q = p^r$ for $r > 1$, the Frobenius automorphism of the field \mathbb{F}_q will produce analogous unexpected behavior. \diamond

Moreover, those “bad” configurations depend strongly on q because the geometries of $(\mathbb{Z}/\langle q-1 \rangle)^2$ also depend strongly on q , not just on the locations of the points from S in \mathbb{Z}^2 or \mathbb{R}^2 .

Example 3.8. Consider the “trapezoid” $S_0 = \{(0, 0), (3, 0), (1, 4), (2, 4)\}$, viewed as a subset of the affine planes over $\mathbb{Z}/\langle 6 \rangle, \mathbb{Z}/\langle 7 \rangle$, and $\mathbb{Z}/\langle 8 \rangle$ in turn. The corresponding toric codes $C_{S_0}(\mathbb{F}_q)$ have parameters as follows:

$$\begin{array}{ll} C_{S_0}(\mathbb{F}_7) & [36, 4, 18] \\ C_{S_0}(\mathbb{F}_8) & [49, 4, 36] \\ C_{S_0}(\mathbb{F}_9) & [64, 4, 40]. \end{array}$$

As a result, the presence of S_0 (or, by Theorem 3.1, any other configuration S_1 obtained from S_0 by an invertible affine transformation of the corresponding plane) in a set S imposes different “penalties” $n - d$ depending on q . The penalty is much larger for $q = 7$ or $q = 9$ than it is for $q = 8$.

The explanation for this behavior comes from the finite geometries. In the geometry over $\mathbb{Z}/\langle 6 \rangle$, the points $(0, 0)$ and $(3, 0)$ are neighbors contained in three distinct lines. We get $d(C_{S_0}(\mathbb{F}_7)) \leq 36 - 3 \cdot 6 = 18$ from Corollary 3.3.

In the plane over $\mathbb{Z}/\langle 8 \rangle$, on the other hand, the situation is more subtle. First, we note that in $(\mathbb{Z}/\langle 8 \rangle)^2$, the configuration S_0 is actually also a “parallelogram.” This is true since $(2, 4) - (0, 0) = (2, 4)$ and $(3, 0) - (1, 4) = (2, 4)$. But we also have

$(2, 4) = 2(1, 2)$ and the vectors $(1, 4) = (1, 4) - (0, 0)$ and $(1, 2)$ do not generate all of $(\mathbb{Z}/\langle 8 \rangle)^2$. As a result, the statement of Theorem 3.4 does not apply and while the bound is still true, it is not sharp. We can understand what is happening in this example algebraically by working in $\mathbb{F}_9[x, y]/\langle x^8 - 1, y^8 - 1 \rangle$, the coordinate ring of the torus $(\mathbb{F}_9^*)^2$. One minimum-weight word in the $C_S(\mathbb{F}_9)$ code comes from evaluating

$$\alpha^7 + \alpha^2 x^3 + \alpha^6 xy^4 + \alpha^3 x^2 y^4 \equiv \alpha^2 (y^2 + x)(\alpha^4 y^2 + x)(\alpha y^4 + x)$$

(recall that $y^8 \equiv 1$). This is a maximally factorizable polynomial in the span of $1, x^3, xy^4, x^2 y^4$. The number of zeroes in $(\mathbb{F}_9^*)^2$ turns out to be $3 \cdot 8 = 24$ in this case, since the curves

$$y^2 + x = 0, \quad 2y^2 + x = 0, \quad \alpha y^4 + x = 0$$

defined by the factors do not intersect at \mathbb{F}_9 -rational points in the torus. \diamond

We believe that the lesson of examples like these is that toric codes over fields such as \mathbb{F}_7 and \mathbb{F}_9 are not automatically bad, but that there are certain configurations of points special to the field \mathbb{F}_q that must be avoided in S in order to find codes $C_S(\mathbb{F}_q)$ with good d . Here is an example where this approach was followed to try to find a good code.

Example 3.9. The following S giving a nearly optimal $C_S(\mathbb{F}_9)$ code with parameters $[64, 8, 45]$ was found by a randomized heuristic search at the MSRI-UP 2009 undergraduate research program by then-students Alejandro Carbonara, Juan Murillo, and Abner Ortiz:

$$S = \{(0, 4), (1, 1), (2, 0), (2, 3), (2, 5), (3, 7), (5, 2), (7, 4)\}.$$

According to [4], the best known d for this n and k over \mathbb{F}_9 is $d = 46$. It is not difficult to check that all but four of the pairwise difference vectors $(a, b) - (c, d)$ for $(a, b), (c, d) \in S$ are contained in the set B considered here (for the field \mathbb{F}_9). Moreover the four that are not in B , such as $(2, 0) - (0, 4) \equiv (2, 4)$, are all elements of order 4 in $(\mathbb{Z}/\langle 8 \rangle)^2$. So the upper bound $d \leq 48$ from Corollary 3.3 or Example 3.5 applies. This is a case where taking one pair of the points in S gives a code with $d = 48$, but then adding six more points decreases d by only an additional 3.

Another observation is that the set of differences $(a, b) - (c, d)$ contains only two pairs of equal vectors (there are 26 different vectors in the set of differences). The two pairs of equal vectors consist of vectors in B . Hence there are two “parallelograms” contained in S , and Corollary 3.4 applies with $N_1 = N_2 = 1$. This gives a less tight upper bound of $d \leq (9 - 1)^2 - 2(9 - 1) + 1 = 49$. \diamond

Computations done by my student Lauren Buckley at Holy Cross in 2014 show that $d = 45$ is *optimal for generalized toric codes* with $n = 64$ and $k = 8$ over \mathbb{F}_9 . But the method requires a detailed (and tedious) case-by-case analysis and we will not attempt to present the details here. The idea was simply to enumerate all the $\text{AGL}(2, \mathbb{Z}/\langle 8 \rangle)$ -equivalence classes of base sets S_0 with $|S_0| = 4$, and then consider all possible ways to “build up” to $k = 8$ by adding 4 additional points to one S_0 in each class. As k increased, it quickly became impossible to avoid some sets dropping d to 45 or less. The examples presented above were all used to recognize when this happened. Needless to say, though, we would like to have a better argument to show $d \leq 45$.

4. FINAL COMMENTS

We will conclude this note by making some further observations regarding the potential of the generalized toric code construction for producing really good codes (say better than those found by other methods and recorded in the database [4]). As we mentioned previously, this construction has been most successful over \mathbb{F}_8 , as shown for example in the new codes found in [2]. The underlying reason for this should be somewhat clear by now – we believe that this is simply a reflection of the fact that the underlying geometry in this case comes from the *field* $\mathbb{Z}/\langle 8-1 \rangle \simeq \mathbb{F}_7$, rather than from a ring with zero divisors, hence neighboring points in the affine plane. All of the properties in (4) of Theorem 2.1 hold in this case, so there are many fewer “bad configurations” to avoid in searches for good codes. While there are isolated examples like the one in Example 3.9 over fields \mathbb{F}_q for which $\mathbb{Z}/\langle q-1 \rangle$ is not also a field, and even a few others where optimal codes have been obtained as generalized toric codes, we believe that these cases will be much rarer and more difficult to find. The best next case to look at will probably be codes over \mathbb{F}_{32} and more generally the other cases where p is a Mersenne prime and $p+1=2^r$. But of course these cases are relatively rare and they lead to large fields where virtually nothing is known yet about optimal codes.

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