MATH 136 – Calculus 2
Retest Problems for the *Core Learning Outcomes*

BP 1 – I understand the definition of the definite integral $\int_a^b f(x)\,dx$ as a limit of Riemann sums

1. The following limit yields the value of a definite integral $\int_a^b f(x)\,dx$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n}+1\right)^3 \frac{3}{n}$$

   Give a possible $f(x)$ and interval $[a, b]$. (There is more than one correct answer here; any one correct answer is OK.)

2. The following limit yields the value of a definite integral $\int_a^b f(x)\,dx$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k-1}{n}+6\right)^2 \frac{1}{n}$$

   Give a possible $f(x)$ and interval $[a, b]$. (There is more than one correct answer here; any one correct answer is OK.)

3. The following limit yields the value of a definite integral $\int_a^b f(x)\,dx$:

$$\lim_{n\to\infty} \frac{4}{n} \sum_{k=1}^{n} \sqrt{\frac{4k}{n}+7}$$

   Give a possible $f(x)$ and interval $[a, b]$. (There is more than one correct answer here; any one correct answer is OK.)

4. The following limit yields the value of a definite integral $\int_a^b f(x)\,dx$:

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{\pi(k-1)}{n}\right)$$

   Give a possible $f(x)$ and interval $[a, b]$. (There is more than one correct answer here; any one correct answer is OK.)

5. The following limit yields the value of a definite integral $\int_a^b f(x)\,dx$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^6}{n^7}$$

   Give a possible $f(x)$ and interval $[a, b]$. (There is more than one correct answer here; any one correct answer is OK.)

BP 2 – I understand the interpretation of the value of $\int_a^b f(x)\ dx$ as a signed area.

1. Sketch the graph $y = -3x + 2$ on the interval $[0, 2]$ and find the value of $\int_0^2 -3x + 2\ dx$ by using the geometry of your graph.

2. Sketch the graph
$$f(x) = \begin{cases} 2x - 2 & \text{if } 0 \le x \le 2 \\ 4 - x & \text{if } 2 \le x \le 6 \end{cases}$$
on the interval $[0, 5]$ and find the value of $\int_0^5 f(x)\ dx$ by using the geometry of your graph.

3. Sketch the graph
$$f(x) = \begin{cases} 2x - 2 & \text{if } 0 \le x \le 2 \\ 4 - x & \text{if } 2 \le x \le 6 \end{cases}$$
on the interval $[1, 6]$ and find the value of $\int_1^6 f(x)\ dx$ by using the geometry of your graph.

4. Sketch the graph
$$f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } 0 \le x \le 1 \\ -\sqrt{4 - (x-3)^2} & \text{if } 2 \le x \le 5 \end{cases}$$
on the interval $[0, 5]$ and use it to determine $\int_0^3 f(x)\ dx$.

5. Sketch the graph
$$f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } 0 \le x \le 1 \\ -\sqrt{4 - (x-3)^2} & \text{if } 2 \le x \le 5 \end{cases}$$
on the interval $[0, 5]$ and use it to determine $\int_0^5 f(x)\ dx$.

BP 3 – I understand the Fundamental Theorem of Calculus (Parts I and/or II).

1. Let
$$F(x) = \int_3^x \frac{t}{t^4 + 4t^2 + 1}\, dt.$$
What is the slope of the tangent line to $y = F(x)$ at $x = 1$?

2. Let
$$F(x) = \int_3^x \frac{t}{t^4 + 4t^2 + 1}\, dt.$$
Does F have a local maximum, a local minimum, or neither at $x = 0$?

3. Let
$$G(x) = \int_x^{10} \frac{t}{t^4 + 4t^2 + 1}\, dt.$$
What is $F'(x)$?

4. Let
$$G(x) = \int_0^{e^x} \frac{t}{t^4 + 4t^2 + 1}\, dt.$$
What is $G'(x)$?

5. Let $\mathrm{Si}(x)$ be the function defined by $\mathrm{Si}(x) = \displaystyle\int_0^x \frac{\sin(t)}{t}\, dt$. Find an antiderivative $G(x)$ of $f(x) = \dfrac{\sin(x)}{x}$ satisfying $G(2) = 0$ and give a formula for $G(x)$ using $\mathrm{Si}(x)$.

6. Let $\mathrm{erf}(x)$ be the function defined by $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2/2}\, dt$. Find an antiderivative $H(x)$ of $f(x) = \dfrac{2}{\sqrt{\pi}} e^{-x^2/2}$ satisfying $H(1) = 0$ and give a formula for $H(x)$ using $\mathrm{erf}(x)$.

7. Compute the definite integral
$$\int_0^1 x^4 - 3\sqrt{x} + \sin(\pi x)\, dx.$$

8. Compute the definite integral
$$\int_1^4 x^{3/2} - \frac{2}{x^2} + \cos(2\pi x)\, dx.$$

9. Compute the definite integral
$$\int_0^3 12x - 2e^x - \frac{1}{1 + x^2}\, dx.$$

BP 4 – I understand how the integral of a rate of change gives the total change and I can apply that idea.

1. The velocity of a car moving along a straight line road at time t is give by $v(t) = 15t^2 - 30t$ (t in hours, v in miles per hour). What is the difference between the car's positions at $t = 0$ and $t = 3$?

2. The velocity of a car moving along a straight line road at time t is give by $v(t) = 15t^2 - 30t$ (t in hours, v in miles per hour). What is the total distance traveled by the car between $t = 0$ and $t = 3$?

3. Water is being poured into large holding tank at a rate of $5 + t$ gallons per minute (t in minutes). What is the difference between the amount of water in the tank at $t = 1$ and $t = 2$? Include the appropriate units in your answer.

4. At time t (in hours), the rate of change of the population of a colony of bacteria is $r(t) = e^{0.1t}$ (in units of 1000 bacteria per hour). How much does the population change between $t = 1$ and $t = 2$ hours?

5. The velocity of a car moving along a straight line road at time t is give by $v(t) = 20\sin(t) + 30$ (t in hours, v in miles per hour). What is the difference between the car's positions at $t = 0$ and $t = \pi$?

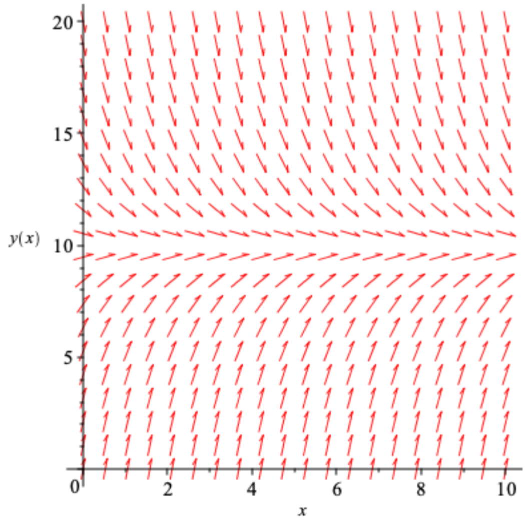

Figure 1: Slope field for $y' = 10 - y$

BP 5 – I understand what a differential equation is, what the order means, what a slope field shows, and what a solution is.

1. Given: The rate of change of the temperature T of a frozen dinner in a 350° oven at time t is proportional to the difference between 350 and the dinner's temperature. Express this statement as a differential equation, including a proportionality constant k. What is the order of your equation? Is the function $T(t) = 350 - 340e^{kt}$ a solution of your equation?

2. What is the order of the differential equation $y'' - 4y' + 4y = 0$ with unknown $y = y(x)$? Is $y = xe^{3x}$ a solution of this equation?

3. What is the order of the differential equation $y'' + y = e^t$ with unknown $y = y(t)$? Is $y = \sin(t) + \frac{e^t}{2}$ a solution of this equation?

4. The plot (see figure above) shows the slope field of the differential equation $y' = 10 - y$ for (x, y) in the rectangle $[0, 10] \times [0, 20]$. On the slope field plot, sketch the graphs of the solutions with $y(0) = 10$ and $y(0) = 5$.

5. The plot (see figure below) shows the slope field of the differential equation $y' = y\left(1 - \frac{y}{10}\right)$ for (x, y) in the rectangle $[0, 10] \times [0, 12]$. On the slope field plot, sketch the graphs of the solutions with $y(0) = 12$ and $y(0) = 5$.

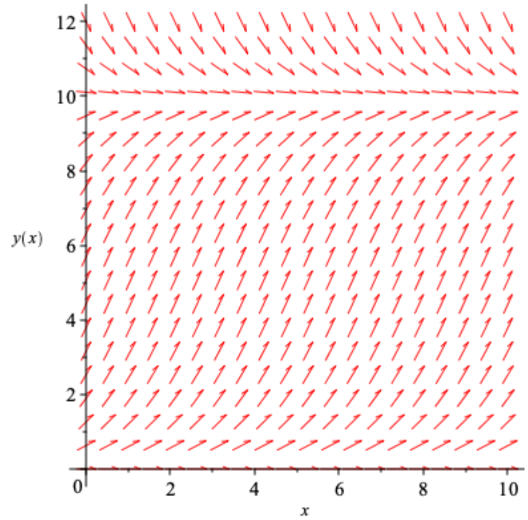

Figure 2: Slope field for $y' = y\left(1 - \frac{y}{10}\right)$

BP 6 – I understand what sequences and series are and what it means for each of them to converge or diverge.

1. Let $a_n = \dfrac{n+1}{n^2+7}$. Is this a sequence or a series? Does it converge? How can you tell?

2. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+7}$. Is this a sequence or a series? Does it converge? How can you tell?

3. Let $b_n = e^{-n}$ Is this a sequence or a series? Does it converge? How can you tell?

4. Consider $\displaystyle\sum_{n=0}^{\infty} e^{-n}$. Is this a sequence or a series? Does it converge? How can you tell?

5. Let $s_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Is this a sequence or a series? Does it converge? How can you tell?

6. Consider $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$. Is this a sequence or a series? Does it converge? How can you tell?

BP 7 – I understand what the Taylor series of a function is, and I know the Taylor series for $e^x, \sin(x), \cos(x)$ expanding at 0.

1. Find the Taylor series for $f(x) = x^4$, expanding at $a = 1$. Use the definition.

2. Find the Taylor series for $f(x) = e^{9x}$, expanding at $a = 0$. How does your answer relate to the Taylor series for e^x?

3. Find the Taylor series for $f(x) = \dfrac{e^x + e^{-x}}{2}$, expanding at $a = 0$. There are a number of different methods for this. Any correct one is OK. How does your answer relate to the Taylor series for $g(x) = \cos(x)$?

4. Find the Taylor series for $f(x) = \dfrac{e^x - e^{-x}}{2}$, expanding at $a = 0$. There are a number of different methods for this. Any correct one is OK. How does your answer relate to the Taylor series for $g(x) = \sin(x)$?

5. Find the Taylor series for $f(x) = \ln(1 - x)$, expanding at $a = 0$. There are a number of different methods for this. Any correct one is OK. How does your answer relate to the geometric series with ratio $r = x$ and first term $c = 1$?

S 1 – I understand how to set up and evaluate Riemann sums.

1. Compute the left- and right-hand Riemann sums for $f(x) = x^3 - 1$ on $[a, b] = [1, 3]$ using the regular subdivision with $n = 4$. Express your answers in exact form as fractions. One of these is an overestimate for $\int_1^3 x^3 - 1\, dx$ and one is an underestimate. Which is which?

2. Compute the midpoint Riemann sum for $f(x) = x^2 + 1$ on $[a, b] = [1, 3]$ using the regular subdivision with $n = 4$. Express your answer in exact form as a fraction. Is your answer an overestimate or an underestimate for $\int_1^3 x^2 + 1\, dx$? How can you tell?

3. What is the difference between the left- and right-hand Riemann sums for $f(x) = e^x$ with $n = 1000$ on the interval $[0, 1]$. (Hint: You do not need to evaluate the left- and right-hand sums separately to find the difference!)

4. Compute the left- and right-hand Riemann sums for $f(x) = e^{-x}$ on $[a, b] = [0, 5]$ using the regular subdivision with $n = 5$. Express your answers in decimal form. One of these is an overestimate for $\int_0^5 e^{-x}\, dx$ and one is an underestimate. Which is which?

5. Compute the midpoint Riemann sum for $f(x) = \sqrt{x}$ on $[a, b] = [2, 3]$ using the regular subdivision with $n = 5$. Express your answer in decimal form. Is your answer an overestimate or an underestimate for $\int_2^3 \sqrt{x}\, dx$? How can you tell?

S 2 – I can find antiderivatives by the basic rules with power functions, exponential and logarithm functions, trigonometric and inverse trigonometric functions.

1. Find the most general antiderivative for
$$f(x) = \sqrt{x} + \frac{1}{x^{3/4}} - e^{5x} + \ln(x) + \cos(3x) - \frac{3}{\sqrt{1-x^2}}.$$

2. Find the most general antiderivative for
$$f(x) = \sqrt[3]{x} + \frac{1}{x^7} - e^{2x} + \ln(x) + \sin(4x) - \frac{2}{1+x^2}.$$

3. Find the most general antiderivative for
$$f(x) = x^{5/7} - \frac{1}{x^6} - e^{-x} + \ln(x) + \sec^2(5x) - \frac{8}{\sqrt{1-x^2}}.$$

4. Find the most general antiderivative for
$$f(x) = \sqrt[5]{x} + \frac{1}{x^4} - e^{-2x} + \ln(x) + \sec(5x)\tan(5x) - \frac{1}{|x|\sqrt{x^2-1}}.$$

5. Find the most general antiderivative for
$$f(x) = x^{1/9} + \frac{1}{\sqrt{x}} + e^{3x} + \ln(x) + \csc(2x)\cot(2x) - \frac{16}{1+x^2}.$$

S 3 – I can find indefinite integrals by the u-substitution method.

1. Integrate:

$$\int (x+3)\sqrt{5x^2+30x+8}\;dx$$

2. Integrate:

$$\int \frac{e^{6x}}{\sqrt[3]{e^{6x}+7}}\;dx$$

3. Integrate:

$$\int \frac{e^{\tan(3x)}}{\cos^2(3x)}\;dx$$

4. Integrate:

$$\int \frac{x+4}{x^2+8x+23}\;dx$$

5. Integrate:

$$\int \frac{1}{x\ln(x)}\;dx$$

S 4 – I can recognize when integration by parts applies and apply the parts formula.

1. Consider integrals of the form $\int x^n \sin(ax)\ dx$ where n is a positive integer and a is a constant. Choose a u and a dv for an integration by parts and apply the parts formula to rewrite this in an equivalent form. You *do not* need to evaluate the resulting integral on the right-hand side of your equation.

2. Consider integrals of the form $\int x^n \cos(ax)\ dx$ where n is a positive integer and a is a constant. Choose a u and a dv for an integration by parts and apply the parts formula to rewrite this in an equivalent form. You *do not* need to evaluate the resulting integral on the right-hand side of your equation.

3. Consider integrals of the form $\int x^n \ln(ax)\ dx$ where n is a positive integer and $a > 0$ is a constant. Choose a u and a dv for an integration by parts and apply the parts formula to rewrite this in an equivalent form. Evaluate the resulting integral on the right-hand side of your equation.

4. Consider integrals of the form $\int e^{ax} \sin(bx)\ dx$ where a, b are constants. Choose a u and a dv for an integration by parts and apply the parts formula to rewrite this in an equivalent form. You *do not* need to evaluate the resulting integral on the right-hand side of your equation.

5. Consider integrals of the form $\int e^{ax} \cos(bx)\ dx$ where a, b are constants. Choose a u and a dv for an integration by parts and apply the parts formula to rewrite this in an equivalent form. You *do not* need to evaluate the resulting integral on the right-hand side of your equation.

S 5 – I can recognize when integration by trigonometric substitution applies, select the appropriate substitution and convert an integral to an equivalent trigonometric form.

1. Does the trigonometric substitution method apply to the following integral:

$$\int \frac{x^2}{\sqrt{16 - x^2}}\, dx?$$

   Why or why not? If so, what trigonometric substitution would you apply to evaluate it? Make the substitution and convert to a trigonometric integral in θ. Simplify as much as possible.

2. Does the trigonometric substitution method apply to the following integral:

$$\int \frac{x^2}{\sqrt{x^3 - 36}}\, dx?$$

   If so, what trigonometric substitution would you apply to evaluate it? Make the substitution and convert to a trigonometric integral in θ. Simplify as much as possible.

3. Does the trigonometric substitution method apply to the following integral:

$$\int \frac{x^2}{\sqrt{x^2 + 49}}\, dx?$$

   Why or why not? Make the substitution and convert to a trigonometric integral in θ. Simplify as much as possible.

4. Does the trigonometric substitution method apply to the following integral:

$$\int \frac{1}{x^2\sqrt{x^2 - 16}}\, dx?$$

   Why or why not? If so, what trigonometric substitution would you apply to evaluate it? Make the substitution and convert to a trigonometric integral in θ. Simplify as much as possible.

5. Does the trigonometric substitution method apply to the following integral:

$$\int \frac{1}{x\sqrt{9 - x^5}}\, dx?$$

   Why or why not? What trigonometric substitution would you apply to evaluate it? Make the substitution and convert to a trigonometric integral in θ. Simplify as much as possible.

S 6 –I can recognize when integration by partial fractions applies, and set up and solve for the coefficients in the partial fractions.

1. Does the partial fractions method apply to the following integral:

$$\int \frac{x^3}{\sqrt{16 - x^2}}\, dx?$$

   Why or why not? If so, divide if necessary, then set up and solve for the coefficients in the partial fractions.

2. Does the partial fractions method apply to the following integral:

$$\int \frac{x^2 + 1}{x^3 + 3x}\, dx?$$

   Why or why not? If so, divide if necessary, then set up and solve for the coefficients in the partial fractions.

3. Does the partial fractions method apply to the following integral:

$$\int \frac{x^2 + x + 2}{x^3 - 9x}\, dx?$$

   Why or why not? If so, divide if necessary, then set up and solve for the coefficients in the partial fractions.

4. Does the partial fractions method apply to the following integral:

$$\int \frac{e^x}{x^2 - 9x - 90}\, dx?$$

   Why or why not? If so, divide if necessary, then set up and solve for the coefficients in the partial fractions.

5. Does the partial fractions method apply to the following integral:

$$\int \frac{x^2}{x^2 - 9x - 90}\, dx?$$

   Why or why not? If so, divide if necessary, then set up and solve for the coefficients in the partial fractions.

S 7 –I can use combinations of the methods in S2 through S6 as appropriate to completely work out indefinite integrals.

1. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$
If you used a trigonometric substitution, convert back to a function of x.

2. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int \frac{x^2+1}{x^2-5x+6}.$$
If you used a trigonometric substitution, convert back to a function of x.

3. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int \frac{x}{\sqrt{1-x^4}}\,dx$$
If you used a trigonometric substitution, convert back to a function of x.

4. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int x^5\cos(5x^2)\,dx$$
If you used a trigonometric substitution, convert back to a function of x.

5. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int \sin^{-1}(x)\,dx$$
If you used a trigonometric substitution, convert back to a function of x.

6. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int \frac{1}{e^x+1}\,dx$$
Hint: Multiply numerator and denominator by e^x to start off. If you used a trigonometric substitution, convert back to a function of x.

7. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral
$$\int x\tan^{-1}(x)\,dx$$
If you used a trigonometric substitution, convert back to a function of x.

8. Use combinations of the methods in S2 through S6 as appropriate to compute the indefinite integral

$$\int x^2 \tan^{-1}(x)\ dx$$

If you used a trigonometric substitution, convert back to a function of x.

S 8 –I understand what makes an integral improper, what it means for an improper integral to converge or diverge, and how to tell by setting and evaluating appropriate limits.

1. Is the integral
$$\int_0^{\infty} e^{-x}\, dx$$
improper? Why or why not? If it is improper, set up and evaluate the limit(s) to determine whether the integral converges or diverges. If it is not improper, evaluate the integral.

2. Is the integral
$$\int_1^3 \frac{1}{x^2 - 3x + 2}\, dx$$
improper? Why or why not? If it is improper, set up and evaluate the limit(s) to determine whether the integral converges or diverges. If it is not improper, evaluate the integral.

3. Is the integral
$$\int_5^{10} \frac{1}{x^2 - 3x + 2}\, dx$$
improper? Why or why not? If it is improper, set up and evaluate the limit(s) to determine whether the integral converges or diverges. If it is not improper, evaluate the integral.

4. Is the integral
$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 9}\, dx$$
improper? Why or why not? If it is improper, set up and evaluate the limit(s) to determine whether the integral converges or diverges. If it is not improper, evaluate the integral.

5. Is the integral
$$\int_0^3 x \ln(x)\, dx$$
improper? Why or why not? If it is improper, set up and evaluate the limit(s) to determine whether the integral converges or diverges. If it is not improper, evaluate the integral.

S 9 – I can recognize separable differential equations and solve them by integration.

1. Is the first-order differential equation
$$y' = xy - 3y + 5x - 15$$
separable? Why or why not? If it is separable, find the general solution and express in explicit form $y =$ function of x.

2. Is the first-order differential equation
$$y' = \frac{x^2}{\cos(y)}$$
separable? Why or why not? If it is separable, find the general solution and express in explicit form $y =$ function of x.

3. Is the first-order differential equation
$$y' = x^3(y^2 + 1)$$
separable? Why or why not? If it is separable, find the general solution and express in explicit form $y =$ function of x.

4. Is the first-order differential equation
$$y' = \frac{3}{x}y + \frac{4}{x}$$
separable? Why or why not? If it is separable, find the general solution and express in explicit form $y =$ function of x.

5. Is the first-order differential equation
$$y' = 4y(1 - y)$$
separable? Why or why not? If it is separable, find the general solution and express in explicit form $y =$ function of x.

S 10 – I can recognize harmonic, geometric, and p-series and determine whether they converge or diverge.

1. Of the following three infinite series, only one is either harmonic, geometric, or a p-series. Indicate the one that falls into at least one of those categories, say which categories it belongs to, and say whether it converges or not by applying what we know about those types of series.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^5} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{1}{2^n + 3^n} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{n+3}{n^2 - n + 7}$$

2. Of the following three infinite series, only one is either harmonic, geometric, or a p-series. Indicate the one that falls into at least one of those categories, say which categories it belongs to, and say whether it converges or not by applying what we know about those types of series.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^3 + 8} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{8^n}{n!} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

3. Of the following three infinite series, only one is either harmonic, geometric, or a p-series. Indicate the one that falls into at least one of those categories, say which categories it belongs to, and say whether it converges or not by applying what we know about those types of series.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{9^n}{8^n} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{n!}{e^n} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{n^2 - 5}$$

4. Of the following three infinite series, only one is either harmonic, geometric, or a p-series. Indicate the one that falls into at least one of those categories, say which categories it belongs to, and say whether it converges or not by applying what we know about those types of series.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n^4}{n^5 + 1} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{\sin(n)}{n}$$

5. Of the following three infinite series, only one is either harmonic, geometric, or a p-series. Indicate the one that falls into at least one of those categories, say which categories it belongs to, and say whether it converges or not by applying what we know about those types of series.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 3n - 1} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{\cos(n)}{e^n} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{n^{5/8}}$$

S 11 – I understand when the nth term, integral, alternating series, or ratio tests apply and I can use them to determine whether a series converges or diverges.

1. Why does the integral test apply to the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5}?$$

   Apply it to determine whether the series converges or diverges. State your conclusion clearly.

2. Why does the integral test apply to the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}?$$

   Apply it to determine whether the series converges or diverges. State your conclusion clearly.

3. According to the nth term test, what can we say about the infinite series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+3n-1}?$$

   State your conclusion clearly.

4. According to the nth term test, what can we say about the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+3n-1}?$$

   State your conclusion clearly.

5. Let $p(n)$ be the nth prime number, so $p(1) = 2$, $p(2) = 3$, $p(3) = 5$, and so on. Why does the alternating series test apply to the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{p(n)}?$$

   Apply the test to determine whether the series converges or diverges. State your conclusion clearly.

6. Why does the alternating series test apply to the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[5]{n}}?$$

   Apply the test to determine whether the series converges or diverges. State your conclusion clearly.

7. What does the ratio test say about the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 8^n}{(2n)!}?$$

Explain what you are doing to apply the test and state your conclusion clearly.

8. What does the ratio test say about the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^4}{e^n}?$$

Explain what you are doing to apply the test and state your conclusion clearly.

S 12 – I can use the ratio test to determine the interval of absolute convergence for a power series, and test convergence at the endpoints of the interval (if there are any).

1. Use the ratio test to determine the interval of absolute convergence for the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

   and test convergence at the endpoints of the interval (if there are any).

2. Use the ratio test to determine the interval of absolute convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n}$$

   and test convergence at the endpoints of the interval (if there are any).

3. Use the ratio test to determine the interval of absolute convergence for the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

   and test convergence at the endpoints of the interval (if there are any).

4. Use the ratio test to determine the interval of absolute convergence for the power series

$$\sum_{n=1}^{\infty} n^2 (x-3)^n$$

   and test convergence at the endpoints of the interval (if there are any).

5. Use the ratio test to determine the interval of absolute convergence for the power series

$$\sum_{n=1}^{\infty} e^n (x-2)^n$$

   and test convergence at the endpoints of the interval (if there are any).

A 1 – I understand how approximations to areas, average values, volumes, moments, etc. lead to Riemann sums and where the integral formulas come from.

1. The area between two graphs $y = f(x)$ and $y = g(x)$ and $a \le x \le b$ can be approximated by subdividing the x-interval into n equal subintervals, picking one point x_k^* in the kth subinterval, and forming the sum

   $$A \doteq \sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\Delta x$$

   if $f(x) \ge g(x)$ on the whole interval. What do we want to do to this sum to get the exact value of the area? What is the corresponding integral formula for the area. What would need to be changed if $f(x) < g(x)$ on part or all of the interval?

2. The average value $\overline{y}$ of $f(x)$ on the interval $a \le x \le b$ can be approximated by subdividing the x-interval into n equal subintervals, picking one point x_k^* in the kth subinterval, and forming the sum

   $$\overline{y} \doteq \frac{1}{n} \sum_{k=1}^{n} f(x_k^*)$$

   How does this need to be rearranged before we can recognize a Riemann sum in this? What do we want to do to this sum to get the exact average value? What is the corresponding integral formula for the average value?

3. Suppose a solid is lined up along the x-axis and occupies the interval $a \le x \le b$. If the area of the cross-section at x is $A(x)$, then the volume of the solid can be approximated by by subdividing the x-interval into n equal subintervals, picking one point x_k^* in the kth subinterval, and forming the sum

   $$V \doteq \sum_{k=1}^{n} A(x_k^*)\ \Delta x.$$

   What do we want to do to this sum to get the exact value of the volume? What is the corresponding integral formula for the volume.

4. Suppose a solid is generated by rotating the region between $y = f(x) \ge 0$ and the x-axis for $a \le x \le b$ *about the x-axis*. What is the area $A(x)$ of the cross-section at x in this case? The volume of the solid can be approximated by subdividing the x-interval into n equal subintervals, picking one point x_k^* in the kth subinterval, and forming the sum

   $$V \doteq \sum_{k=1}^{n} A(x_k^*)\ \Delta x.$$

   What do we want to do to this sum to get the exact value of the volume? What is the corresponding integral formula for the volume, expressed using the function $f(x)$?

5. Suppose a solid is generated by rotating the region between $y = f(x) \geq 0$ and the x-axis for $a \leq x \leq b$ *about the y-axis.* The volume of the solid can be approximated by subdividing the x-interval into n equal subintervals, picking one point x_k^* in the kth subinterval, and forming the sum

$$V \doteq \sum_{k=1}^{n} 2\pi\, x_k^*\, f(x_k^*)\ \Delta x.$$

Where does the $2\pi\, x_k^*\, f(x_k^*)$ come from here? (What is the shape of the part of the solid we get by rotating one strip of the region under $y = f(x)$ about the y-axis?) What do we want to do to this sum to get the exact value of the volume? What is the corresponding integral formula for the volume, expressed using the function $f(x)$?

A 2 – I can find the area between two graphs for some specified range of x-values or by finding where the graphs cross.

1. Find the area below $y = e^{-x}$ and above $y = 1/2$ for $x > 0$.

2. Find the area below $y = \frac{1}{1+x^2}$ and above $y = \frac{1}{5}$.

3. Find the area between $y = x^2$ and $y = 2x$ for $-1 \leq x \leq 3$. Note: These graphs cross at least once in the given interval.

4. Find the area between $y = \sin(x)$ and $y = \cos(x)$ for $0 \leq x \leq \pi$. Note: These graphs cross at least once in the given interval.

5. Find the area between $y = x^3$ and $y = x/3$ for $-1 \leq x \leq 1$. Note: These graphs cross at least once in the given interval.

6. Set up an integral or integrals to find the area inside the unit circle centered at $(0, 0)$ and above the parabola $y = x^2$. Do not evaluate.

7. Set up an integral or integrals to find the total area between upper branch of the hyperbola $y^2 - x^2 = 1$ and its asymptote line $y = x$ for $x > 0$. Do not evaluate. What kind of integral is this?

A 3 – I can find volumes by slicing and by cylindrical shells.

1. A solid has a base equal to the unit circle in the xy-plane. The cross-sections by planes perpendicular to the x-axis are squares extending the full width of the base at that x. What is the volume of the solid?

2. A solid has a base equal to the unit circle in the xy-plane. The cross-sections by planes perpendicular to the x-axis are equilateral triangles extending the full width of the base at that x. What is the volume of the solid?

3. A solid is generated by rotating the region between $y = \sqrt{x^2 + 1}$ and $y = x$ for $0 \le x \le 2$ about the x-axis. What is its volume? (Any correct method is OK.)

4. A solid is generated by rotating the region between $y = \sqrt{x^2 + 1}$ and $y = x$ for $0 \le x \le 2$ about the y-axis. What is its volume? (Any correct method is OK.)

5. A solid is generated by rotating the region below $y = e^{-x}$ and above $y = 1/2$ for $x > 0$ about the x-axis. What is its volume? (Any correct method is OK.)

6. A solid is generated by rotating the region below $y = e^{-x}$ and above $y = 1/2$ for $x > 0$ about the y-axis. What is its volume? (Any correct method is OK.)

7. A solid is generated by rotating the region bounded above by $y = \sqrt{1 - x^2}$ and below by $y = 1/3$ about the x-axis. What is its volume? (Any correct method is OK.)

8. Is it true that

$$\int_{-2\sqrt{2}/3}^{2\sqrt{2}/3} \pi \cdot (\sqrt{1 - x^2})^2 - \pi \left(\frac{1}{3}\right)^2 \, dx = 2 \int_{1/3}^{1} 2\pi x \sqrt{1 - x^2} \, dx?$$

   Explain, either by interpreting the value of each integral as a certain volume or by computing both sides.

A 4 – I can solve modeling problems about exponential growth/decay, heating and cooling, and logistic population growth using solutions of differential equations.

1. In radioactive decay, the rate of change of the amount of an isotope present is proportional to the amount present at all times. A sample of radioactive carbon-14 contained 100 grams at time $t = 0$ years. After $t = 5730$ years, the amount had decayed to 50 grams. How long after $t = 0$ will it take for the amount to decay to 15 grams?

2. The population of a colony of bacteria is growing at a rate proportional to the population size. The population was 1000 bacteria at $t = 0$ and by $t = 1$ hour it had grown to 1500 bacteria. How long will it take for the population to reach 5000 bacteria?

3. A frozen pizza is taken out of 0°F freezer at $t = 0$ and placed into an oven held at a constant 400°F. After 5 minutes, the temperature of the pizza has increased to 50°F. How long will it take for the pizza to reach 300°F? Assume that the situation is governed by Newton's Law of Heating and Cooling: the rate of change of the temperature of the pizza is proportional to the difference between its temperature and the oven temperature.

4. A pizza with temperature 400°F is taken out of the oven at $t = 0$ and placed on a counter a kitchen held at a constant 70°F to cool before it is served. After 5 minutes, the temperature of the pizza has decreased to 250°F. How much longer will it take for the pizza to reach 200°F? Assume that the situation is governed by Newton's Law of Heating and Cooling: the rate of change of the temperature of the pizza is proportional to the difference between its temperature and the room temperature.

5. A population of rabbits is living in a habitat that can support at most 100 of them in the long run. At time $t = 0$ months there are 30 rabbits and the population has grown to 40 rabbits by time $t = 3$ months. How long will it take for the population to reach 95 rabbits? Assume logistic growth – that is, if $P = P(t)$ is the population, then $P' = rP(1 - P/K)$, where K is the carrying capacity, and r is a proportionality constant.

6. At $t = 0$, a population of 120 rabbits is introduced into a habitat that can only support at most 100 of them in the long run. The population has fallen to 110 by time $t = 3$ months. How long will it take for the population to reach 105 rabbits? Assume logistic growth – that is, if $P = P(t)$ is the population, then $P' = rP(1 - P/K)$, where K is the carrying capacity, and r is a proportionality constant.