MATH 357 – Combinatorics
Solutions for Problem Set 5
March 17, 2017

4.3.13. (Recall the directions for the problem I gave said to interchange the 16 and the 9 so there are 16 balls and 9 urns!) Also refer to Table 4.6 on page 109.

(i) This is the number of *partitions* of 16 things into 9 or fewer parts, $\sum_{k=1}^{9} p(16,k)$. This form by itself was acceptable. If you compute the value, you want to use the recurrence for these partition numbers given in Theorem 4.3.4 in Beeler. (These are numbers of *partitions, not permutations!*) The correct numbers are:

$$\begin{aligned} p(16,1) &= 1,\ p(16,2) = 8,\ p(16,3) = 21,\ p(16,4) = 34,\ p(16,5) = 37, \\ p(16,6) &= 35,\ p(16,7) = 28,\ p(16,8) = 22,\ p(16,9) = 15. \end{aligned}$$

The total should be $\sum_{i=1}^{9} p(16,i) = 201$. I computed these with a *recursive procedure* in Maple based on the recurrence:

```
pnk := proc(n, k)
if n < k then
  return(0)
elif n = k then
  return(1)
elif k = 1 then
  return(1)
else
  return(pnk(n-1,k-1)+pnk(n-k,k))
end if
end proc:
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(ii) This is $p(16,9) = 15$. (Note that $p(9,16) = 0$ so this question is slightly silly as stated!)

(iii) This is 9^{16} by the Multiplication Principle (9 choices for the urn the first ball goes to, then 9 choices again for the second ball, etc.)

(iv) This is the Stirling number of the second kind $S(16,9) = 820784250$. The $S(n,k)$ can be computed in Maple using a procedure with the same outline as the one above, but using the recurrence from Theorem 4.3.8:

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$

(v) This is $9! \cdot S(16,9)$.

4.3.17 (i) $p_2(n,k)$ is the number of partitions of n into k integers each at least 2. If the smallest part in the partition is a 2, then removing one of the 2's gives a partition of $n-2$ with $k-1$ parts, each of which is at least 2. Conversely, starting from such a partition of $n-2$ and adding 2 gives a partion of n of the type we are considering. If the smallest part in the partition is a 3, then similarly, removing one of the 3's gives a partition of $n-3$ with $k=1$ parts, each of which is at least 3 and conversely. Otherwise, each part is at least 4. We subtract two from each of the parts, yielding a partition of $n-2k$ into k parts, all of which are at least two. Every partition counted by $p_2(n,k)$ falls into one of these categories, and the categories are pairwise disjoint. Hence by the Addition Principle,

$$p_2(n,k) = p_2(n-2,k-1) + p_3(n-3,k-1) + p_2(n-2k,k).$$

4.3.18. (The idea here is similar to the proof of Theorem 4.3.8 – the recurrence relation for the Stirling numbers of the second kind.) We have that $S(n+1,k)$ gives the number of ways to distribute $n+1$ balls into k urns where the balls are labeled, the urns are unlabeled and no urn is empty. The idea is to break up the possible distributions based on which other balls end up together with ball 1 (recall, they're labeled!).

- If ball 1 is alone, the other n balls get distributed among the remaining $k-1$ urns with none empty, so $\binom{n}{n}S(n,k-1)$ such distributions;
- If ball 1 is with one other ball, there are $\binom{n}{n-1}$ possible choices for the ones not with ball 1, and those other $n-1$ balls get distributed among the remaining $k-1$ urns with none empty, so $\binom{n}{n-1}S(n-1,k-1)$ of these;
- If ball 1 is with j other balls, there are $\binom{n}{n-j}$ possible choices for the remaining balls, and those other $n-j$ balls get distributed among the remaining $k-1$ urns with none empty, so $\binom{n}{n-j}S(n-j,k-1)$ of these.

Hence summing over the numbers j by the Addition Principle, we have

$$S(n+1,k) = \sum_{j=0}^{n} \binom{n}{n-j} S(n-j,k-1) = \sum_{i=0}^{n} \binom{n}{i} S(i,k-1)$$

where the last equality just comes from re-indexing the sum with $i=n-j$. (Comments:

1. Not all of the terms in this sum will be nonzero, of course.
2. You can think of the i in the binomial coefficient as *either* the number of balls $2,\ldots,n+1$ besides the balls with ball 1, or as the number of balls $2,\ldots,n+1$ with ball 1. One set is the complement of the other inside the set of balls other than ball 1. Recall the binomial coefficient identity:
$$\binom{n}{n-j} = \binom{n}{j}.$$
I explained this both ways with different people in office hours, so don't get confused if it looks as though it's reversed from the way you were thinking about it!)

4.3.20. We have that $S(n,k)$ gives the number of ways to distribute n balls into k urns where the balls are labeled, the urns are unlabeled and no urn is empty. Suppose we *temporarily label* the urns using the integers in $[k]$. Then the numbers of balls in the urns, $\lambda_1, \ldots, \lambda_k$ must sum to n and they must all be greater than or equal to 1. By the labeled urns case, we know that the number of different distributions where the kth urn gets λ_k balls is given by the multinomial coefficient

$$\binom{n}{\lambda_1, \ldots, \lambda_k}$$

and the total number of possible distributions is

$$\sum \binom{n}{\lambda_1, \ldots, \lambda_k}$$

where the sum is over all the k-tuples $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 + \cdots + \lambda_k = n$ and $\lambda_i \geq 1$ for all i. Now *take the labels off the urns*. The only thing we can distinguish now is when urns have different numbers of balls. So if a_i is the number of times the integer i appears in the list $\lambda_1, \ldots, \lambda_k$ from before, then there are $a_1!a_2!\cdots a_n!$ possible different labelings in the first step yielding the same eventual unlabeled distribution. Hence

$$S(n,k) = \sum \binom{n}{\lambda_1, \ldots, \lambda_k} \cdot \frac{1}{a_1!a_2!\cdots a_n!}$$

where the summation is over the k-tuples $\lambda_1, \ldots, \lambda_k$ as before.

5.1.10. (i) We have

$$4x^2 - 20x + 24 = 24\left(1 - \frac{5}{6}x + \frac{1}{6}x^2\right) = 24\left(1 - \frac{1}{3}x\right)\left(1 - \frac{1}{2}x\right).$$

(ii) We have

$$6 - 5x - 2x^2 + x^3 = 6\left(1 - \frac{5}{6}x - \frac{1}{3}x^2 + \frac{1}{6}x^3\right)$$

We see (e.g. by the Rational Roots Test) that $x = 1$ is a root and that starts the factorization:

$$= 6(1-x)\left(1 + \frac{1}{6}x - \frac{1}{6}x^2\right) = 6(1-x)\left(1 - \frac{1}{3}x\right)\left(1 + \frac{1}{2}x\right).$$

5.1.11. We write

$$\frac{5x-12}{x^2-5x+6} = \frac{5x-12}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Then clearing denominators gives

$$5x - 12 = A(x-3) + B(x-2) = (A+B)x + (-3A-2B).$$

Hence $A+B=5$ and $-3A-2B=-12$, so $A=2$ and $B=3$. This gives

$$\frac{5x-12}{x^2-5x+6} = \frac{2}{x-2} + \frac{3}{x-3} = \frac{-1}{1-x/2} + \frac{-1}{1-x/3}$$

The reason for writing it in the final form is explained in §5.2(!)

5.2.8. Start from the geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This series converges for all x with $|x| < 1$ to the function on the left, so we can differentiate twice with respect to x to yield

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$$

Since the terms on the right with $n = 0, 1$ now are zero we can reindex the sum with $n-2 = m$ or $n = m+2$. This yields, after dividing both sides by 2:

$$\frac{1}{(1-x)^3} = \sum_{m=0}^{\infty} \frac{(m+2)(m+1)}{2} x^m.$$

This series converges on the same interval, namely for all $|x| < 1$.

5.2.10. We first factor the bottom as in 5.1.10 above, then compute the partial fraction decomposition, then apply the geometric series formula:

$$\begin{aligned}
\frac{3}{1-5x+6x^2} &= \frac{3}{(1-2x)(1-3x)} \\
&= \frac{-6}{1-2x} + \frac{9}{1-3x} \\
&= (-6) \cdot \sum_{k=0}^{\infty} (2x)^k + 9 \cdot \sum_{k=0}^{\infty} (3x)^k \\
&= \sum_{k=0}^{\infty} \left(-6 \cdot 2^k + 9 \cdot 3^k \right) x^k.
\end{aligned}$$