Abstract

Three geometric construction problems—the duplication of the cube, trisection of an angle, and quadrature of a circle—fueled the mathematical enterprise of the ancient Greeks and the many generations of mathematicians that followed. In the strictest interpretation, solutions were restricted to those using only two tools: a straightedge and compass. In their search for solutions, however, the Greeks and early modern geometers developed a variety of mechanical and other solutions to the problems, extending this restriction. Despite these great advancements, Pappus, a commentator of ancient Greek mathematics, insisted that the planar problems of construction should be solved by a planar solution. This sparked a fruitful debate among mathematicians that came with the revival of these problems in the Western world in the late sixteenth and early seventeenth centuries. Descartes would provide the necessary algebraic foundation to complete the proofs of impossibility, which would be published in the nineteenth century by Wantzel and Lindemann. The proofs rely on Descartes' notions of polynomials and can be interpreted by the more modern concepts of field theory.

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Chapter 1

Introduction

The civilization of ancient Greece remains a beacon for its advancements in literature, architecture, philosophy, and mathematics. Indeed, the advancements of Greek geometers provided a foundation for the entire mathematical enterprise for the next two millennia, from the Classical period, through the Middle Ages, and into modernity in both the Islamic world and Europe. This progress is manifested not only by the content of the Greek works—their brilliance is evident—but also in how the ancients went about their research. They opened schools, championed reason and rationality, and were intrigued by the physical and natural complexities of the world. Such is the context for one of the greatest mathematical endeavors history has ever witnessed.

Historians have divided the Greek mathematical enterprise into two distinct periods. The first of which, the Classical period, dates from 600 to 300 BCE, roughly the three centuries from the demonstrative works of Thales to the extraordinary *Elements* of Euclid.[1] During this time, Athens flourished as the cultural hub, culminating in the establishment of Plato's Academy around 387 BCE. The Hellenistic, or Alexandrian, period ranges from 300 BCE to roughly 415 CE, beginning with the establishment of the University of Alexandria, of which Euclid served as the first mathematical chair.[2] Throughout the entirety of the epoch, there remained three clear features of Greek mathematics: the importance of deductive reasoning, the emphasis on abstract mathematics, and the prominence of geometry to solve problems.[3] Nowhere are these characteristics more apparent than the three construction

[1] Eves, p.84.
[2] Eves, p.112.
[3] Hollingdale, p.113.

problems that dominated the mathematical field in antiquity. These three problems are described as follows:

1. Duplication of the cube: constructing a cube having twice the volume of a given cube.

2. Trisection of an angle: constructing an angle with one-third the measure as that of a given angle.

3. Quadrature of the circle: constructing a square with the same area as that of a given circle.[4]

The duplication of the cube, tradition suggests, likely began from pronouncement of the citizens of Delos around 430 BCE. A plague had stricken the island of Delos, and a delegation of concerned citizens sought the oracle of Apollo to determine how the plague could be alleviated.[5] The oracle responded that the altar to Apollo—a cubical shrine—should be doubled in volume. Ignorant of geometry, the obedient Delians merely doubled the edges of the altar, increasing the volume eightfold.[6] Apollo then cursed the city by strengthening the plague. Plato claimed that this divine problem was conceived not to double the size of an altar, but to make the Greeks aware of their ignorance of geometry.[7] The mathematician and historian Howard Eves gives a similar tale, where King Minos commands that the cubical tomb of his son be doubled.[8] The former description appears to have been more common, as the duplication of the cube has long been referred to as the Delian problem.

The origins of the other two construction problems are not associated with such romantic tales. Historians believe that the trisection of an angle likely arose as a natural progression to the multi-section of a given angle, after bisections of both an angle and a segment were proved by Euclid.[9] Or perhaps, some argue, the trisection of an angle stemmed from attempts to construct a regular nine-sided polygon, after the construction of a regular pentagon had been discovered.[10] The origin of the quadrature of the circle is attributed to Anaxagoras, who attempted the problem while passing the

[4] Eves, p.89.
[5] Burton, p.119.
[6] Boyer, p.64.
[7] Burton, p.119.
[8] Eves, p.90.
[9] Eves, p.92.
[10] Heath, p.235.

time in prison.[11] Collectively, these three problems would dominate the field of geometry not just for the ancient Greeks, but for many mathematicians over the following centuries.

In solving these problems, however, the Greeks also initiated a deeper study of mathematical methodology by asking for solutions using particular tools. In fact, many commentators of Greek mathematics assert that the Greeks restricted themselves to the use of only two tools: a straightedge and a compass, though the origin of such a restriction is still greatly debated. Knorr, for example, writes that scholars agree that Euclid "would become the one responsible for the formal project of effecting geometric constructions within the restriction of employing compass and straightedge alone."[12] Similarly, Eves asserts that the postulates of Euclid's *Elements* restrict geometers to the use of the straightedge and compass.[13] These tools, therefore, are often called Euclidean tools, named after the geometer who begins his *Elements* with three postulates on how lines and circles were to be constructed.

Other scholars assert that it is Plato who is responsible for imposing the restriction to the Euclidean tools. In his work *Quaestiones Convivales*, Plutarch notes Plato condemned Greek mathematicians who endeavored to "bring down" geometry to "mechanical operations; for by this means all that was good in geometry would be lost and corrupted."[14] James Gow, a nineteenth-century historian of Greek mathematics, therefore states that "to Plato we owe the strict limitation of geometrical instruments to the ruler and compass."[15] David Burton, author of *The History of Mathematics*, agrees, stating "tradition has it that Plato insisted that the task be performed with straightedge and compass only."[16]

No matter the origin of this restriction, it is important to specify how these tools were to be used in problems of geometric constructions. With a straightedge, geometers could draw a straight line through any two points.

[11] Boyer, p.64.

[12] Knorr, p.15. Knorr goes on to say that while the restriction to these tools stems from Euclid's postulates, it is Oenopides of Chios, a fifth century BCE geometer, who "set about the task of regularizing and classifying problems according to the means of construction adopted," (p.16).

[13] Eves, p.90.

[14] An online version of the works of Plutarch can be found at the Perseus Digital Library. This passage is consulted from http://data.perseus.org/citations/urn:cts:greekLit:tlg0007.tlg112.perseus-eng1:8.2.1

[15] Gow, p.181.

[16] Burton, p.116.

Similarly, with a compass they could draw a circle with any point as its center and passing through any other point. Transferring distance was prohibited—thus the straightedge could not be marked or graduated and the compass must be "regarded as collapsing as soon as either point is lifted off the paper."[17] In the fourth century BCE, nearly seven hundred years after Euclid, the mathematician Pappus comments on the classification of geometric problem solving, stating that "there are three kinds of problems in geometry."[18] The first are called planar problems, so named because they can be solved by straight lines and circles and therefore their construction has its origin in the plane. There are also linear geometric problems, which are generated from more complex lines, such as "spirals and quadratrices and conchoids and cissoids."[19] Finally, Pappus classifies problems as solid if their solution involves the use of curves defined using solid figures, such as plane sections of a cone.

Pappus' influence on geometry and the legitimacy of constructions extended into the early modern period, over one thousand years after his death. Indeed, the Latin translation of his *Collection* in 1588 by Commandino would provide the spark for many mathematicians to take up these concerns.[20] The mathematician and historian Henk J.M. Bos describes how Christopher Clavius, Johannes Kepler, Johannes Molther, Francois Viète, and René Descartes each addressed the matter of demarcating between legitimate and illegitimate geometrical constructions and classifying the different geometrical procedures. Each, in turn, offered his own solutions to these three problems of antiquity, extending the field of mathematical knowledge. Perhaps most significantly, Descartes made advancements in the field of algebra that would ultimately provide the bridge between Classical geometry and more modern applied and analytical mathematics.

Moreover, although they were unable to solve the three construction problems using just the Euclidean tools, ancient Greek geometers did produce remarkable solutions using other methods. Hippias of Elis, for example, developed the quadratrix curve in the fifth century BCE that would later be used to both square the circle and trisect the angle.[21] Archimedes, the third century BCE geometer and inventor, developed a spiral that could produce

[17] Burton, p.116.
[18] Knorr, p.341
[19] Ibid, p.342.
[20] Bos, p.37.
[21] Boyer, p.69.

solutions to the angle trisection and the squaring of the circle.[22] Around 350 BCE, the Greek mathematician Menaechmus employed certain conic sections to duplicate the cube, while nearly a century later Nicomedes achieved the same result using conchoidal curves.[23] Some geometers even invented their own apparatuses to construct these curves, stopping at no limits to provide solutions to these problems, and advancing the mathematical field beyond what their predecessors had established.

While mechanical solutions to the three construction problems flourished throughout antiquity and the early modern period, solutions using just the Euclidean tools were not proved to be impossible until the nineteenth century. In 1837, for example, the French mathematician Pierre Wantzel detailed the proofs of impossibility of trisecting an angle and duplicating the cube using just a straightedge and compass.[24] Rooted in these proofs is the concept of reducing geometric problems to algebraic equations, first set forth by early sixteenth and seventeenth century mathematicians, particularly Descartes. These two problems reduce to cubic equations that do not yield rational roots and do not have a degree of a power of 2, and hence their construction using the Euclidean tools is impossible.[25] The quadrature of the circle was proved impossible in an 1882 paper by Ferdinand von Lindemann. He proved that π is a transcendental number that cannot be the root of an algebraic equation, hence it follows that a circle cannot be squared using just a straightedge and compass.[26]

This thesis will study all the stages in this long and tangled story, with the aim of showing how the Greeks' work on these problems stimulated later developments.

[22] Ibid, p.126.

[23] Burton, p.123.

[24] Wantzel's work, *On the means of recognizing whether a geometric construction can be made with straightedge and compass*, was translated from French to English by John Little.

[25] Ibid, p.121.

[26] Boyer, p.573.

Chapter 2

Mechanical and Other Solutions to the Construction Problems

The search for solutions to the three famous problems fueled the mathematical enterprise of the ancient Greeks. Yet, early on in their attempts, the Greeks realized the restrictions of using only the Euclidean tools were hindering their abilities to develop solutions.[1] Many mathematicians, therefore, developed their own techniques to solve the construction problems, using methods and tools beyond the use of straightedge and compass. In this regard, these geometers were successful. Yet attempts to provide solutions to the three construction problems using only the Euclidean tools persisted through the Dark Ages and into the early modern period.

The first mechanical method used to solve the construction problems was developed by Hippias of Elis in the fifth century BCE. The only surviving sources on the life of Hippias come from the dialogues of Plato, in which he is cast in a mostly negative light.[2] Born about 460 BCE, Hippias was part of a group of itinerant teachers, called "sophists," who would travel throughout Greek cities, sharing their mathematical and philosophical knowledge in exchange for money. While many sophists were well-informed and reputable scholars serving as effective tutors, others were mere imposters seeking money, and thus "accusations of shallowness directed against the sophists"

[1]Burton states that "early investigators must have suspected that the allowable means were inadequate," (p.116).

[2]Burton notes that Plato describes Hippias as "an arrogant, boastful buffoon," (p.125).

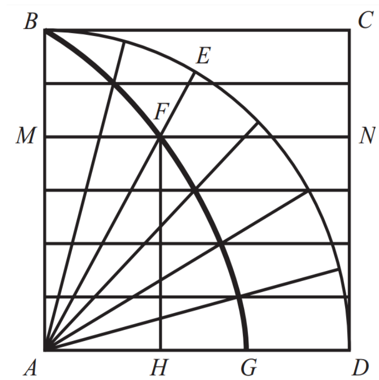

Figure 2.1: The quadratrix curve of Hippias, the curve BFG.

were, in some ways, warranted.[3] Perhaps not surprisingly, Plato and other skeptics castigated the sophists, and the term acquired a negative, cynical connotation.

Leaving aside accusations of his morals, it is true that Hippias' quadratrix would become a critical mathematical contribution to the solutions of the three geometric problems.[4] The construction of the quadratrix curve, pictured in Figure 2.1, can be described as follows.[5]

Given a square $ABCD$, let segment BC move down with a constant velocity toward segment AD. At the same moment that BC leaves its initial position, let side AB rotate clockwise with a constant velocity, so that both segments coincide at exactly the same time with AD. At any given moment, define the positions of the two moving lines to be MN and AE, intersecting at point F. The locus of these intersection points forms the quadratrix curve BFG. This definition, however, does not locate any point on AD. If the moving segment MN and the rotating radius AE coincide with AD simultaneously, then they intersect everywhere along AD. The point on the quadratrix, labelled as G, can only be defined by the modern notion of a limit of points F on the sector of the curve.

The quadratrix is therefore the first curve beyond the straight line and

[3]Boyer, p.68.

[4]The definition of the quadratrix curve can be found on Burton, p.126 and Boyer, p.68-9.

[5]This construction can be found in the seventh edition of Burton's *The History of Mathematics: An Introduction* on p.131.

circle curve "that could not be drawn by the traditionally required straightedge and compass" but rather must be plotted point by point.[6] Hippias had also developed an apparatus which could effectively draw a quadratrix curve, leading many geometers to further reject its use on the grounds of its mechanical origins. Plutarch, for example, cites Plato as chastising mechanical constructions, for they make "sensible things more powerful over us than intelligible, and by forcing the understanding to determine rather according to passion than reason."[7] To Plato, geometry was a divine gift, and it is through mechanical operations that one "loses that instrument and light of the soul, which is worth a thousand bodies, and by which alone the Deity can be discovered."[8]

The quadratrix curve of Hippias can easily be used to trisect a given angle. But, it is important to first consider the case for trisecting a given line segment, which is used in the proof. This construction, however, does not follow immediately from any proposition Euclid provides; rather, it uses the result of a construction he gives in Proposition 10 of Book IV and facts about similar triangles. Given a segment AB to be trisected, draw another line, CA, intersecting AB at A, forming an $\angle BAC$. Construct a circle with center A that intersects both AB and AC, and denote by X the intersection with AC. Now, draw another circle with center X and radius AX, and let Y denote the point of intersection with AC. Draw a third circle with center Y and radius YX and call Z the point of intersection with AC. Now, join Z and B. Construct a line from Y to AB parallel to ZB, with S as the point of intersection. Finally, draw a line from X to AB parallel to both YS and ZB, with R being the point of intersection with AB. The claim is that $AR = RS = SB = \frac{1}{3}AB$, and this result follows from the facts that lines ZB, YS, and XR are all parallel, $AX = XY = YR$, and properties of similar triangles.

With this construction in hand, the trisection of the angle, shown in Figure 2.2, can be described as follows.[9]

Suppose, for example, that $\angle XAY$ is the angle to be trisected. Position this angle within a square, with the initial ray coinciding with the base of the

[6] Burton, p.125. Boyer also states that that we owe Hippas "the introduction into mathematics of the first curve beyond the circle and straight line," (p.68).

[7] Plutarch, http://data.perseus.org/citations/urn:cts:greekLit:tlg0007.tlg112.perseus-eng1:8.2.1.

[8] Ibid. For Plato, "God always plays the geometer."

[9] Burton, 7th ed. p.132.

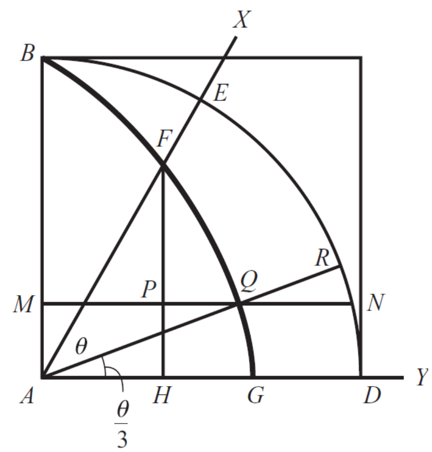

Figure 2.2: The quadratrix curve to trisect an angle.

square. Construct a quadratrix curve, BFG, which intersects the terminal ray, AX, at F. Erect a perpendicular from AD to F, with the point on AD denoted by H. Trisect segment FH to get the point P, and draw the parallel through P to AD, labelled as MN. This parallel then intersects the quadratrix curve at Q, and $\angle QAD$ is one-third of angle $\angle XAY$.[10]

The use of the quadratrix to square the circle—which ultimately gave the curve its name—is given by Dinostratus in the mid-fourth century BCE. It should be noted, however, that this work of Dinostratus has survived only because of the preservation of his work in later texts, such as those by Pappus. This solution, however, requires the use of the point G. If this point is assumed to be found, Pappus provides a proposition involving three line segments and the circular arc BD, stated as

$$\widehat{BD}/BC = BC/AG.$$

This is proved by a double *reductio ad absurdum* argument, where the alternatives to the proposition are proven to be false.[11] Assume that

$$\widehat{BD}/BC = BC/AK,$$

[10]Boyer, p.69.

[11]Burton notes that this is one of the "earliest examples in Greek mathematics of the indirect method of reasoning Euclid used so extensively," (p.127).

where $AK > AG$. Let the circle with center A and radius AK intersect the quadratrix at F and side AB at L. From F draw FH perpendicular to AD. As Burton states, "since corresponding arcs of a circle are proportional to their radii,"

$$\widehat{BD}/BC = \widehat{LK}/AK,$$

and from the hypothesis, it follows that

$$\widehat{LK} = BC.$$ [12]

Yet it is known that

$$\widehat{LK}/\widehat{FK} = BC/FH$$

from the definition of the quadratrix. Since

$$\widehat{LK} = AB,$$

it follows that

$$\widehat{FK} = FH,$$

which cannot be true, since "the perpendicular is shorter than any other line or curve" from point F to AD.[13]. A similar argument will show that AK cannot be less than AG, thus $AK = AG$ and

$$\widehat{BD}/BC = BC/AG.$$

Therefore, if $BC = 1$, $AG = \frac{2}{\pi}$. It is thus possible to construct a segment with length $\frac{\pi}{2}$, which can then be doubled to produce a segment with length equal to π. To construct a square with area equal to π requires the construction of a segment of length $\sqrt{\pi}$. To do this, construct a segment AB with length π, and extend from B a unit segment BC of length 1, to get segment AC equal to $\pi + 1$. Now, construct a circle with diamter AC. From B, erect a perpendicular to AC, and let D denote the point of intersection with the circle. Join AD and DC to make a right triangle, $\triangle ADC$, which is similar to both $\triangle ABD$ and $\triangle DBC$. Therefore,

$$\frac{x}{1} = \frac{\pi}{x}$$

[12] Burton, p.127

[13] Boyer, p.97

and thus $x^2 = \pi$, so $x = \sqrt{\pi}$. Hence, given a segment of length π, a square with area π can be constructed, and the squaring of the circle is complete.

Dinostratus' brother Menaechmus was also a famed mathematician of the fourth century BCE, eventually becoming a tutor to Alexander the Great.[14] Prior to Menaechmus, Hippocrates of Chios made considerable progress in solving the duplication of the cube, reducing it finding two mean proportionals of a and $2a$; that is, lengths x and y satisying

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a} \tag{2.1}$$

While Hippocrates himself did not find the mean proportionals using the Euclidean tools, it was nevertheless a "significant achievement" to reduce "a problem in solid geometry to one in plane geometry."[15] Menaechmus, however, anticipating work with conic sections, discovered that "the cutting of a right circular cone by a plane perpendicular to an element of the cone" produces a "family of appropriate curves" that satisfy the property expressed by Hippocrates.[16]. These curves, in modern terms, would become known as the ellipse, parabola, and hyperbola, which are names given by Apollonius.

With modern algebraic notation, the discovery articulated by Menaechmus can provide a theoretical solution to the duplication of the cube.[17] Given a cube of edge a, construct two parabolas, one with latus rectum a and the other with latus rectum $2a$, sharing a common vertex and perpendicular axes, as pictured in Figure 2.3.[18]

Thus, the two equations corresponding to the parabolas are

$$x^2 = ay, \tag{2.2}$$

$$y^2 = 2ax \tag{2.3}$$

Isolating x in (2.3) gives

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[14]Burton, p.123.

[15]Burton goes on to say: "From this time on, the duplication of the cube was always attacked in the form in which Hippocrates stated it," (p.123).

[16]Boyer, p.93.

[17]A thorough explanation of this solution can be found on Burton, p.123 and Boyer, p.95.

[18]Burton, 7th ed. p.128.

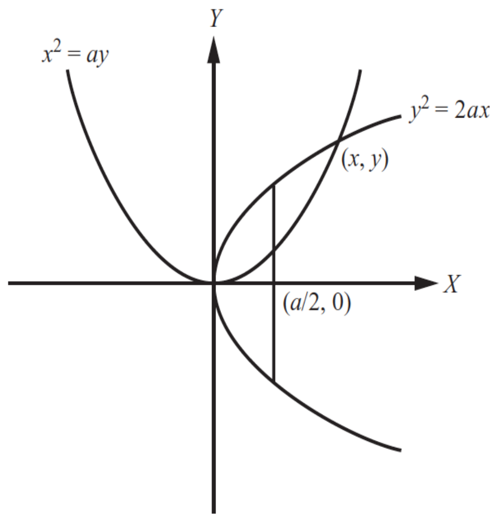

Figure 2.3: Conic sections to duplicate the cube.

$$x = \frac{y^2}{2a}$$

which substituted into (2.2) yields

$$\frac{y^4}{4a^2} = ay$$

or

$$y = \sqrt[3]{4}a$$

Substituting this result into (2.3)

$$(\sqrt[3]{4}a)^2 = 2ax$$

or

$$x = \sqrt[3]{2}a$$

Therefore, the x-coordinate of the intersection of the two parabolas is the desired edge of the cube sought. This solution is not mechanical in the same sense as the quadratrix curve, which uses moving segments to construct an entirely new curve besides those the Euclidean tools provide. Menaechmus' solution, rather, is purely theoretical. However, the desired curves and segments cannot be constructed with a straightedge and compass, so this approach also goes beyond the strictest interpretation of the construction problems.

About a century after Menaechmus, the third-century BCE geometer Nicomedes attempted to find solutions to the problems of the angle trisection and cube duplication using a *neusis* construction. The word *neusis* is Greek, and it roughly means "sloping" or "verging towards." Bos defines the *neusis* construction as follows: "Given two lines, a point O and a segment a, to draw a straight line through O intersecting the two lines in points A and B such that $AB = a$."[19] Therefore, the mechanical nature of this type of construction lies in marking a straightedge to transfer a segment of length a, which was in violation of the use of Euclidean tool. If marking a straightedge were permissible, then the transferring of distances would be legitimate, and thus this construction would be accepted as planar.

Burton describes Nicomedes' use of the *neusis* to trisect an angle, pictured in Figure 2.4.[20]

Let $\angle AOB$ be a given angle. Through B, draw a line perpendicular to segment OA at C and another line parallel to OA. Using a straightedge, mark the length $a = 2OB$ and slide the straightedge so that it passes through O, with endpoints of segment a as P on BC and Q on BD (so $PQ = a$). If M is the midpoint of PQ, then $OB = \frac{a}{2}$ by construction, $PM = MQ = \frac{a}{2}$ by definition of a midpoint, and $BM = \frac{a}{2}$ since "the midpoint of the hypotenuse of a right triangle is equidistant from the endpoints of its sides."[21] Therefore, $\angle MOB = \angle BMO$, since the base angles of an isosceles triangle are equal. In $\triangle BMQ$, $\angle BMP = \angle MBQ + \angle MQB$ from the exterior angle theorem, and thus $\angle BMP = 2\angle MQB$ since $\triangle MQB$ is isosceles. Since BD and OA

[19] Bos, p.168.

[20] Burton, 7th ed. p.129.

[21] Burton, p.123. The corresponding right triangle in the figure is $\triangle BPQ$.

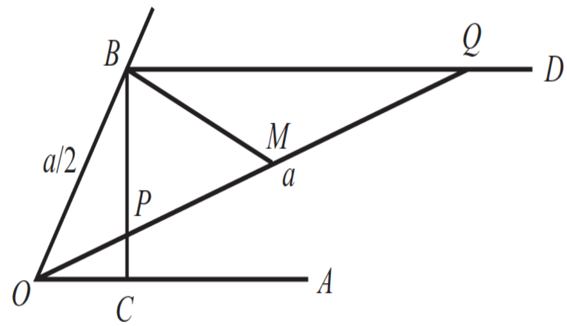

Figure 2.4: The neusis construction to trisect an angle.

are parallel, the interior angles $\angle AOQ$ and $\angle BQO$ are equal. Thus, since $\angle AOB = \angle AOQ + \angle QOB$, $\angle AOB = 3\angle AOQ$, and therefore $\angle AOQ = \frac{1}{3}\angle AOB$, and the angle is trisected.

Nicomedes also approached the duplication of the cube by means of finding the two mean proportionals, much like Menaechmus.[22] His solution is described as follows, referring to Figure 2.5.[23]

Construct a rectangle $ABCD$ with side $AB = a$ and $AD = 2a$, with M being the midpoint of AD and N the midpoint of AB. Extend the segments CM and BA to meet in G. Choose the point F on the perpendicular FN to be such that $FB = a$. Draw a segment BH parallel to GF and a segment FP to cut segment AB at p, with P chosen such that $HP = a$. Extend the segments PC and AD until they meet at Q. Let $DQ = x$, $BP = y$, and $FH = z$.

With this construction established, one can see that $\triangle PAQ$, $\triangle PBC$, and $\triangle CDQ$ are similar. Each triangle contains a right angle, $\angle PCB = \angle PQA$ since they are alternate interior angles, and $\angle APC = \angle DCQ$ since they are also alternate interior angles. This similarity establishes the relationship

$$\frac{a}{x} = \frac{y}{2a} = \frac{a+y}{2a+x}. \tag{2.4}$$

Likewise, $\triangle PBH$ and $\triangle PGF$ are similar, since both contain $\angle P$, $\angle PBH =$

[22]See Burton, p.123.
[23]Burton, 7th ed. p.129.

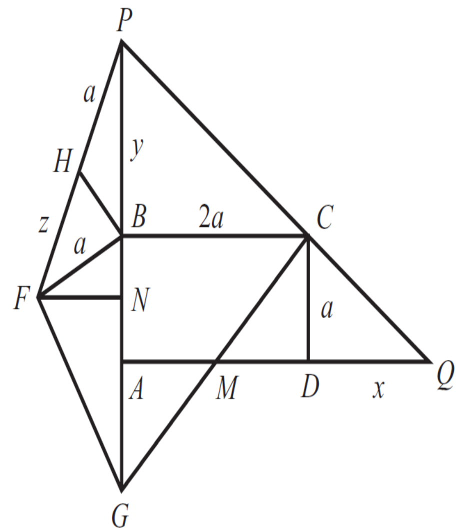

Figure 2.5: Two mean proportionals to duplicate the cube.

$\angle PGF$, and $\angle PHB = \angle PFG$ by definition of alternate interior angles. This similarity establishes the relationship

$$\frac{a}{y} = \frac{z+a}{y+2a}.$$

Cross-multiplication yields $ay + 2a^2 = yz + ay$. Subtracting ay from both sides leaves $2a^2 = yz$, or $\frac{a}{z} = \frac{y}{2a}$. Substituting this into (4) shows that $\frac{a}{x} = \frac{a}{y}$, and thus $x = z$. Since $\triangle FNB$ and $\triangle FNP$ have FN as a common side, the Pythagorean Theorem produces

$$a^2 - \frac{a}{2}^2 = (x+a)^2 - \left(y + \frac{a}{2}\right)^2,$$

or

$$\frac{x}{y} = \frac{y+a}{x+2a}.$$

From (2.4), this yields

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$$

which is precisely the equation (1) that Hippocrates of Chios discovered in the fourth century BCE. Thus, the segments $DQ = x$ and $BP = y$ are the two mean proportionals between a and $2a$, which implies that given a cube of edge a, x is the edge of the duplicated cube.

The third century BCE also saw the flourishing of Archimedes, considered by many scholars as the greatest mathematician of all antiquity. Few details of his life are known, though it is speculated that he was born about 287 BCE in Syracuse, a Greek city on the coast of Sicily.[24] While it is likely that he studied at the Museum in Alexandria under former students of Euclid, he spent most of his productive years in Syracuse, where he devoted his time to studying and experimenting.[25] Tradition has it that Archimedes was killed in 212 BCE during a siege of Syracuse by Roman troops in the Second Punic War.

In addition to mathematics, Archimedes' interests also included "astronomy, hydraulics, mechanics and general engineering."[26] It was indeed his mechanical inventions that earned Archimedes great fame throughout his life. His Archimedean screw, for example, a device still in use today, serves to raise water, primarily in irrigation systems. He also invented a number military defenses used to fight off the siege of the Romans, and his work with pulleys and levers led him to boast: "Give me a place to stand and I will move the earth."[27].

Despite his mechanical achievements, Archimedes "remained firmly within the Greek philosophical tradition" in championing theoretical studies and abstract thought.[28] He was, therefore, attracted to the three famous construction problems of geometry, and invented a new type of curve that would provide solutions to both the angle trisection and quadrature of the circle. The curve, known as the Archimedean spiral, is pictured in Figure 2.6 and described in his treatise *On Spirals*:[29]

> If a straight line [half-ray] one extremity of which remains fixed be made to revolve at a uniform rate in the plane until it returns to the position from which it started, and if, at the same time as the

[24] Burton, p.186. Burton also notes that, according to Plutarch, Archimedes' family was of the same royalty as King Hieron II, ruler of Syracuse.

[25] Ibid, p.187.

[26] Hollingdale, p.65.

[27] Burton, p.187

[28] Hollingdale, p.66.

[29] Burton, 7th ed. p.204.

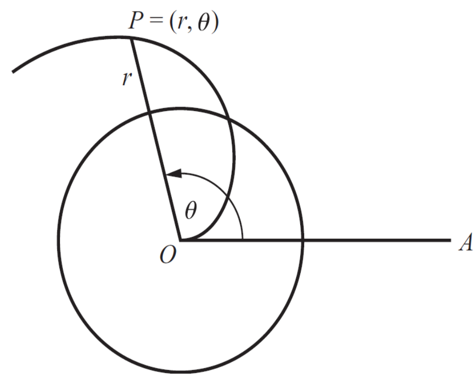

Figure 2.6: Construction of the Archimedean spiral.

> straight line is revolving, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane.[30]

As Boyer explains, the spiral is "the plane locus of a point which, starting from the end point of a ray or half line, moves uniformly along this ray while the ray in turn rotates uniformly about its end point."[31] Yet, while Archimedes' spiral was, like many of his developments, inspired by practical considerations, it was nevertheless considered a mechanical solution. Like the quadratrix, the spiral is a curve constructed from means other than a straightedge and compass, and therefore could not provide a legitimate solution to the construction problems, according to the strictest interpretation.

However, with the use of the Archimedean spiral, the trisection of an angle can be carried out with ease.[32]

Refer to Figure 2.7.[33] Take $\angle POA$ to be trisected. Position the angle so that the vertex and initial side of the angle coincide with initial point O of the spiral and the initial position OA of the rotating ray. The point P is given to be the intersection of the terminal side of the angle and the spiral. Trisect OP at Q and R, and construct two circles with centers at O and OQ and OR as radii. If these points intersect the spiral at points U and V, lines

[30]Burton, p.196.
[31]Boyer, p.126.
[32]A description of this solution can be found on Boyer, p.126 and Burton, p.200.
[33]Burton, 7th ed. p.210.

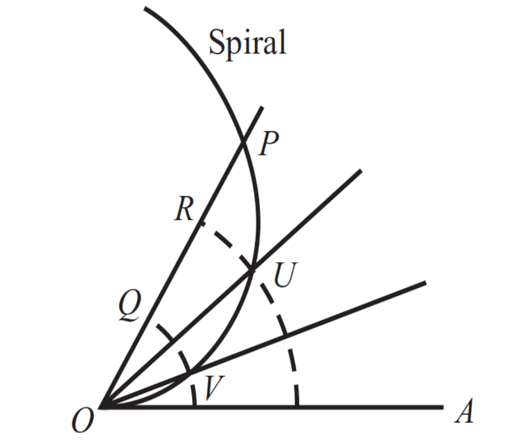

Figure 2.7: The Archimedian spiral to trisect an angle.

OU and OV trisect $\angle POA$.

The Archimedean spiral also provides a clever solution to the squaring of the circle, pictured in Figure 2.8.[34]

Given a circle with center O and radius OA, construct a spiral from O such that the end of its first revolution coincides with A. Construct a tangent to the spiral at A, and extend a perpendicular to OA from O, where the tangent and the perpendicular intersect at B. Using Cartesian coordinates, label the point O as the origin, $(0,0)$, and call the radius of the circle x, where A is labelled as $(x,0)$. The slope of the tangent line can be derived using a parameterization:

$$\alpha(\theta) = \left(\frac{\theta x}{2\pi}\cos(\theta), \frac{\theta x}{2\pi}\sin(\theta)\right).$$

The tangent line, therefore, has a parameterization

$$\alpha'(\theta) = \left(\frac{x}{2\pi}\cos(\theta) - \frac{\theta x}{2\pi}\sin(\theta), \frac{x}{2\pi}\sin(\theta) + \frac{\theta x}{2\pi}\cos(\theta)\right),$$

and thus

$$\alpha'(2\pi) = \left(\frac{x}{2\pi}, x\right).$$

[34]Burton, 7th ed. p.210.

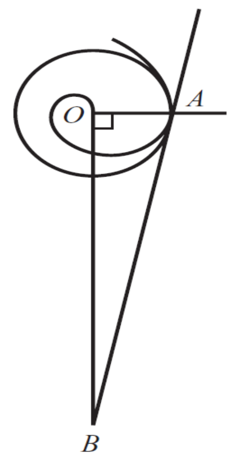

Figure 2.8: The Archimedian spiral to square the circle.

The slope of the tangent at 2π is given as

$$\frac{x}{\frac{x}{2\pi}},$$

which is 2π. Hence, the coordinates of B must be $(0,-2\pi x)$, and the length of the segment OB is given as $2\pi x$, which equals the circumference of the circle. The area of $\triangle OAB$, thererefore, is given as $\frac{1}{2}(x)(2\pi x)$ or πx^2. The area of $\triangle OAB$ is thus equal to the area of the circle.[35]

To transform the triangle into a square requires some algebraic manipulation. First, construct a line segment, AB equal in length to the area of the triangle and circle, say y. From point B, extend the segment a unit length to produce a new segment, AC, with length $y+1$. At B, erect a perpendicular BD to AC with length x. Bisect segment AC at O and construct a circle with center O and radius OA, with D on the circle. From propositions given in Book I of the *Elements*, $\triangle ADC$ makes a right triangle, which is similar to $\triangle ABD$ and $\triangle CBD$. This similarity establishes the relationship

$$\frac{x}{1} = \frac{y}{x},$$

[35]Heath, in his translation of Euclid in *The Thirteen Books of The Elements*, cites Proclus: "Archimedes actually proved that any circle is equal to the right-angles triangle which has one of its sides about the right angle [the perpendicular] equal to the radius of the circle and its base equal to the perimeter of the circle. But of this elsewhere," (p.347).

or $x^2 = y$. A square with side length x thus has the same area as the triangle and circle, and the squaring of the circle is accomplished.

It should be noted, however, that this algebraic technique would not be how the Greeks—Euclid and Archimedes in particular—would approach the problem. Heath, in his translation of the *Elements*, cites two propositions from Euclid that allow one to transform the triangle into a square with equal area: Proposition 45 from Book I, which enables one to construct a parallelogram equal to a given rectilinear figure, and Proposition 14 from Book II, which states that one can construct a square equal to a given rectilinear figure.[36]

The search for solutions for the construction problems did not end with the fall of the Greek world. Rather, it took two thousand years for valid proofs of impossibility to be published, cementing the claim that the three geometric problems of antiquity could not be solved by straightedge and compass alone. The mathematical developments in the early modern period that led to the breakthroughs of algebra and calculus would, ultimately, provide the missing link that was needed for the proofs.

[36]Heath, p.347. Heath adds a comment by Proclus: "I conceive that it was in consequence of this problem that the ancient geometers were led to investigate the squaring of the circle as well."

Chapter 3

The Transmission of Greek Work on the Construction Problems to Renaissance Europe

The Golden Age of Greek mathematics had reached its twilight at the end of the third century BCE. Over the next two centuries, Greece fell to Roman conquest, with many schools, texts, and intellectual ideas being destroyed and defeated.[1] Yet, during the third and fourth centuries CE, the mathematical field experienced a brief revival, where many commentators of Greek mathematics expounded on the works of their predecessors. The majority of these commentators, including scholars such as Hero, Eutocius, and Proclus, contributed very little in the way of original work in their publications.[2] These men, therefore, are not regarded as having advanced the development of geometrical ideas or introducing new concepts on technical solutions. They are, however, praised for the transmission of the textual works of their precursors, since most of our knowledge of ancient Greek mathematics stems from their commentary. Additionally, these commentators offered evaluative

[1] Burton notes that the Romans were more interested in the practical use of mathematics for engineering purposes, disregarding the theory behind it (p. 205).

[2] As Knorr asserts, "the prospect of new and interesting discoveries was discouraging," (p.340).

analysis of mathematical works of both their time and earlier.[3] Through the lens of these commentators, modern scholars are able to glimpse the field of geometrical problem solving as it advanced through ancient times. Nowhere is this more apparent than in the work of Pappus of Alexandria.

Pappus is the most renowned mathematician of his time, spanning the late third and first half of the fourth century CE. His most famous work is his *Mathematical Collection*, a series of eight books, in which he originally intended to consolidate much of the geometrical advancements that had taken place in the previous centuries.[4] Yet Pappus also offers original proofs, theorems, and concepts, extending and clarifying results of earlier geometers. One of his most interesting contributions is his classification of problems, where he attempts to assign geometric problems into categories based on the constructions required for their solutions. In doing so, Pappus has helped "preserve a shadow of ancient metamathematical thinking," where modern scholars can gain a better understanding of the context of the ancients' search for solutions of geometric construction problems.[5]

Despite the significance of Pappus as a commentator on ancient geometry, there is little scholarship that coherently traces the *Collection* from its fourth-century CE origin. Gow notes that Pappus "is not cited by any of his successors," while the works of others, like Ptolemy and Diophantus, were successfully transmitted through a "continuous history of progress."[6] It is not until Commandino's Latin translation of the *Collection* became available in 1589 that many early modern mathematicians gained a greater insight into the scope, methods, and results of Greek mathematics.[7]

In two nearly identical passages regarding the problems of the cube duplication and angle trisection, Pappus classifies geometric problems into three distinct categories: planar, solid, and line-like. Planar problems were those

[3]For example, Theon and Eutocius both contributed to the collection of editions of the works of Euclid, Archimedes, and Apollonius, three mathematical giants of Ancient Greece (Knorr, p.341).

[4]Burton describes that the first and second books of the *Collection* are missing, while Pappus' commentary on Ptolemy's *Almagest* is his only other surviving work (p.221).

[5]Knorr, p.341.

[6]Gow, p.308. He goes on to say that "no Indian or Arab ever studied Pappus or cared in the least for his style or his matter," (p.309).

[7]Commandino (1509-1575) was a sixteenth century Italian humanist and mathematician. He is most notable for translating the works of Greek mathematicians and commentators, including Aristarchus, Heron, Euclid, and Pappus. His translation of Pappus' *Collection* was published fourteen years after his death.

that could be solved by straight lines and circles, and therefore just by the use of the Euclidean tools. A solid problem, Bos translates, "employs surfaces of solid figures, namely the conic surfaces."[8] Finally, line-like, or linear, problems involve more complex lines and curves than these Euclidean tools or conics provide, such as the quadratrix, spiral, and cissoids. Several discrepancies in such a classification lead historians to suggest that this was largely Pappus' own categorization rather than that of previous mathematicians.[9] For example, from the modern point of view, if the planar class consists exclusively of lines having their origins in the plane, than it should also include the spiral, quadratrix, and conchoids.[10] Pappus also is clear that the solid class refers only to conic sections and not to constructions derived from curves or other surfaces studied in solid geometry. Such inconsistencies with respect to terminology suggest that even Pappus himself did not appreciate some of these subtleties in the classification he laid out.

These discrepancies also imply that Pappus' influence would likely have been negligible if his *Collection* had only provided these classifications. Yet, there are two reasons why his work would become so influential. First, Pappus provides an extensive discussion of geometric solutions by earlier mathematicians. For many of these geometers, their work survives only in Pappus, and thus he provides essential information explaining the scope and content of Greek mathematics. Second, he combines his classification with, as Bos states, a "methodological precept: problems should be constructed with the means appropriate to their class."[11] This, indeed, is where Pappus' originality is evident. Previous classifications of problems, whether by Pappus or other geometers, were merely descriptive—problems were differentiated based on the method of construction. Pappus, however, provides a normative claim to distinguish between problems, insisting that problems be solved with a solution appropriate to their class.

Ironically, there is also a discrepancy among different translations of this precept by historians, leading to significantly different interpretations not

[8]Knorr, p.341.

[9]For example, Knorr cites that Apollonius is attributed to classifying loci as "ephectic," "diexodic," and "anastrophic" based on the dimension required to construct the locus (p.343).

[10]The definition and constructions of these and other curves will be provided in Chapter 2. For now, suffice it to say they are other means the Greeks developed to create curves other than a compass and straightedge.

[11]Bos, p.49.

only of the passage itself, but also of its impact on the entire mathematical enterprise of problem solving. Knorr translates the precept as follows:

> "The following sort of thing somehow appears to be no small error to geometers: whenever a planar problem is found by someone via conic or linear [lines], and on the whole whenever [some problem] is solved from a class other than its own, for instance, the problem on the parabola in the fifth book of Apollonius' *Conics* and the solid *neusis* toward a circle assumed by Archimedes in the book *On the Spiral*; for by using no solid one is able to find the theorem proved by him."[12].

He goes on to call such a precept a "rather timid pronouncement" and virtually the only surviving statement that provides a formal preference for planar constructions over others.[13] If this precept were indeed timid, as Knorr suggests, it is unlikely such a restriction would have any strong influence over geometers, either in Antiquity or the early modern period.

Commandino, however, provides a much more dramatic translation of the original Greek text. As Bos translates Commandino's passage:

> Among geometers it is in a way considered to be a considerable *sin* [emphasis added] when somebody finds a plane problem by conics or line-like curves and when, to put it briefly, the solution of a problem is of an inappropriate kind.[14]

This translation, therefore, carries a much more compelling tone than Knorr's "no small error." This "strict directive" would therefore have a wide influence on the field of problem solving. Indeed, Bos provides several examples of later geometers who cite Commandino's translation of this precept in

[12]Knorr, p.345

[13]Yet, Knorr describes, this pronouncement is "almost invariably presented in modern accounts as the principal objective of problem solving throughout the ancient tradition," (p.345).

[14]Bos, p.49. Bos also gives Commandino's Latin translation, in which he uses the word *peccatum* as a translation of the original Greek $\alpha\mu\acute{\alpha}\rho\tau\eta\mu\alpha$, which Bos then translates as "sin." Actually, the full semantic range of the Greek word also includes failure, fault, and error (see the Greek-English Lexicon by Liddell and Scott). That Commandino chooses the term *peccatum* or "sin" is perhaps a reflection of sixteenth-century Italy, when Latin was used almost exclusively by scholars and in the Church.

their works.[15]

While the influence on later geometers may be apparent, much evidence refutes the assumption that ancient geometers at the height of the Greek enterprise were guided by this rule. To impose a restriction on problem solving would have served as a detriment to the search for solutions. In the early stages of geometric development, Knorr states "the ambition is to find solutions by whatever means one can."[16] This they did—Hippocrates constructs quadratures of lunules using a *neusis*, Menaechmus duplicates the cube via conics, and Archimedes trisects the angle with a spiral. Even Euclid, Knorr argues, should not be considered as having followed Pappus' precept. Although the postulates in Book I of the *Elements* lead only to planar constructions in Pappus' scheme, Euclid still pursued the study and constructions of solid problems, especially from conic sections, in many of his later works.[17] Additionally, when geometers would criticize others for their methods of construction, these criticisms were rooted in issues of practicality and not if they were solved by the means according to their class. In this light, even if it were a common opinion, the influence of the precept articulated by Pappus seems negligible during the Golden Age of Greek mathematics.

It is also possible, of course, that this precept was not consciously ignored, but completely unknown to the early geometers. Knorr states that a formal restriction to the use of compass and straightedge constructions would be premature at the time of Hippocrates in the fifth century BCE.[18] The body of mathematical work had not yet evolved sufficiently to create such a formal restriction as to narrow the solutions to the use of the Euclidean tools. Knorr asserts that until the "geometric corpus had attained a size and diversity meriting" a restriction, it would be unwise to engage "in these formal inquiries."[19] Such a size, he believes, was reached by the time of Apollonius in the third century BCE. While previous geometers had not provided a clas-

[15]Ibid. p.50. Bos provides three examples of early modern mathematicians citing Pappus' "sin" or "error" passage. Most notably, Descartes states it would be "an error in geometry...to try in vain to construct some problem by a simpler kind of curves than the nature of the problem allows" [(Descartes 1637], p. 371).

[16]Knorr, p.345

[17]Euclid wrote several other mathematical treatises besides his *Elements*, but many of these works have not survived. For example, his treatise *Conics* consists of four books that were completed by Apollonius, who added four books himself (Heath, p.438).

[18]Knorr goes on to say that the "enterprise of discovering the solutions to problems could hardly be well served by the imposition of a restriction at this early stage," (p.40).

[19]Ibid, p.40.

sification of distinct classes of problem-solving methods, Apollonius appears to have provided such a distinction in several of his later works.[20] Apollonius had also sparked a contentious environment from his theory of conics, and it seems evident that such an environment merited the formal distinction of appropriate solutions. For example, Knorr states that Apollonius was "bound to clash" with his contemporaries working closely on similar matters and mentions how Apollonius was "notably ungenerous" in commenting on the achievements of his predecessors, like Euclid, and his "grudging acknowledgement of others' merit must surely have been an enormous irritant to many."[21]

Both Bos and Knorr agree that even Pappus himself did not subscribe to his own rule. He critiques the works of Archimedes and Apollonius, especially the solid construction of Archimedes' *neusis*.[22] Pappus goes on, however, to provide an alternative construction using conic sections, which are therefore solid by his distinction. Elsewhere Pappus gives constructions via the quadratrix and other more complex curves. He formally provides a construction of a *neusis* from the intersection of conics and uses this construction to trisect the angle.[23] Knorr also details that it is not only what Pappus includes in his *Collection* that generated confusion, but also what he omitted. To classify a problem as solid, a geometer must prove that no planar construction is possible.[24] Yet neither Pappus, nor any other Greek mathematician, provides such proofs; he merely claims that cube duplication and angle trisection problems are solid in nature.

Pappus' inconsistencies in claim and practice have led many geometers and historians astray. It is possible, Knorr speculates, that Pappus had extrapolated the division of loci—which had previously been classified in descriptive accounts—to the entire field of geometric problem solving.[25] He

[20]Knorr, p.344. Knorr cites that the terms "planar locus" and "solid locus" are attributed to Apollonius as evidence that the field of geometry warranted a classification by his time. In fact, Knorr states, his *On Solid Loci* gives considerable attention to the "formalization of the results attained within the analytic tradition up to his time," (p.348).

[21]Ibid, p.329.

[22]Ibid, p.345.

[23]Bos adds that Pappus provides this construction after first demonstrating that that a "*neusis* between perpendicular lines could be performed by means proper to solid problems," (p.55).

[24]But, a planar construction does indeed establish that the problem is planar (Knorr, p.347).

[25]Knorr adds that the perspective Pappus articulates was "far from being the "standard"

further believes that Pappus is responsible for this "defective conception" of the classification of problems, calling it a "misconception of the nature of the 3rd-century [BCE] enterprise of problem-solving."[26]

Yet the works of Pappus had a compelling influence on later geometers, particularly after Commandino's Latin translation at the end of sixteenth century.[27] His accounts of Greek mathematics sparked a fruitful debate in the early modern period of how to classify geometrical problems and distinguish between legitimate and illegitimate constructions.

One of these later mathematicians was Christopher Clavius, a German Jesuit geometer and astronomer of the late sixteenth and early seventeenth centuries. He was an admired publisher and composer of textbooks, teaching for several decades at the esteemed Roman College in Italy. In his treatise on the quadratrix curve first published in 1589, Clavius claims he had successfully provided the legitimately geometrical solution for the quadrature of the circle from his distinct construction of the quadratrix.[28] Although other geometers had used the quadratrix curve centuries before, Clavius provided a unique, pointwise construction that differed from the uniform-motion procedure of Hippias. Clavius' construction involves plotting an arbitrary number of points evenly distributed along the curve, with a close approximation for the endpoint, or the point labelled as G in Figure 2.2.[29] Additionally, Clavius justified his claim by asserting that his construction was more accurate than that of Pappus and equally as valid as the pointwise constructions of Apollonius, Menaechmus, and Nicomedes.[30]

From Clavius' quadtratrix construction and claim of a legitimate solution

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ancient view," but rather a "minority opinion held by a few of the late commentators," (p.348).

[26]Ibid, p.348.

[27]Bos provides several quotes by early modern geometers to illustrate this point. One is by Pierre de Fermat, a seventeenth-century French mathematician, who states "it has been often declared already, by Pappus and by more recent mathematicians, that it is a considerable error in geometry to solve a problem by means that are not proper to it," (p.50). Additionally, Alexander Anderson, a Scottish mathematician, says "it was considered no light offence for someone to solve a plane problem by means of conics or line-like curves," (p.219).

[28]Bos notes that the constructions Clavius provides were essentially the same as Pappus', with Clavius adding "an argument as to why these constructions should be considered as genuinely geometrical." (p.161).

[29]Bos translates a claim by Clavius that this approximation is "without...an error which can be detected by the senses," (p.162).

[30]Ibid, p.163-4.

to the problem of the squaring of the circle, one can speculate that he did not seem to think it a "considerable sin" to provide non-planar solutions to the construction problems. He does, however, change his stance on the validity of his construction. While in 1589 Clavius authoritatively asserted that he solved the squaring of the circle, he later relaxed his claim and adopted a more cautious perspective. The reason for his change of mind is unknown, yet it can be speculated that perhaps the impracticality of the construction led other mathematicians to critique his work. Yet, undoubtedly, the extent of Clavius' influence was great, ultimately providing the foundation for Descartes' work on algebraic analysis.

While Clavius was teaching and writing in Rome, the French mathematician Francois Viète was developing his own contributions to the field of geometric constructions. In 1593, just four years after Clavius' edition of Euclid's *Elements*, Viète published his *Supplement of geometry* in which he suggested that *neusis* constructions be admitted as a postulate. Indeed, the opening sentence of the work addresses this issue. As Bos translates

> [To supply the defect of geometry, let it be conceded] To draw a straight line from any point to any two given lines, the intercept between these being any possible predefined distance.[31]

This claim would therefore make a *neusis* construction as legitimately geometrical as the use of the Euclidean tools. The very title of his work reflects this necessity—geometry, to this point, was deficient and thus needed a supplement, one capable of allowing for solutions beyond just straightedge and compass. By admitting the *neusis* as a postulate, Viète was effectively avoiding the issue of its construction, asserting it was as obvious as drawing straight lines and circles. As a result of this new postulate, Viète proved that any problem producing third- or fourth-degree equations could be reduced to finding a mean proportion or by trisecting an angle.[32] Hence, all non-planar problems reducible to third- or fourth-degree equations, "legitimated by a new geometrical postulate," could now be constructed by a *neusis*.[33] In this way, Viète demonstrated how algebraic analysis could extend the field of Euclidean geometry.

[31]Ibid. p.168.

[32]Ibid, p.169.

[33]Ibid. p.176. Bos also states that with the acceptance of the *neusis* postulate, "all problems leading to equations of degree less than five were duly brought within the power of legitimate geometry," (p.169).

Viète, however, not only asserted his new postulate as geometrical, but also commented on other methods of construction that were resurfacing at the turn of the seventeenth century. In regards to shifting rulers, for example, Viète described that they are mechanical and not geometrical, despite their accuracy.[34] Viète also gives a pointwise construction of the quadratrix curve yet does not, unlike Clavius, claim it as a geometrical solution. In his 1600 publication *Apollonius from Gaul*, Viète comments on non-planar constructions from several ancient geometers. Bos translates Viète as saying that Dinostratus was able to square the circle via the quadratrix curve and Archimedes via the spiral, "but is the circle thereby geometrically squared? No geometer would make that proposition."[35] Here, Viète is not only classifying problems as planar and non-planar, but definitively stating that non-planar solutions are not geometrical. Viète, therefore, not only classified problems based on their construction, but also demarcated between legitimate and illegitimate geometrical constructions as a consequence of his supplemental postulate.

Johannes Molther, another seventeenth century mathematician, offered his own interpretation of geometrical exactness in his 1619 work, *The Delian Problem.* Molther begins by criticizing the work of other scholars—he condemns pointwise constructions of the quadratrix curve and he regards many constructions as approximate at best, therefore impractical and not geometrical, for geometrical constructions to Molther were rooted in precision and accuracy. In attempting to construct two mean proportions, Molther proposes the acceptance of the *neusis* postulate as the necessary solution, precisely echoing the work of Viète. Yet Viète provided no argument as to why the *neusis* postulate should be accepted, he only claims it as necessary. Molther, however, justifies his claim, believing it to be just as acceptable as the use of the Euclidean tools. Indeed, Molther's construction of the *neusis* uses both a straightedge and compass, using both motion and the judgement of the senses. This, he argues, is exactly what the Euclidean postulates assume, for in constructing both a line and a circle the geometer must move the required tools and succumb to the judgement of his senses. While Molther's point is true, he does not address the issue of demarcating between valid and invalid geometric constructions, for surely not all geometric constructions of

[34]For example, Viète describes the construction of a regular heptagon by Francois Foix de Candale as "accurate, but not geometrical" for its use of a shifting ruler (Bos, p.176).

[35]Ibid, p.178.

the motion and judgement of the senses were to be accepted.

Perhaps no early modern geometer had a more rigid interpretation of geometrical exactness than Johannes Kepler, the famed mathematician and astronomer. The underlying principle of Kepler's philosophical view of constructions was harmony—a harmonious ratio was the quotient of two lengths constructible by the Euclidean tools.[36] These ratios, Bos asserts, occurred "in nature as signs of God's deliberate choices," and hence to extend the legitimacy of construction beyond straightedge and compass was a philosophical, almost religious wrongdoing.[37] In this sense, Kepler imposes a strict interpretation of Commandino's "considerable sin," insinuating that any inappropriate solution is both geometrically illegitimate and morally wrong. Yet Kepler does not provide a detailed argument as to why his strong demarcation rests solely on the Euclidean tools. Bos states that Kepler sought to "appeal to authority and tradition," namely the works of Euclid and Proclus.[38] The entirety of Euclid's *Elements* stems from the restriction of constructions of straight lines and circles, and therefore to extend this restriction would derail not only Euclid's work, but the greater field of geometry.

Kepler, like other mathematical scholars, went on to criticize his contemporaries for their studies of non-planar constructions. He believed Pappus' method of hyperbolic constructions, Apollonius' procedure of forming conics, and the pointwise construction of the quadratrix curve were both impractical and imprecise, and therefore rejected these methods of construction as geometrically illegitimate.[39] Additionally, Kepler opposed the incorporation of algebra into geometric problems. Although reducing these problems to equations of varying degrees was a powerful and useful tool, this would, at best, provide an approximation, where the search for precise solutions would be compromised.

The fifty years between Commandino's 1589 *Collection* and Descartes' 1637 *Géométrie* generated much mathematical activity as geometrical prob-

[36]Bos calls such ratios "knowable." The Greeks, however, championed whole numbers, with little acknowledgment of other types of ratios, such as irrational numbers. Kepler is therefore taking his interpretation of harmony one step further than the Greeks (p.183).

[37]Ibid, p.184

[38]As Bos describes, Euclid provided the proper geometrical constructions through the restriction to straightedge and compass, and Proclus detailed the arguments to this restriction in his commentary of the *Elements* (p.194).

[39]The flaw with the conic constructions of Apollonius rests in finding the hyperbola from the intersection of the cone with the plane (p.188).

lem solving flourished among early modern mathematicians. Yet, on the verge of the breakthroughs Descartes provided, Bos notes that the field of geometrical problem solving still contained three issues regarding the interpretations of legitimacy and exactness. First, the works of the early modern geometers sparked a growth in the number of constructions beyond the use of the Euclidean tools. Still, there lacked a precedence among these order, with hardly any available criteria to distinguish genuinely geometrical constructions.[40] Thus, Bos asserts, "the objective of problem solving had become opaque and the practice lacked a clear direction."[41] Second, while many geometers recognized the use of algebraic methods in relating problems, equations, and solutions, the nature of these relations was not well articulated or understood before Descartes. Third, the field required a more "definite and refined interpretation of the exactness of constructional procedures" to help both clarify the differing perspectives and progress the field of problem solving.[42] The necessity to clarify these issues and advance the geometrical field was the impetus for the mathematical breakthroughs of Descartes.

René Descartes is considered by many to be the catalyst behind modern mathematics. Born in 1596 in La Haye, France, Descartes received an elite education at the prestigious Jesuit College in La Flèche, where he was properly trained in the Classics and humanities.[43] He spent much of his life as a solider and a philosopher, embracing a Catholic faith despite believing in what were then considered radical and heretical claims. He applied his philosophical inquiries to his mathematics, where he was primarily concerned with issues of methodology and exactness.[44] Consequently, his results in determining geometrical exactness were much more thorough than his predecessors'. His 1637 publication of *La Géométrie* as an appendix to his *Discourse on*

[40]Bos notes Willebrord Snellius, the seventeenth-century Dutch mathematician and astronomer, as an exception. Snellius often cites ruler and compass constructions as legitimately geometrical. In regards to the squaring of the circle, he states, a solution had not been found "authoratively and by ruler and compass according to the rules of the art," despite the many solutions that existed (p.220).

[41]Bos, p.221.

[42]Ibid, p.221.

[43]Hollingdale notes that Descartes attended the Jesuit College in France, where his best philosophical and mathematical insights came not from his classes, but when he was lying in bed in the morning (p.126).

[44]Bos notes that the famed *La Géométrie* was in fact an appendix to his philosophical treatise *Discourse on Method*, where he tackled the method of "rightly conducting one's reason and seeking the truth in the sciences," (p.228).

Method was necessitated, as he saw it, for "a more precise and reasoned definition of exactness in geometry."[45] Written in the vernacular of French rather than Latin, Descartes addressed his work to the general public rather than a small portion of the educated elite. Indeed, the influence of his work lead many to assert his career as the "the turning point between medieval and modern mathematics."[46]

La Géométrie, divided into three books, served to unify the disciplines of algebra and geometry, bridging the gap between the mathematics of the ancient Greeks and the more recent works of Descartes' contemporaries.[47] The first book introduces algebraic terms and notations that are still used today. Additionally, Descartes provided an algebraic solution to a problem that appeared in the works of Pappus, where the fourth-century geometer claimed without proof that the required locus was designed from conic sections.[48] One form of the problem, for instance, was to find the locus of points such that the product of the distances to two given lines was equal to the product of the distances to two other given lines. Bos describes Pappus' problem as "an indeterminate problem whose infinitely many solutions form a one-dimensional locus."[49] Descartes' approach was to find a pointwise construction of the locus by assigning variables, x and y, to the two unknowns in the final indeterminate equation, with the pairs of variables being "coordinates of points on the locus."[50] In algebraic terms, the geometric condition is equivalent to an equation of second degree in x and y that defines a conic section. The second book, titled *On the Nature of Curved Lines*, provides a distinction between two types of curves. Geometrically acceptable curves were those that were constructed by the intersection of two lines, "each moving parallel to the one coordinate axis with "commensurable" velocities."[51] Legitimate geometrical curves were those that produced

[45]Ibid, p.225.

[46]Burton also claims that Descartes "laid the foundations for the growth of mathematics in modern times;" indeed, shortly after his death, Newton and other mathematicians advanced Descartes' work to lead to the development of calculus (p.337).

[47]Indeed, Hollingdale provides the context of Descartes' motivation with the opening sentence of *La Géométrie*: "Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction," (p.131).

[48]Burton, p.342.

[49]Bos, p.313.

[50]Ibid, p.313.

[51]Burton, p. 343. Perhaps Hollingdale provides a simpler definition: "Geometric curves

algebraic equations of finite degree. Other curves, such as the quadratrix and spiral, were deemed mechanical and therefore illegitimate, for they required "two simultaneous motions whose relation does not admit of precise determination."[52] Descartes then provides instructions for constructing the roots of equations of degrees less than six. The third book of *La Géométrie* introduces the solutions of equations, with several consciously omitted proofs to "give others the pleasure of discovering for themselves."[53]

Thus the mathematical commentary of both the fourth and fifth centuries and early modern period stimulated a revival of the search for solutions for the construction problems of antiquity. Not only did these problems demonstrate the intellectual fortitude of the Greeks, but they also established the distinction between legitimate and illegitimate geometric constructions that would last for the next two millennia.

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are those that can be expressed by an algebraic equation (of finite degree) in x and y," (p.134).

[52]Such pointwise methods only provide accurate constructions of a finite number of points on the curve (Burton, p.343).

[53]Ibid, p.348.

Chapter 4

The Proofs of Impossibility

It is known today that none of the three construction problems can be solved using only the straightedge and compass. The proofs of impossibility of the three construction problems come from Pierre Wantzel and Ferdinand Lindemann, two nineteenth-century mathematicians whose publications brought the search for solutions to its end. These proofs, however, did not solely come from two mathematical geniuses who happened to discover what two millennia of previous mathematicians did not have the right ideas to prove; rather, they build on previous advancements from both antiquity and the early modern period. In fact, it was the translation of the three geometric problems into algebra that ultimately enabled Wantzel and Lindemann to demonstrate that their required constructions cannot be achieved by ruler and compass alone.

The earliest argument for the impossibility of the cube duplication and angle trisection is attributed to Descartes in the seventeenth century, two hundred years before Wantzel's proof was published.[1] While it was apparently believed among mathematicians since late antiquity that the two problems could not be solved by the ruler and compass constructions, no one formalized the notion into a proof. While Pappus, for example, was prepared to consider the impossibility of the geometric problems in his *Collection*, he "nowhere stated the necessity, desirability, or even possibility of giving a mathematical proof of this impossibility."[2] The early modern geometers, de-

[1] For a thorough history of the proofs of the cube duplication and angle trisection, see Jesper Lützen's paper *The Algebra of Geometric Impossibility: Descartes and Montucla on the Impossibility of the Duplication of the Cube and the Trisection of the Angle.*

[2] Lützen, p.6.

spite providing their own solutions to the problems, also did not attempt to provide a proof of impossibility. Why, then, was Descartes the first to address this issue?

One possible explanation is that a proof of impossibility, by nature, requires a different sequence of logic than proving a theorem or proposition. Since early antiquity, the mathematical enterprise could be viewed primarily as a problem-solving activity, and proving a statement is impossible is contrary to this end. As Lützen states, "such an impossibility is not a proper mathematical result, but a sort of meta-statement regarding the problem solving activity."[3] As both a mathematician and a philosopher, Pappus was the first of the Greek commentators to blur this distinction—he "mixed solutions of mathematical problems with discussions about methodological issues."[4] Thirteen centuries later, the philosopher-mathematician Descartes would also be interested in the solvability of the geometric problems. Influenced by Pappus, Descartes classified problems based on the methods required for their solutions, suggesting it was prohibited to solve a planar problem—by Pappus's definition—by other means. Thus, to clearly demonstrate that his construction using conic sections was the correct method for the solution, he needed to argue that a ruler and compass construction does not exist.

Yet, to carry out this kind of proof, the geometric problems needed to be translated into algebra. The method of using algebra to prove a theorem was not, however, a "trivial matter of course" or a "natural line of argument."[5] Most early modern mathematicians were reluctant to use algebra in proofs. For example, even in the early seventeenth century there was still an incomplete understanding of concepts like negative and complex numbers. Indeed, when algebraic methods were used in problems, they were used for finding the proper construction of the problem, but geometric methods were used for the proof of the correctness of the construction. As Bos states, "an algebra beset with such restrictions and uncertainties held little promise for geometry."[6] In the early modern period of mathematics, algebra problems usually contained a geometric proof, so the concept of using an algebraic proof for geometric problems was, in a sense, revolutionary.

[3] Lützen, p.6,

[4] Lützen, p.7. He adds that it is therefore "not so surprising that Pappus took up meta-mathematical questions such as the (un)solvability of problems with given means."

[5] Lützen, p.9, 10.

[6] Bos, p.133.

The proofs Descartes provides do not, however, accurately give an algebraic synthesis of the impossibility of a ruler and compass construction to the cube duplication and angle trisection.[7] His first step was to assign letters to the known and unknown line segments and to formulate the relations of the problem into algebraic equations. He would then reduce and rearrange the equations, leaving an equation in terms of just the unknowns, precisely the method that led Descartes to the correct equation corresponding to the two mean proportionals, as given by Hippocrates. Descartes demonstrated that if the final equation is quadratic than the unknown segments can be constructed by the Euclidean tools, and thus the problem was planar by Pappus' definition. "If the equation is of degree higher than two," it was "according to Descartes impossible to solve...by plane means."[8] Later in his book, Descartes showed that the problems of the cube duplication and angle trisection lead to cubic equations. Therefore, since problems constructible by planar means lead to quadratic equations and since the two geometric problems lead to cubic equations, Descartes asserted that the two problems cannot be solved by ruler and compass alone. Yet, his logic for this argument is somewhat convoluted: he went on to prove the converse of the statement, that if "the final equation is quadratic then the problem can be solved by" the Euclidean tools.[9]

Much of the mathematics Descartes puts forth in his *La Géométrie* would become fundamental in the development of the theory of solutions of polynomial equations, which in turn helped Wantzel formulate his proof of the impossibility of the planar constructions. He extensively describes the reducibility of polynomials and the factorization of equations. Specifically, he considers the case where a given polynomial can be factored into two quadratic factors. To do this, Descartes instructs to set up a resolvent of the form $p(x) = x^4 + px^2 + qx + r$ and to seek binomial factors considered as third degree equations.[10] If such binomial factors can be found, the roots of the two quadratic factors can be found by straightedge and compass constructions, and thus the problem is plane. Lützen summarizes Descartes' method as follows: when the polynomial is determined for the "principal unknown" with coefficients in terms of the knowns, one carries out several ruler and compass constructions to determine "new line segments from the givens,"

[7] Thorough discussions of Descartes' method can be found in Lützen and Bos.

[8] Lützen, p.12.

[9] Lützen, p.14.

[10] Lützen, p.16.

reducing the polynomial as much as possible.[11] If this produces a quadratic equation the problem is planar. While Descartes' results are remarkable, he nevertheless "did not develop an explicit algebraic formulation of the process of construction by ruler and compass."[12] Moreover, Descartes lacked all the algebraic tools that would eventually be developed to finish the proofs.

It is important to take stock of the many contributions Descartes gave to impossibility of geometric constructions, as well as some of his shortcomings.[13] His contributions, among many, include formulating the impossibility of the cube duplication and angle trisection as theorems that should be proved, rather than just empirical statements. Additionally, he attempted his own geometric proof, "translated the geometric problems into algebraic ones concerning the solution of polynomial equations," and studied several methods to factorize polynomials.[14] For these advancements, the mathematical field is indebted to Descartes and his discoveries. Yet, he did not provide the algebraic proof necessary to complete the study of the geometric problems. He "did not establish in a satisfactory way that...[the geometric]constructions could only solve quadratic equations," and he "did not stress the relation between the algebraic process of factorization and geometric constructions but only focused on the construction of the final equation."[15] It was these deficiencies, together with the lack of extensive knowledge of field theory, which led to an incomplete proof of impossibility.

Two centuries after Descartes, the French mathematician Pierre Wantzel would use these algebraic advancements in his proofs of impossibility of the cube duplication and angle trisection constructions. What follows is an account of Wantzel's work translated into modern mathematical language. Yet, before turning to Descartes' coordinate geometry in the proofs of impossibility for the construction problems, it is important to first examine several algebraic definitions and theorems that will be used later in the proofs. Many of these properties stem from the German mathematician Ernst Steinitz, whose influential paper on the algebriac theory of fields would become an important step in the development of abstract algebra.[16] First, a field is an

[11]Lützen, p.19.

[12]Lützen, p.21.

[13]Lützen gives an extensive list of Descartes' contributions and failures, p.26-7.

[14]Lützen, p.26.

[15]Lützen, p.27.

[16]Steinitz's paper, *Algebraische Theorie der Körper*, was published in 1910 in the German periodical *Journal für die reine und angewandte Mathematik.*

algebraic structure given by $(F, +, \cdot)$, a set F with two binary operations $+$ and $\cdot$, that satisfies the following three conditions:[17]

- $(F, +)$ is an abelian group
- $(F - \{0\}, \cdot)$ is an abelian group
- The distributive law $a \cdot (b + c) = a \cdot b + b \cdot c$ holds for all $a, b, c, d \in F$.

This leads to the following definitions.

Definition: A field E is an *extension field* of a field F if $F \subseteq E$ and the field operations on E restrict to the field operations on F.

A field E is said to be a vector space over a field F if E is an Abelian group under addition and the following conditions hold.[18]

1. $a(v + u) = av + au$
2. $(a + b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$

for all $a, b \in F$ and $u, v \in E$.[19] The field axioms show that these properties hold whenever E is an extension field of F.

Definition: Let E be an extension field of a field F. Then E has degree n over F, written as $[E : F]= n$, if E has dimension n as a vector space over F.

Definition: Let E be an extension field of a field F and let $a \in E$. Then a is *algebraic over* F if a is the zero of some polynomial in $F[x]$. Also, a is transcendental over F if there is no polynomial $p(x) \in F[x]$ with $p(a) = 0$.

Proposition: Let $[E : F]= n$. Then all elements in E are algebraic over F.

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[17]Gallian defines a field as "simply an algebraic system that is closed under addition, subtraction, multiplication, and division (except by 0)," where all the usual properties of these operations hold, p.176.

[18]Steinitz, p.172.

[19]Gallian, p.241.

Proof: Let $[E : F] = n$ with $a \in E$. Then the set $\{1, a, a^2, \dots, a^n\}$ has $n+1$ elements and therefore is linearly dependent. Thus, there exists coefficients $c_0, \dots, c_n \in E$ such that $c_n a^n + c_{n-1}a^{n-1} + \cdots + a_0 = 0$, so a is a root of the polynomial $p(x) = c_n x^n + \cdots + c_0 \in F[x]$. Hence all elements in E are algebraic over F.

Definition: Given any field $F \subseteq \mathbb{R}$ with $a \in \mathbb{R}$, $F(a)$ is the smallest field containing F and a.

Proposition: If $\Delta \in F$, with Δ not a square in F, then $F(\sqrt{\Delta}) = \{f + g\sqrt{\Delta} | f, g \in F\}$.

Proof: First $\{f + g\sqrt{\Delta} | f, g \in F\}$ is contained in $F(\sqrt{\Delta})$ by the definition of a field. So it suffices to show that $\{f + g\sqrt{\Delta} | f, g \in F\}$ is itself a field. To demonstrate closure under multiplication, consider the product of

$$(f + g\sqrt{\Delta})(f' + g'\sqrt{\Delta})$$

which, when expanded, gives

$$ff' + fg'\sqrt{\Delta} + f'g\sqrt{\Delta} + gg'\Delta$$

which equals

$$ff' + gg'\Delta + (fg' + f'g)\sqrt{\Delta}$$

which is in $F(\sqrt{\Delta})$. Thus $F(\sqrt{\Delta})$ is closed under multiplication. The additive inverse is given by $-(f + g\sqrt{\Delta})$, since $f + g\sqrt{\Delta} + (-f - g\sqrt{\Delta}) = 0$. To demonstrate a multiplicative inverse, it might be tempting to use $(f - g\sqrt{\Delta})$. However,

$$(f + g\sqrt{\Delta})(f - g\sqrt{\Delta}) = f^2 - g^2\sqrt{\Delta}$$

which, if equal to 0, would imply that $f^2 = g^2\Delta$, or $\Delta = (\frac{f}{g})^2$, which contradicts Δ not being a square. So the correct multiplicative inverse is given by

$$\frac{1}{f^2 - g^2\Delta}(f - g\sqrt{\Delta}) = \frac{f}{f^2 - g^2} - \frac{g}{f^2 - g^2}\sqrt{\Delta}$$

since

$$(f+g\sqrt{\Delta})\left(\frac{1}{f^2-g^2\Delta}(f-g\sqrt{\Delta})\right)=1$$

and thus the multiplicative inverse is in $F(\sqrt{\Delta})$.

Corollary: If $F_1 = F(\sqrt{\Delta})$ with $\Delta \in F$ not a square, then $[F_1 : F] = 2$.

Proof: $\{1, \sqrt{\Delta}\}$ spans F_1. A typical element is $a + b\sqrt{\Delta}$, with $a, b \in F$. If $a+b\sqrt{\Delta} = 0$ with $b \neq 0$, then $\sqrt{\Delta} = -\frac{a}{b}$, or $\Delta = (-\frac{a}{b})^2$, which contradicts Δ not being a square. Thus $\{1, \sqrt{\Delta}\}$ is linearly independent over F, and hence forms a basis. Thus dim $F_1 = 2$, or $[F_1 : F]$=2.

Definition: A "tower" of fields $F_1 \subseteq F_2 \subseteq ... \subseteq F_{k-1} \subseteq F_k$ is a finite sequence of field extensions $F_j \subseteq F_{j+1}$ where $j = 1, \ldots, k-1$.

These lead to the following theorems.

Theorem 1 *If $E \subseteq F \subseteq K$ is a "tower" of fields, then $[K : E] = [K : F][F : E]$.*[20]

Proof: Say $[F : E] = r$ and $\{\alpha_1, \ldots, \alpha_r\}$ is a basis and $[K : F] = s$ with $\{\beta_1, \ldots, \beta_s\}$ a basis. Let $\gamma \in K$, then $\gamma = C_1\beta_1 + \cdots + C_s\beta_s$ for some $C_i \in F$. Then $C_i = b_{i1}\alpha_1 + \cdots + b_{ir}\alpha_r$ for some $b_{ij} \in F$. So $\gamma = (b_{11}\alpha_1 + \cdots + b_{1r}\alpha_r)\beta_1 + (b_{s1}\alpha_1 + \cdots + b_{sr}\alpha_r)\beta_s$, which, when expanded, equals

$$b_{i1}\alpha_1\beta_1 + b_{i2}\alpha_2\beta_2 + \cdots + b_{ir}\alpha_r\beta_1 + \cdots + b_{sr}\alpha_r\beta_s$$

So $\{\alpha_i\beta_j | 1 \leq i \leq r, 1 \leq j \leq s\}$ spans K over E. If

$$b_{11}\alpha_1\beta_1 + \cdots + b_{1r}\alpha_r\beta_1 + \cdots + b_{sr}\alpha_r\beta_s = 0$$

then

$$(b_{11}\alpha_1 + \cdots + b_{1r}\alpha_r)\beta_1 + \cdots + (b_{s1}\alpha_1 + \cdots + b_{sr}\alpha_r)\beta_s = 0.$$

[20] As Gallian states, if K is a finite extension field of the field F and F is a finite extension of the field E, then K is a finite extension of field E, p.259.

Since $\{\beta_1, \ldots, \beta_s\}$ is a basis and thus linearly independent, $b_{11}\alpha_1 + \cdots + b_{1r}\alpha_r = 0, \ldots, b_{s1}\alpha_1 + \cdots + b_{sr}\alpha_r = 0$. Since $\{\alpha_1, \ldots, \alpha_r\}$ is linearly independent in F over E, it follows that $b_{ij} = 0$.
Conclusion: $\{\alpha_i\beta_j | 1 \leq i \leq r, 1 \leq j \leq s\}$ is a basis for K over E. Thus $[K : E] = r \cdot s = [K : F][F : E]$. This proves the theorem.

Theorem 2 *(Primitive Element Theorem) If F_n is a finite extension of a field $F \supseteq \mathbb{Q}$, then there exists an element a in F_n such that $F_n = F(a)$.*[21]

The link between this field theory and the proofs of the geometric constructions comes in the form of irreducible polynomial equations.

Definition: The irreducible polynomial of an algebraic element a over $\mathbb{Q}$ is $p(x) \in \mathbb{Q}[x]$ of smallest degree with $p(a) = 0$.

This can be illustrated with the following example: Say $k = \sqrt{1 + \sqrt{2}}$ So $k^2 = 1 + \sqrt{2}$ $(k^2 - 1) = \sqrt{2}$, so $(k^2 - 1)^2 = 2$, which expanded gives $k^4 - 2k^2 + 1 = 2$. Therefore $k^4 - 2k^2 - 1 = 0$. The claim is that $p(x) = x^4 - 2x^2 - 1 \in \mathbb{Q}[x]$ is the irreducible polynomial. It remains to show that $p(x)$ cannot be factored and thus is the irreducible polynomial. First, try factoring $p(x)$ into the form $(x + a)(x^3 + bx^2 + cx + 1)$.Expanding this gives

$$x^4 + bx^3 + cx^2 + x + ax^3 + abx^2 + acx + a$$

which is equivalent to

$$x^4 + (a + b)x^3 + (ab + c)x^2 + (ac + 1)x + a$$

and therefore $a + b = 0$, $ab + c = -2$, $ac + 1 = 0$, and $a = -1$. Substituting backwards obtains $c = -1$ and $b = 3$, but $-1+3 \neq 0$, and thus a factorization of this kind has no rational roots. Now, try factoring $p(x)$ into the form $(x^2 + ax + b)(x^2 + cx + d)$. Expanding this gives

$$x^4 + cx^3 + dx^2 + ax^3 + acx^2 + adx + bx^2 + bcx + bd$$

which is equivalent to

$$x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd$$

[21] The proof of the Primitive Element Theorem is quite extensive, but a thorough examination can be found in Iain T. Adamson's *Introduction to Field Theory*.

and therefore $a+c=0$, $b+d+ac=-2$, $ad+bc=0$, $bd=-1$. This implies $a=-c$, which substituted into the third equation gives $-cd+bc=0$, or $(b-d)c=0$. This implies either $b-d=0$ or $c=0$. If $b-d=0$, then $b=d$, which cannot be true since $bd=-1$. If $c=0$, then $b+d=-2$, and since $bd=-1$, then $-2b+b^2=-1$. Applying the quadratic formula gives

$$b = \frac{-2 \pm \sqrt{4+4}}{2}$$

which is not an element of $\mathbb{Q}$. Therefore a factorization of this kind has no rational roots. Thus $p(x)=x^4-2x^2-1 \in \mathbb{Q}[x]$ is the irreducible polynomial.

Proposition: If a is algebraic over F with irreducible polynomial $p(x)$ of degree n, then $[F(a):F]=n$.

Proof: $\left\{1, a, a^2, \ldots, a^{n-1}\right\}$ is a basis for $F(a)$ over F.

With this algebraic background in hand, one can supply a modern account of the solutions provided by Wantzel and Lindemann. First, it is important to characterize what points can be constructed by a straightedge and compass in the Cartesian plane. That is, addressing the question: Given a unit of distance, or a segment of length one, and the coordinate axes with the origin, what points can be constructed by a straightedge and compass?[22]

The first claim is that any point (a,b) can be constructed with $a,b \in \mathbb{Q}$. To show this, it is sufficient to construct $(a,0),(0,b)$, or even all $a \in \mathbb{Q}$ along one line. Thus, suppose $a=\frac{m}{n}$ in lowest terms. Since a unit length is assumed, a segment AB of length one exists and hence segments of all integer lengths m can be constructed. It therefore remains to show that a segment of length $\frac{1}{n}$ exists. This has been proved in Chapter 2 for the case $n=3$, but the general argument can be described as follows. Take segment AB. From A, lay off points $C_1, C_2, \ldots, C_n$ such that $AC_1 = C_1C_2 =, \ldots, = C_{n-1}C_n$. Join BC_N. Construct parallels $C_1B_1, C_2B_2, \ldots, C_{n-1}B_{n-1}$ to BC_n, with $B_1, B_2, \ldots, B_{n-1}$ on AB. AB_1 then has length equal to $\frac{1}{n}$. Thus, it is possible to construct any $(a,b) \in \mathbb{Q}^2$. This leads to the following theorem.

[22]A thorough examination of the abstract algebra and field theory that led to the proofs of impossibility can be found in Gallian's *Contemporary Abstract Algebra* and Morandi's *Field and Galois Theory*.

Theorem 3 *If α is constructible over $\mathbb{Q}$, then there exists a tower of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ where for each i either $F_{i+1} = F_i$ or $F_{i+1} = F_i(\sqrt{\Delta_i})$ where $\Delta_i \in F_i$ not a square and $\alpha \in F_n$.*

Proof: Each step of a construction will involve new points generated by the intersections of either two lines, a line and a circle, or two circles defined by previously constructed points. Considering each of these cases in turn yield the following.

Case 1: The Intersection of Two Lines

If it is known that one can construct with the Euclidean tools any $(a, b) \in F$, assume L_1 and L_2 are two lines such that $L_1 : (a_1, b_1)$ and (c_1, d_1) and $L_2 : (a_2, b_2)$ and (c_2, d_2) with $a_i, b_i, c_i, d_i \in F$. The lines can be expressed in point-slope form as

$$L_1 : y - b_1 = \left(\frac{d_1 - b_1}{c_1 - a_1}\right)(x - a_1)$$

$$L_2 : y - b_2 = \left(\frac{d_2 - b_2}{c_2 - a_2}\right)(x - a_2)$$

with each having the form

$$A_1 x + B_1 y = C_1$$

$$A_2 x + B_2 y = C_2.$$

Using determinants to solve for the point of intersection, (x, y), gives

$$x = \frac{\det\begin{pmatrix} C_1 & B_1 \\ C_2 & B_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}}$$

and

$$y = \frac{\det\begin{pmatrix} A_1 & C_1 \\ A_2 & C_2 \end{pmatrix}}{\det\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}}$$

and therefore both $x, y \in F$ and thus $(x, y) \in F$.

Thus, the intersection of two lines has rational coordinates in the same field as the two lines. Or, as Gallian states, "two lines in F intersect in a point in the plane of F."[23]

Case 2: The Intersection of a Line and a Circle

Assume L is a line such that $L : (a, b)$ and (c, d) with $a, b, c, d \in F$ and C is a circle with center (p, q) and radius r with $p, q, r \in F$

So L can be expressed as

$$L : Ax + By - C \tag{4.1}$$

for some $A, B, C, D \in F$ and C can be expressed as

$$C : (x - p)^2 + (y - q)^2 = r^2 \tag{4.2}$$

In (4.1), assuming $B \neq 0$ and solving for y gives

$$y = -\frac{A}{B}x + \frac{C}{B} \tag{4.3}$$

which substituted into (4.2) gives

$$(x - p)^2 + \left(-\frac{A}{B}x + \frac{C}{B} - q^2\right) = r^2$$

Expanding this gives

$$\alpha x^2 + \beta x + \gamma = 0$$

for some $\alpha, \beta, \gamma \in F$. Applying the quadratic formula gives

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

Substituting this back into (4.3) gives

$$y = -\frac{A}{B}\left(\frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\right) + \frac{C}{B}$$

[23] Gallian, p.264.

Define Δ as $\beta^2 - 4\alpha\gamma$.

So if Δ is not a square in F, one needs to go to a bigger field to get the coordinates of the intersection.

Thus, for the intersection of a line and circle, if Δ is not a square in F it is required to go to some field $F(\sqrt{\Delta})$.

Case 3: The Intersection of Two Circles

The intersection of two circles will produce at most two points of intersection, unless the circles are tangent, in which case they intersect at one point. Suppose the two circles, C_1 and C_2, intersect at two points. Define the circles as

$$C_1 : (x - h_1)^2 + (y - k_1)^2 = r_1^2$$

$$C_2 : (x - h_2)^2 + (y - k_2)^2 = r_2^2$$

Expanding these equations gives

$$C_1 : x^2 + y^2 - 2h_1x - 2k_1y + h_1^2 + k_1^2 = r_1^2 \tag{4.4}$$

$$C_2 : x^2 + y^2 - 2h_2x - 2k_2y + h_2^2 + k_2^2 = r_2^2 \tag{4.5}$$

Subtracting (4.4) from (4.5) gives

$$2(h_1 - h_2)x + 2(k_1 - k_2)y - h_1^2 + h_2^2 - k_1^2 + k_2^2 = r_2^2 - r_1^2$$

which, together with (4.4), gives the intersection of a circle and a line. Thus, the intersection of two circles yields the same results as the intersection of a line and a circle. $\square$

Let $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ be a tower as in Theorem 3. For each i, F_{i+1} is a vector space over F_i, so $[F_{i+1} : F_i]$=1 if $F_{i+1} = F_i$ and $[F_{i+1} : F_i] = 2$ if $F_{i+1} = F_i(\sqrt{\Delta_i})$.

Corollary: If α is constructible, $[\mathbb{Q}(\alpha) : \mathbb{Q}]$= 2^k for some nonnegative integer k.

Proof: Say

$$F_0 = \mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F_n$$

comes from a construction and $\alpha \in F_n$. It is known that $[F_{i+1} : F_1]$=1 or 2 for all i. By Theorem 3,

$$[F_n : \mathbb{Q}] = [F_n : F_{n-1}][F_{n-1} : \mathbb{Q}]$$

which equals

$$[F_n : F_{n-1}][F_{n-1} : F_{n-2}][F_{n-2} : \mathbb{Q}]$$

which equals

$$[F_n : F_{n-1}][F_{n-1} : F_{n-2}]...[F_1 : \mathbb{Q}],$$

with each dimension equalling 1 or 2. So $[F_n : \mathbb{Q}]$ is thus 2^l for some l. Then $\mathbb{Q} \subseteq \mathbb{Q}(\alpha) \subseteq F_n$, so $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ divides 2^l, and hence $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^k$ for some nonnegative integer k, with $k \leq l$. □

An application of this field theory can be used to demonstrate the impossibilty of the three construction problems.[24] First, consider the duplication of the cube. Given a cube of volume 1, the problem compels the geometer to construct a cube of volume 2, thus having an edge of length equal to $\sqrt[3]{2}$. The problem is thus reduced to determining if a length of $\sqrt[3]{2}$ is constructible with a straightedge and compass. In this case, $x^3 - 2 = 0$ is the irreducible equation. To show this equation is irreducible involves the use of the Rational Roots Test, described as follows.

Claim: If $a_n x^n + ... + a_1 x + a_0 = 0$ with each $a_i \in \mathbb{Z}$ has a rational root x, then $x = \frac{p}{q}$ in lowest terms where q divides a_n and p divides a_0.

Proof: Substitute $x = \frac{p}{q}$ into the polynomial equation. Then

$$a_n \left(\frac{p}{q}\right)^n + ... + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides by q^n gives

$$a_n p^n + a_{n-1} p^{n-1} q + ... + a_0 q^n = 0, \tag{4.6}$$

[24]An explanation of these proofs can be found in Gallian, p.265 and Morandi, p.145.

which is equivalent to

$$a_n p^n = -q\left(a_{n-1}p^{n-1} + ... + a_0 q^{n-1}\right).$$

Therefore, since q and p are relatively prime, q must divide a_n. Likewise, (7) is equivalent to

$$a_0 q^n = -p\left(a_n p^{n-1} + ... + a_1 q^{n-1}\right).$$

Again, since q and p are relatively prime, p must divide a_0.

In the case of the given equation, it follows from the Rational Roots Test that the only rational roots would be $x = \pm\ 1$ or ± 2. Substituting each of these into $x^3 - 2 = 0$ yields: $x = 1$ gives $-1 \neq 0$, $x = -1$ gives $-3 \neq 0$, $x = 2$ gives $6 \neq 0$, and $x = -2$ gives $-10 \neq 0$. Since none of the potential rational roots work, $x^3 - 2 = 0$ is the irreducible equation. Thus $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$=3, which is not equal to 2^s for some s. Therefore $\sqrt[3]{2}$ is not constructible, and the duplication of the cube is impossible with straightedge and compass.

The impossibility of the trisection of an angle can be proved with a particular example, because if there were a general construction, it would have to hold for this case. In order to trisect an angle of 60°, the number $\cos 20°$ must be constructible. The proof makes use of the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

with $\theta = 20°$. Therefore

$$\frac{1}{2} = 4\cos^3 20° - 3\cos 20°$$

Substituting $\cos 20° = x$ gives $\frac{1}{2} = 4x^3 - 3x$, or $1 = 8x^3 - 6x$. Thus, $8x^3 - 6x - 1 = 0$ is a polynomial equation over $\mathbb{Q}$ satisfied by x. To show this equation is irreducible also requires the Rational Roots Test, which gives the potential rational roots as $x = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. Substituting each of these into the polynomial yields: $x = 1$ gives $1 \neq 0$, $x = -1$ gives $-3 \neq 0$, $x = \frac{1}{2}$ gives $-3 \neq 0$, $x = -\frac{1}{2}$ gives $-1 \neq 0$, $x = \frac{1}{4}$ gives $-\frac{19}{8} \neq 0$, $x = -\frac{1}{4}$ gives $-\frac{3}{8} \neq 0$, $x = \frac{1}{8}$ gives $-\frac{111}{64} \neq 0$, and $x = -\frac{1}{8}$ gives $\frac{17}{64} \neq 0$. Since none of the potential rational roots work, $8x^3 - 6x - 1 = 0$ is the irreducible equation.

Therefore, $[\mathbb{Q}(\cos 20°) : \mathbb{Q}] = 3$, which is not equal to 2^s for some s. Thus $\cos 20°$ is not constructible with the Euclidean tools, and the general trisection of an angle problem is impossible.

The proof of the squaring of the circle involves the use of theorem and a corresponding corollary:

Theorem 4 *(Lindemann-Weierstrass Theorem)*[25] *Let $\alpha_1, \ldots, \alpha_m$ be distinct algebraic numbers. Then the exponentials $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are linearly independent over $\mathbb{Q}$.*[26]

Corollary: The numbers π and e are transcendental over $\mathbb{Q}$.

Proof of the corollary: Suppose e is algebraic over $\mathbb{Q}$. Then there exist $r_i \in \mathbb{Q}$ such that $r_0 + r_1 e^1 + \cdots + r_n e^n = 0$. This implies $e^0, e^1, \ldots, e^n$ are linearly dependent over $\mathbb{Q}$. But, by choosing $m = n + 1$ and $\alpha_i = i - 1$, this dependence contradicts the Lindemann-Weierstrass Theorem, and hence e is transcendental over $\mathbb{Q}$. If π is algebraic over $\mathbb{Q}$, then so is πi. This implies e^0 and $e^{\pi i}$ are linearly independent over $\mathbb{Q}$, which cannot be true since $e^{\pi i} = -1$. Therefore, π is transcendental over $\mathbb{Q}$. $\square$

Hence, the impossibility of the squaring of the circle can also be proved with a specific example. Consider a circle of radius 1. The problem compels the geometer to construct a square with area π, thus having sides of length $\sqrt{\pi}$. But, since π is transcendental over $\mathbb{Q}$, a segment with length $\sqrt{\pi}$ cannot be constructed by the Euclidean tools since it is not algebraic of degree of a power of 2.[27]

At last, the three construction problems of antiquity had reached their end. It is tempting to assign Wantzel and Lindemann the credit of completing the search for planar solutions. While this bears some truth—they did, in fact, publish the proofs of impossibility—their work serves as a culmination of mathematical progress over the course of nearly two thousand years, from the beginnings in ancient Greece, through the early modern period, and into the nineteenth century. Countless mathematicians helped advance the mathematical enterprise from its roots in Euclidean geometry to the field theory necessary to state the proofs of impossibility sketched in this thesis.

[25] The proof of this theorem is also quite extensive, but a comprehensive account can be found in *An Alternative Proof of the Lindemann-Weierstrass Theorem* by Beukers et al., 1990.

[26] This is an alternative version of the theorem given by Morandi, since it is "a little easier to prove than the original," (p.134). He states the original, equivalent theorem on p.138 as follows: If $\alpha_1, \ldots, \alpha_m$ are $\mathbb{Q}$-linearly-dependent algebraic numbers, then the exponentials $e^{\alpha_1}, \ldots, e^{\alpha_m}$ are algebraically independent, hence there is no non-zero polynomial $f(x_1, \ldots, x_m) \in \mathbb{Q}[x_1, \ldots, x_m]$ with $f(e^{\alpha_1}, \ldots, e^{\alpha_m}) = 0$.

[27] In fact, it is not algebraic at all.

Their contributions are a testament that mathematics transverses cultures, languages, and time. The search for solutions, in any discipline, is not a finite process but an ongoing pursuit of knowledge and truth.

It is important to assess exactly what the solutions of these geometric constructions have taught the mathematical world. First, mathematics is a communal activity. Indeed, Plato's Academy was founded as a place where scholars could gather and discuss ideas, educating many great mathematicians in the process. Yet this mathematical community comes with interesting dynamics, where many of the best advancements result from challenging, contested environments. In ancient Greece, for example, scholars chastised each other over the use of mechanical constructions. This dispute would continue in the early modern period, where mathematicians disagreed over issues such as the acceptance of a *neusis* postulate and a new construction of a quadratrix curve. Despite these disagreements, or perhaps because of them, the mathematical field rapidly developed, attracting the attention of many of the world's best scholars.

Interpreted literally, the three problems of antiquity prove three seemingly basic facts about geometric constructions: using only the Euclidean tools, a cube cannot be doubled, a general angle cannot be trisected, and a circle cannot be squared. Yet the importance of these problems does not rest in these answers, but rather in the mathematics they generated. The search for solutions sparked the mathematical enterprise of the ancient Greeks, with each scholar seeking to advance the work of the preceding generation. The revival of this enterprise in the Western world led to the developments of calculus, field theory, linear algebra, and more. While twenty-first century students may never encounter the three geometric construction problems, they will undoubtedly come to know the mathematical fields that grew out of the many centuries of progress. The spirit of these geometric constructions lives on today, educating and inspiring generation after generation of mathematicians, just as they had two thousand years ago.

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