Mathematics 357 – Combinatorics
Midterm Examination 2 – April 7, 2017

Directions

Do any combination of parts of the following problems in the blue exam book. (The exam will be scored out of 100 points, but you have the possibility of accumulating up to 110 points if you do all the parts.) Explain your reasoning fully and do not place any work on the exam sheet. *Relax, and read each question carefully before starting to work!*

I.
A) (5) What is the definition of the Stirling number of the second kind, $S(n, k)$, in terms of distribution problems?
B) (15) State and prove the recurrence relation for the $S(n, k)$.
C) (5) Determine the Stirling number $S(6, 2)$.

II. For all parts of this question, express the number of distributions in terms of partition numbers, Stirling numbers, binomial coefficients, etc. *You do not need to evaluate any of these to a single number.*
A) (5) How many ways are there to distribute 10 identical unmarked pads of paper into 4 piles, with no restrictions on the sizes of the piles?
B) (5) How many ways are there if the piles of pads are supposed to go to 4 numbered tables and at least one pad goes to each table?
C) (5) How many ways are there if the pads have 10 different colors of paper but we're back in the situation of part A (i.e. just 4 piles, not the 4 numbered tables)?
D) (5) What changes in part C) if every table gets at least one pad?

III. (15) 100 pencils are divided into four lots of sizes x_1, x_2, x_3, x_4, *each a multiple of 5*, and placed in 4 classrooms. Room 1 receives *at least* 20 pencils, room 2 receives *no more than* 40 pencils, room 3 receives *between* 15 *and* 25 pencils, and room 4 receives a number of pencils that is a *multiple of* 10. How many different vectors (x_1, x_2, x_3, x_4) of numbers of pencils satisfy these conditions? Set up an appropriate *generating function* and explain how you would use it to answer this question.

IV. (15) For each $n \geq 1$, let T_n be the number of ways of arranging n "triominoes" (like dominoes, but 3×1) in a $3 \times n$ tray with no overlapping. Find a recurrence relation of order 3 for the T_n sequence and the initial values T_1, T_2. (*Don't try to solve the recurrence.*)

V.
A) (20) A sequence R_n is defined by the recurrence $R_n = 6R_{n-1} - 8R_{n-2}$. Express the *generating function* of the sequence R_n as a rational function of x if the initial terms are $R_0 = 3, R_1 = 1$ and use it to derive a formula for R_n as a function of n. (Partial credit 10 for solution by the "shortcut" method.)
B) (15) Now suppose we wanted to solve $R_n = 6R_{n-1} - 8R_{n-2} + 2^n$. What would be the appropriate "good guess" for the particular solution? See table from Beeler on back. Determine the solution of this recurrence with the initial conditions $R_0 = 1$, $R_1 = 1$.