Toric Codes

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Outline of Talk

1. Coding theory basics

2. Toric codes

3. Multivariate Vandermonde matrices and estimating the minimum distance

4. Codes from simplices

Joint work with Hal Schenck (Texas A+M), and undergrad students (Ryan Schwarz HC '05, Alex Simao, HC '08).

[1] -, R. Schwarz, *On m-dimensional toric codes*, `arXiv/cs.IT/0506102` (to appear, *AAECC*)

[2] -, H. Schenck, *Toric surface codes and Minkowski sums*, `arXiv/math.AG/0507598` (to appear, SIAM J. Disc. Math.)

§1. Coding Theory Basics

A fundamental problem in coding theory is the construction of codes with "good" error-control properties.

- We'll consider "linear block codes" − vector subspaces C of $\mathbb{F}_q^n$ for some n.

- parameters: n, $k = \dim_{\mathbb{F}_q}(C)$,

  $$d = \min_{x \neq y \in C} d(x, y) = \min_{x \neq 0 \in C} \mathrm{weight}(x)$$

  (Hamming minimum distance/weight)

- $t = \lfloor \frac{d-1}{2} \rfloor \Rightarrow$ all errors of weight $\leq t$ can be corrected by "nearest neighbor decoding"

- Good codes: k/n not too small (so not extremely redundant), but at same time d or d/n not too small.

Reed-Solomon codes

Pick a primitive element α for $\mathbb{F}_q$ (i.e. generator of the cyclic multiplicative group of field), and write the nonzero elements of $\mathbb{F}_q$ as

$$1, \alpha, \ldots, \alpha^{q-2}.$$

Let $L_k = \{f \in \mathbb{F}_q[x] : \deg f < k\}$. Then

$$\begin{array}{rcl} ev : L_k & \to & \mathbb{F}_q^{q-1} \\ f & \mapsto & (f(1), f(\alpha), \ldots, f(\alpha^{q-2})) \end{array}$$

is linear and one-to-one if $k < q$. The image is called $RS(k, q)$.

All f of degree $< k$ have at most $k-1$ roots in $\mathbb{F}_q$ (and some have exactly that many)

$$\Rightarrow d = (q-1) - (k-1) = n - k + 1.$$

(Singleton bound: $d \leq n - k + 1$.)

An Example

Using the standard monomial basis for L_k:

$$\{1, x, x^2, x^3, \ldots, x^{k-1}\}$$

The Reed-Solomon code $RS(3, 16)$ (parameters: $n = 15, k = 3, d = 13$ over $\mathbb{F}_{16}$, so $16^3 = 4096$ distinct codewords) has generator matrix:

$$G = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^7 & \alpha^8 & \cdots & \alpha^{14} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{14} & \alpha & \cdots & \alpha^{13} \end{pmatrix}$$

(means: the rows of G form a basis for $C = RS(3, 16)$).

How Reed - Solomon Codes are Used

Reed-Solomon codes are among the most useful codes in engineering practice in situations where errors tend to occur in "bursts" rather than randomly.

$RS(3, 16)$ has $d = 13$, corrects any error vector of weight $\leq \lfloor \frac{13-1}{2} \rfloor = 6$ in a received word over $\mathbb{F}_{16} \cong \mathbb{F}_2^4$. A "burst" of up to 20 consecutive bit errors would affect *at most 6* of the symbols of the message thought of as elements of $\mathbb{F}_{16}$. $RS(3, 16)$ can correct any 20 or fewer *consecutive* bit errors in a codeword.

Also very efficient algebraic decoding algorithms (Berlekamp-Massey).

Basis for the error-control coding used, for example, in the CD audio system, in communications with deep-space exploration craft like *Voyager*, etc.

§2. Toric codes

Introduced by J. Hansen $\sim$ 1997. Elementary description:

- Let P be an integral convex polytope in $\mathbb{R}^m$, $m \geq 1$.

- Points β in the finite set $P \cap \mathbb{Z}^m$ correspond to monomials x^β (multi-index notation)

- Let $L_P = \text{Span}\{x^\beta : \beta \in P \cap \mathbb{Z}^m\}$.

- Define
$$\begin{aligned} ev : L_P &\to \mathbb{F}_q^{(q-1)^m} \\ f &\mapsto (f(\gamma) : \gamma \in (\mathbb{F}_q^*)^m) \end{aligned}$$
Image is the toric code $C_P(\mathbb{F}_q)$.

Note $RS(k, q)$ is the case $P = [0, k-1] \subset \mathbb{R}$ since $L_k = \text{Span}\{1, x, \ldots, x^{k-1}\}$.

Why are these interesting?

- Have many properties parallel to RS codes, e.g. they are "m-dimensional cyclic" codes (set of codewords is closed under a large automorphism group).

- Computer searches by D. Joyner (USNA) $\sim$ 2000 showed that some very good $m = 2$ toric codes exist (better than any previously known codes in standard databases).

- A number of other isolated very good examples found too.

Searching for good toric codes?

Theorem 1 ([1) *] Let P, P' be polytopes as above.*

1. *If P and P' are lattice equivalent polytopes then $C_P(\mathbb{F}_q)$ and $C_{P'}(\mathbb{F}_q)$ are monomially equivalent codes.*

2. *Similarly, viewing $[0, q-2]^m \cap \mathbb{Z}^m$ as $(\mathbb{Z}_{q-1})^m$, if $S = P \cap \mathbb{Z}^m$ and $S' = T(S)$ for some $T = \mathrm{AGL}(m, \mathbb{Z}_{q-1})$, the resulting evaluation code from S' is monomially equivalent to $C_P(\mathbb{F}_q)$.*

Monomial equivalence: There is an $n \times n$ permutation matrix Π and a $n \times n$ invertible diagonal matrix Q such that $G' = GQ\Pi$; implies that parameters are the same.

Note: In the second case, S' may not be $P' \cap \mathbb{Z}^m$ for a convex polytope P'.

Small needles in huge haystacks

For $m = 3$, $q = 5$, for instance, using the usual *cycle index* polynomial for $G = \mathrm{AGL}(3, \mathbb{Z}_4)$ we can compute the generating function for the number of G-orbits on subsets of $\mathbb{Z}_4^3$ of size k:

$$1 + x + 2x^2 + 4x^3 + 16x^4 + 37x^5 + 147x^6 + 498x^7 + 2128x^8 + 8790x^9 + 39055x^{10} + 165885x^{11} + 678826x^{12} + 2584627x^{13} + \cdots$$

The "middle term" here is:

$$333347580600x^{32}$$

"Most" of these subsets give quite uninteresting codes. But for instance, *one* of the 2128 orbits of size $k = 8$ consists of codes with $d = 42$ (better than best previously known $d = 41$ according to Brouwer's table). Clearly need some other tools(!)

Tools from Algebraic Geometry

The case $m = 2$ is connected with the theory of toric surfaces.

Main results of paper *Toric surface codes and Minkowski sums* ([2]) show that for q sufficiently large, $d(C_P(\mathbb{F}_q))$ can be bounded above and below by looking at subpolygons $P' \subseteq P$ that decompose as *Minkowski sums*.

Theorem 2 *Let ℓ be the largest positive integer such that there is some $P' \subseteq P$ that decomposes as a Minkowski sum $P' = P_1 + P_2 + \cdots + P_\ell$ with nontrivial P_i. For all $q >> 0$, there is some $P' \subseteq P$ of this form such that*

$$d(C_P(\mathbb{F}_q)) \geq \sum_{i=1}^{\ell} d(C_{P_i}(\mathbb{F}_q)) - (\ell - 1)(q - 1)^2.$$

Ideas behind this

The polygon P specifies a normal fan $\Sigma = \Sigma(P)$, hence an abstract toric variety $X = X_\Sigma$, and a line bundle $\mathcal{L}$ on X. Subpolytopes P_i correspond to subspaces of $H^0(X, \mathcal{L})$.

Minkowski-reducible subpolygons $\leftrightarrow$ *reducible sections* (Newton polygon of product of polynomials is a Minkowski sum).

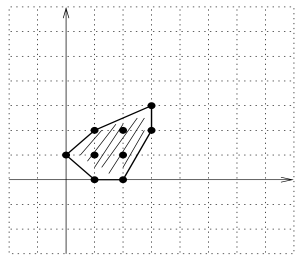
Hasse-Weil upper and lower bounds for a curve Y:

$$q + 1 - 2g(Y)\sqrt{q} \leq |Y(\mathbb{F}_q)| \leq q + 1 + 2g(Y)\sqrt{q}$$

$\Rightarrow$ when $q >$ (a crude but explicit lower bound), reducible curves with ℓ components must have more $\mathbb{F}_q$-rational points than those with $m < \ell$ components.

An Example

Consider P as below



$P \subset [0, q-2]^2$ for all $q \geq 5$.

Note that P contains

$$P' = \text{conv}\{(1,0),(2,0),(1,2),(2,2)\}$$

$(= P_1 + P_2 + P_3$, P_i line segments) and

$$P'' = \text{conv}\{(1,0),(1,1),(3,2),(3,3)\}$$

(similar). No other decomposable $Q \subset P$ with more than three Minkowski summands, and no Minkowski summands with interior lattice points. Theorem 1 above $\Rightarrow$

$$d(C_P(\mathbb{F}_q)) \geq (q-1)^2 - 3(q-1)$$

for $q > \#(P) + 3 = 12$.

Example, cont.

Both subpolygons give rise to reducible curves on the corresponding toric surface. From P' we obtain curves $x(x-a)(y-b)(y-c) = 0$. If $a, b, c \in \mathbb{F}_q^*$ and $b \neq c$, then s has $3(q-1) - 2$ zeroes in $(\mathbb{F}_q^*)^2$. Hence,

$$d(C_P(\mathbb{F}_q)) \leq (q-1)^2 - 3(q-1) + 2.$$

Computations using Magma show that

$$\begin{aligned} d(C_P(\mathbb{F}_5)) = 6^{(*)} &\quad vs. \quad 4^2 - 3 \cdot 4 + 2 = 6 \\ d(C_P(\mathbb{F}_7)) = 20 &\quad vs. \quad 6^2 - 3 \cdot 6 + 2 = 20 \\ d(C_P(\mathbb{F}_8)) = 28 &\quad vs. \quad 7^2 - 3 \cdot 7 + 2 = 30 \\ d(C_P(\mathbb{F}_9)) = 42 &\quad vs. \quad 8^2 - 3 \cdot 8 + 2 = 42 \\ d(C_P(\mathbb{F}_{11})) = 72 &\quad vs. \quad 10^2 - 3 \cdot 10 + 2 = 72. \end{aligned}$$

The dimension is $k = \#(P) = 9$ in each case ($^{(*)}$ code over $\mathbb{F}_5$ is best known for $n = 16, k = 9$).

The case $q = 8$

We may ask: Where does a codeword with $49-28=21$ zero entries come from? Magma: exactly 49 such words. One of them comes, for instance, from the evaluation of

$$\begin{aligned} x + x^3y^3 + y^2 &\equiv x(1 + x^2y^3 + x^6y^2) \\ &\equiv x(1 + x^2y^3 + (x^2y^3)^3) \end{aligned}$$

Here congruences are mod $\langle x^7 - 1, y^7 - 1 \rangle$, the ideal of the $\mathbb{F}_8$-rational points of the 2-dimensional torus. So $1 + x^2y^3 + (x^2y^3)^3$ has exactly the same zeroes in $(\mathbb{F}_8^*)^2$ as $x + x^3y^3 + y^2$.

The case $q = 8$, continued

$1 + u + u^3$ is one of the two irreducible polynomials of degree 3 in $\mathbb{F}_2[u]$, hence

$$\mathbb{F}_8 \cong \mathbb{F}_2[u]/\langle 1 + u + u^3 \rangle.$$

If β is a root of $1 + u + u^3 = 0$ in $\mathbb{F}_8$, then $1 + x^2y^3 + (x^2y^3)^3 =$

$$(x^2y^3 - \beta)(x^2y^3 - \beta^2)(x^2y^3 - \beta^4)$$

and there are exactly $3 \cdot 7 = 21$ points in $(\mathbb{F}_8^*)^2$ where this is zero. Still a sort of *reducibility* that produces a section with the largest number of zeroes here, even though the reducibility only appears when we look modulo the ideal $\langle x^7 - 1, y^7 - 1 \rangle$ (!).

Similar phenomena in many other cases for small q.

§3. Enter the Vandermonde matrices

Now turn to m-dimensional toric codes, any $m \geq 2$.

Square submatrices of the generator matrix G for a Reed-Solomon code are usual (one-variable) Vandermonde matrices:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha^{j_1} & \alpha^{j_2} & \cdots & \alpha^{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha^{j_1})^{k-1} & (\alpha^{j_2})^{k-1} & \cdots & (\alpha^{j_k})^{k-1} \end{pmatrix}$$

(Well-known and standard observation for studying these codes – implies the rows of G are linearly independent, for instance.)

Multivariate generalizations

Let P be an integral convex polytope, and suppose $P \cap \mathbb{Z}^m = \{e(i) : 1 \le i \le \#(P)\}$, listed in some particular order. Let $S = \{p_j : 1 \le j \le \#(P)\}$ be any set of $\#(P)$ points in $(\mathbb{F}_q^*)^m$, also ordered.

Define $V(P; S)$, the *Vandermonde matrix* associated to P and S, to be the $\#(P) \times \#(P)$ matrix

$$V(P; S) = \left(p_j^{e(i)}\right),$$

where $p_j^{e(i)}$ is the value of the monomial $x^{e(i)}$ at the point p_j.

An Example

Let $P = \text{conv}\{(0,0),(2,0),(0,2)\}$ in $\mathbb{R}^2$, and $S = \{(x_j, y_j)\}$ be any set of 6 points in $(\mathbb{F}_q^*)^2$. For one particular choice of ordering of the lattice points in P, we have $V(P;S) =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \end{pmatrix}$$

Estimating d of a toric code

We have the following result:

Theorem 3 *Let $P \subset \mathbb{R}^m$ be an integral convex polytope. Let d be a positive integer and assume that in every set $T \subset (\mathbb{F}_q^*)^m$ with $|T| = (q-1)^m - (d-1)$ there exists some $S \subset T$ with $|S| = \#(P)$ such that $\det V(P;S) \neq 0$. Then the minimum distance satisfies $d(C_P) \geq d$.*

Idea of proof: For all S, $\det V(P;S) \neq 0 \Rightarrow$ the homogeneous linear system obtained the generator matrix, in columns corresponding to S, has only the trivial solution so there are no nonzero codewords with $(q-1)^m - (d-1)$ zero entries. Hence every nonzero codeword has $\geq d$ nonzero entries.

§4. Codes from simplices

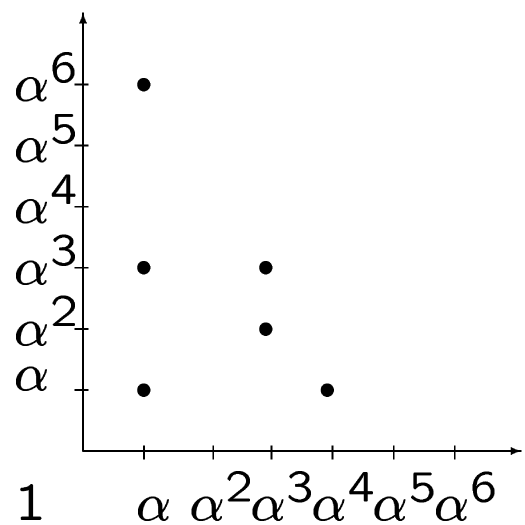
Consider $C_{P_\ell(m)}$ for $P_\ell(m)$ an m-dimensional simplex of the form

$$P_\ell(m) = \text{conv}\{\mathbf{0}, \ell\mathbf{e}_1, \ldots, \ell\mathbf{e}_m\},$$

where the $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{R}^m$. The monomials corresponding to the $\binom{m+\ell}{\ell}$ integer lattice points in $P_\ell(m)$ are all of the monomials in m variables of total degree $\leq \ell$. (The corresponding Vandermonde matrices arise in the study of multivariate Lagrange interpolation using polynomials of bounded total degree.)

Simplicial configurations – an example

Need to identify S for which $\det(V(P_\ell(m); S)) \neq 0$.



A 2-dimensional simplicial configuration of order 2 in $(\mathbb{F}_8^*)^2$.

Definition

Definition 1 *If $m = 1$, an ℓth order* **simplicial configuration** *is any collection of $\binom{1+\ell}{\ell}$ distinct points in $\mathbb{F}_q^*$. For $m \geq 2$, we will say that a collection S of $\binom{m+\ell}{\ell}$ points in $(\mathbb{F}_q^*)^m$ is an m-dimensional ℓth order* **simplicial configuration** *if the following conditions hold:*

1. *For some i, $1 \leq i \leq m$, there are hyperplanes $x_i = a_1, x_i = a_2, \ldots, x_i = a_{\ell+1}$ such that for each $1 \leq j \leq \ell + 1$, S contains exactly $\binom{m-1+j-1}{j-1}$ points with $x_i = a_j$. (Note that*

$$\binom{m+\ell}{\ell} = \sum_{j=1}^{\ell+1} \binom{m-1+j-1}{j-1}$$

   by a standard binomial coefficient identity.)

2. *For each j, $1 \leq j \leq \ell+1$, the points in $x_i = a_j$ form an $(m-1)$-dimensional simplicial configuration of order $j - 1$.*

Some observations

Let S be an m-dimensional ℓth order simplicial configuration consisting of $\binom{m+\ell}{\ell}$ points, in hyperplanes $x_m = a_1, \dots, x_m = a_{\ell+1}$. Write $S = S' \cup S''$ where S' is the union of the points in $x_i = a_1, \dots, a_\ell$, and S'' is the set of points in $x_i = a_{\ell+1}$. Also, let $\pi : \mathbb{F}_q^m \to \mathbb{F}_q^{m-1}$ be the projection on the first $m-1$ coordinates. By the definition, it follows that both S' and $\pi(S'')$ are themselves simplicial configurations, with S' of dimension m and order $\ell - 1$, and $\pi(S'')$ of dimension $m-1$ and order ℓ.

A recurrence

Theorem 4 *Let* $P_\ell(m)$ *be as above and let* S *be an* ℓ*th order simplicial configuration of* $\binom{m+\ell}{\ell}$ *points as in the paragraph above. Then writing* $p = (p_1, \ldots, p_m)$ *for points* $p \in (\mathbb{F}_q^*)^m$,

$$\begin{aligned} \det V(P_\ell(m); S) \;=\; \pm \prod_{p \in S'} & (p_m - a_{\ell+1}) \\ & \cdot \det V(P_{\ell-1}(m); S') \\ & \cdot \det V(P_\ell(m-1); \pi(S'')) \end{aligned}$$

(The recurrence was suggested by a computation of the determinant in a paper on multivariate interpolation by Chui and Lai, where corresponding sets of points in $\mathbb{R}^m$ are identified as "poised sets" for interpolation by polynomials of degree bounded bounded by ℓ.)

An illustrative example

Consider all polynomials of degree ≤ 2 in three variables and the Vandermonde matrix $V(P_2(3); S)$. For notational simplicity, write points in a 3-dimensional simplicial configuration $S \subset (\mathbb{F}_q^*)^3$ of order 2 as (x_i, y_i, z_i), for $i = 1, \ldots, 10 = \binom{3+2}{2}$. Here S' consists of the first four points in S, and S'' consists of the other six points. Under the hypothesis that S is a simplicial configuration, we have $z_5 = z_6 = \cdots = z_{10} = c$ for some c.

Consequences

Corollary 1 *Let* $P_\ell(m)$ *be as above and let* S *be an* ℓ*th order simplicial configuration of* $\binom{m+\ell}{\ell}$ *points. Then* $\det V(P_\ell(m); S) \neq 0$.

Theorem 5 *Let* $\ell < q-1$, *and let* $P_\ell(m)$ *be the simplex in* $\mathbb{R}^m$ *defined above. Then the minimum distance of the toric code* $C_{P_\ell(m)}$ *is given by*

$$d(C_{P_\ell(m)}) = (q-1)^m - \ell(q-1)^{m-1}.$$

The result on Vandermondes is used to show $d(C_{P_\ell(m)}) \geq (q-1)^m - \ell(q-1)^{m-1}$ via Theorem 3. A pigeon-hole principle argument constructs simplicial configurations $S \subset T$ for every T with $|T| = \ell(q-1)^m + 1$. Other inequality comes from reducibles $(x_m - a_1)\ldots(x_m - a_\ell)$.

Concluding Comments

- Can get similar results for other families of polytopes (e.g. parallelotopes see [1])

- *But* the results on toric codes from simplices and parallelotopes show that d is often quite *small* relative to k.

- It is an interesting problem to determine criteria for polytopes (or subsets of the lattice points in a polytope) that yield good evaluation codes.