MATH 357 – Combinatorics
Solutions for Problem Set 1
February 3, 2017

1.5.10. As is true in many cases, there is *more than one correct approach* on this problem. The most direct argument is probably this one:

Solution 1: Thinking of the pigeonhole principle, let the pigeons be the n people, and the pigeonholes be the numbers of people known, which can only be $1, 2, \ldots,$ or n. Arguing by contradiction, suppose no two people know the same number of people. Then exactly one person knows 1 person (him- or herself), exactly one person knows 2 people, ... , and exactly one person knows n people. But this is a contradiction because if one person knows all n people, he must know the person who only knew one person. But note that the problem said if A knows B then B must also know A. So that person would have known at least two people including him- or herself.

Solution 2: Here's another way, using the pigeonhole principle again. Everyone in the group knows at least him- or herself. If two or more people know only themselves, then we have two people in the group who know just one person. If there is exactly one person who knows only him or herself, then the other $n-1$ people can know only $2, 3, \ldots, n-2$ or $n-1$ people. (Noone can know n people because none of them can know the person who knows only him- or herself.) So we have $n-1$ pigeons placed into $n-2$ pigeonholes, and there must be two people who know the same number of people. Finally, if everyone knows at least one other person, then they can only only $2, 3, \ldots, n$ people. This gives n pigeons placed in $n-1$ pigeonholes again and at least two people know the same number of people.

1.5.11 (Extra Credit) Consider the "slots" between consecutive men around the table as the pigeonholes and the $n+1$ women as the pigeons. If any two consecutive slots both have at least one woman, we have a man seated between two women. The remaining case to consider is where no two consecutive slots have women. If n is even, this means that only $n/2$ of the slots could be used. But the generalized pigeonhole principle (Theorem 1.5.5) implies that some one of those slots has strictly more than $\left\lfloor \frac{(n+1)-1}{n/2} \right\rfloor > 2$ women. With three women together, a woman is sitting between two women. If n is odd, there can be at most $(n-1)/2$ slots that contain women (as long as no two consecutive slots do). But then again some one of those slots has strictly more than $\left\lfloor \frac{(n+1)-1}{(n-1)/2} \right\rfloor > 2$, women, so at least 3 women. So some woman is sitting between two other women again.

2.1.14. The set of possible 5-person committees is in 1-1 correspondence with the Cartesian product:

$$\text{Clergy} \times \text{Scientists} \times \text{Lawyers} \times \text{Doctors} \times \text{Lay}.$$

By the Multiplication Principle, the number of possible committees is $8 \cdot 4 \cdot 5 \cdot 3 \cdot 10 = 4800$. *Comment:* Notice that the problem *does not say* to continue and form a second and third committee with the people who are left over after the first committee is formed.

2.1.15. The number of possible combinations of n toppings is 2^n (it's the same as the collection of subsets of the set of toppings). We want the smallest n such that $2^n > 1,000,000$, which is $n = 20$.

2.1.19. If n is even, the first half of the word can be chosen arbitrarily and that determines the last half to get a palindrome. This gives $26^{n/2}$ different palindromes and hence (Addition Principle) $26^n - 26^{n/2}$ nonpalindromes, since no word is both a palindrome and a nonpalindrome. Similarly, if n is odd, then the number is $26^n - 26^{(n+1)/2}$ since now the middle letter of the word can also be chosen arbitrarily. *Comment:* These are two separate cases, so you do not want to combine them. One is valid if n is an even number; the other is valid if n is odd.

2.1.23. We use the Multiplication Principle for all of these:

(i) There are $9 \cdot 10^4$ five-digit numbers (the leading digit cannot be zero).

(ii) To count numbers that have no two equal digits, there are 9 choices for the leading digit (not 0), then 9 again for the second (which must be different from the first digit, but can be 0), then 8 for the third, 7 for the fourth, and 6 for the fifth. So there are $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216$ such five-digit numbers.

(iii) The ones-digit must be either $1, 3, 5, 7$, or 9 if the number is odd, then whatever that digit is, there are 8 choices for the leading digit (not 0 and not equal to the ones-digit), 8 choices for the second, 7 choices for the third, and 6 choices for the fourth. This gives $8 \cdot 8 \cdot 7 \cdot \cdot 5 = 13440$ such numbers.

(iv) *Solution 1:* By the Addition Principle the number of even numbers with no two equal digits is equal to the number of all numbers with no two equal digits, minus the number of odd numbers with no two equal digits, so $27216 - 13440 = 13776$.

Solution 2: These can also be counted as follows. There are two cases. If the final digit is a zero, we have $9 \cdot 8 \cdot 7 \cdot 6 \cdot 1 = 3024$ numbers ending in zero with no two equal digits. If the final digit is not zero, then there are four other even numbers it could be, the first digit could be any one of 8 possibilities, etc. giving $8 \cdot 8 \cdot 7 \cdot 6 \cdot 4 = 10752$ possibilities. Then we get the total number of even numbers with distinct digits by the Addition Principle: $10752 + 3024 = 13776$ as before. (Note that Solution 1 is certainly simpler!)

2.2.7. Let's take the sentence "No two restaurants serve the same menu" to mean that the menus are pairwise disjoint (equivalently, no single dish appears on any two different menus). For instance, if tacos appear on the Mexican menu, they are not also on the American menu. Then by the Addition Principle, there are $6 + 7 + 10 + 8 = 31$ different choices for lunch. (*Comment:* Just saying the menus are different is ambiguous here because we can't really tell whether that means they are just different as sets, or whether they don't have any items in common. If they were just different as sets, we would not have enough information to answer the question!)

2.2.8. By the Multiplication Principle, there are 26^6 seven-letter words starting with a. (See the comment in 2.1.19.) Similarly there are 25^7 words that do not contain a. Since no word appears in both collections, by the Addition Principle, the total number is $26^6 + 25^7 > 6.4 \times 10^9$, a large number of such words :)

2.2.11. In order to answer this, we need to assume that no one menu item appearing on any one menu appears on any other menu. That is the sets of appetizers, entrees, and desserts are all pairwise disjoint. By the Multiplication and Addition Principles, the total number of possible dinners is

$$5 \cdot 11 \cdot 4 + 5 \cdot 9 \cdot 6 + 4 \cdot 10 \cdot 8 + 2 \cdot 7 \cdot 5 + 7 \cdot 30 \cdot 9 = 2770.$$