MATH 357 -- Combinatorics
A Computation With the Cycle Index Polynomial
April 21, 2017

Say $G = C\_8$ acting on the wheel with 8 wedge-shaped sectors. As we just saw

$$Z := \frac{(x1^8 + x2^4 + 2 \cdot x4^2 + 4 \cdot x8)}{8};$$

$$\frac{1}{8}\, x1^8 + \frac{1}{8}\, x2^4 + \frac{1}{4}\, x4^2 + \frac{1}{2}\, x8 \qquad \textbf{(1)}$$

By the ***Polya Enumeration Theorem***, to find the number of distinct colorings of the sectors using 3 colors we compute

$$Zsub := subs(\{x1 = c1 + c2 + c3, x2 = c1^2 + c2^2 + c3^2, x4 = c1^4 + c2^4 + c3^4, x8 = c1^8 + c2^8 + c3^8\}, Z);$$

$$\frac{1}{8}\,(c1 + c2 + c3)^8 + \frac{1}{8}\,(c1^2 + c2^2 + c3^2)^4 + \frac{1}{4}\,(c1^4 + c2^4 + c3^4)^2 + \frac{1}{2}\, c1^8 + \frac{1}{2}\, c2^8 + \frac{1}{2}\, c3^8 \qquad \textbf{(2)}$$

$$expand(Zsub);$$

$$\begin{aligned} & c1^8 + c1^7\, c2 + c1^7\, c3 + 4\, c1^6\, c2^2 + 7\, c1^6\, c2\, c3 + 4\, c1^6\, c3^2 + 7\, c1^5\, c2^3 \\ & + 21\, c1^5\, c2^2\, c3 + 21\, c1^5\, c2\, c3^2 + 7\, c1^5\, c3^3 + 10\, c1^4\, c2^4 + 35\, c1^4\, c2^3\, c3 \\ & + 54\, c1^4\, c2^2\, c3^2 + 35\, c1^4\, c2\, c3^3 + 10\, c1^4\, c3^4 + 7\, c1^3\, c2^5 + 35\, c1^3\, c2^4\, c3 \\ & + 70\, c1^3\, c2^3\, c3^2 + 70\, c1^3\, c2^2\, c3^3 + 35\, c1^3\, c2\, c3^4 + 7\, c1^3\, c3^5 + 4\, c1^2\, c2^6 \\ & + 21\, c1^2\, c2^5\, c3 + 54\, c1^2\, c2^4\, c3^2 + 70\, c1^2\, c2^3\, c3^3 + 54\, c1^2\, c2^2\, c3^4 \\ & + 21\, c1^2\, c2\, c3^5 + 4\, c1^2\, c3^6 + c1\, c2^7 + 7\, c1\, c2^6\, c3 + 21\, c1\, c2^5\, c3^2 \\ & + 35\, c1\, c2^4\, c3^3 + 35\, c1\, c2^3\, c3^4 + 21\, c1\, c2^2\, c3^5 + 7\, c1\, c2\, c3^6 + c1\, c3^7 + c2^8 \\ & + c2^7\, c3 + 4\, c2^6\, c3^2 + 7\, c2^5\, c3^3 + 10\, c2^4\, c3^4 + 7\, c2^3\, c3^5 + 4\, c2^2\, c3^6 + c2\, c3^7 \\ & + c3^8 \end{aligned} \qquad \textbf{(3)}$$

$$coeff(coeff(coeff(Zsub, c3), c2^4), c1^3)$$

$$35 \qquad \textbf{(4)}$$

This is the coefficient of $c1^3 \cdot c2^4 \cdot c3$, which represents the number of distinct ways to color the wheel using three colors, so that exactly three wedges are colored red (color 1), 4 are colored green (color 2), and 1 is colored black (color 3).

The count in this particular case can be seen by ***other methods too***, of course to check our work. Since there is only one black wedge, we can rotate it to position 1. Then the other 7 wedges must be colored 3 red and 4 green and they are fixed in positions 2, ..., 8. (We used the initial rotation to "standardize" the positions to be counted and pick one particular element out of each C_8-equivalence class (one out of each orbit. The number of ways to color the remaining 7 wedges using two colors is $\binom{7}{3} = \binom{7}{4}$ = 35.

binomial(7, 3);

$$35 \tag{5}$$