MATH 357 – Combinatorics
Solutions for Problem Set 8
April 21, 2017

7.1.19. *Solution 1 – the "brute force" way*: Let A be the set of distributions of the 25 gumdrops where Alice's requirement *is satisfied*, B the set where Bob's requirement *is satisfied*, etc. We want to count the number of elements in the *complement of the union* $A \cup B \cup C \cup D$. By the Inclusion-Exclusion Principle,

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|. \end{aligned}$$

The total number of ways to distribute the 25 gumdrops is $\binom{25+3}{3} = 3276$ (unlabeled balls and labeled urns).

Then $|A|$ is the coefficient of x^{25} in the expansion of $\frac{x+x^5}{(1-x)^3}$, which is $|A| = 556$. $|B|$ is the coefficient of x^{25} in $\frac{1}{(1-x)^3(1-x^2)}$, which is $|B| = 1729$. $|C|$ is the coefficient of x^{25} in $\frac{x^4}{1-x)^4}$, which is $|C| = 2024$. $|D|$ is the coefficient of x^{25} in $\frac{1+x+\cdots+x^6}{(1-x)^3}$, which is $|D| = 1946$.

Similarly,

$$\begin{aligned} |A \cap B| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)}{(1-x)^2(1-x^2)} = 290 \\ |A \cap C| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)x^4}{(1-x)^3} = 384 \\ |A \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)(1+\cdots+x^6)}{(1-x)^2} = 280 \\ |B \cap C| &= \text{coeff. of } x^{25} \text{ in } \frac{x^4}{(1-x)^3(1-x^2)} = 1078 \\ |B \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{1+\cdots+x^6}{(1-x)^2(1-x^2)} = 1014 \\ |C \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{(1+\cdots+x^6)x^4}{(1-x)^3} = 1344. \end{aligned}$$

Then,

$$\begin{aligned} |A \cap B \cap C| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)x^4}{(1-x)^2(1-x^2)} = 202 \\ |A \cap B \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)(1+\cdots+x^6}{(1-x)(1-x^2)} = 144 \\ |A \cap C \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{(x+x^5)x^4(1+\cdots+x^6}{(1-x)^2} = 224 \\ |B \cap C \cap D| &= \text{coeff. of } x^{25} \text{ in } \frac{x^4(1+\cdots+x^6)}{(1-x)^2(1-x^2)} = 706. \end{aligned}$$

Finally $|A \cap B \cap C \cap D|$ is the coefficient of x^{25} in $\frac{(x+x^5)x^4(1+\cdots+x^6)}{(1-x)(1-x^2)}$, which is 116. So the final answer is

$$3276-556-1729-2024-1946+290+384+280+1078+1014+1344-202-144-224-706+116 = 251.$$

Solution 2 – "the lazy, clever way" – Note: several of you had this idea, one did it completely correctly this way, others were very close, and a few others were on the right track. In some cases, something must have gone wrong in entering stuff into Maple because your numerical answer was not correct – I cannot tell what went wrong without seeing what you entered into Maple, though; in other papers I couldn't tell what you meant at first and I had to change the scoring, making a mess in the process. Sorry for that! *But please explain what you are doing so I don't have to try to read your mind!!!!!*

In any case we can also count what we want by counting

$$|\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}|$$

(here $\overline{A}$ is the complement of A as above inside the set of all distributions; in words it is the set of distributions where Alice's requirement is *not satisfied*, and the others are similar). This count can be done with a single generating function computation. We want the coefficient of x^{25} in the product

$$(1 + x^2 + x^3 + x^4 + x^6 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3)(x^7 + x^8 + \cdots)$$

The first factor has all the x^k except x and x^5, the second has all but the even exponents, the third has all the terms x^k with $k < 4$ and the last has all the terms x^k with $k > 6$. This way also gives $\cdots + 251x^{25} + \cdots$ so the number we want is 251.

7.3.6. The number is

$$S(n, k-m) = \frac{1}{(k-m)!} \sum_{i=1}^{k-m} (-1)^i \binom{k-m}{i} (k-m-i)!.$$

(Think about the terms in the sum $\sum_{i=1}^{k} S(n,k)$ for the labeled balls, unlabeled urns, no restrictions entry in Table 4.6; the second equality comes from the equation at the top of page 211.)

7.3.7. There are $\binom{n}{k}$ ways to choose the fixed points, and then for each such choice the permutation must be a derangement of the other $n-k$ numbers. By the Multiplication Principle, the number of permutations of $[n]$ with exactly k fixed points is

$$\binom{n}{k} \cdot D_{n-k} = \binom{n}{k} \cdot (n-k)! \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = \frac{n!}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}.$$

8.2.12. As permutations of [5], they are first the rotations:

$$(1)(2)(3)(4)(5), (12345), (13524), (14253), (15432),$$

then the reflections

$$(1)(25)(43), (2)(13)(45), (3)(24)(15), (4)(35)(12), (5)(14)(23).$$

The other information asked for is as follows

g	$Inv(g)$	cyc(g)
(1)(2)(3)(4)(5)	[5]	5
(12345)	$\emptyset$	1
(13524)	$\emptyset$	1
(14253)	$\emptyset$	1
(15432)	$\emptyset$	1
(1)(25)(43)	{1}	3
(2)(13)(45)	{2}	3
(3)(24)(15)	{3}	3
(4)(35)(12)	{4}	3
(5)(14)(23)	{5}	3

and

x	$st(x)$
1	{(1)(2)(3)(4)(5), (1)(25)(43)}
2	{(1)(2)(3)(4)(5), (2)(13)(45)}
3	{(1)(2)(3)(4)(5), (3)(24)(15)}
4	{(1)(2)(3)(4)(5), (4)(35)(12)}
5	{(1)(2)(3)(4)(5), (5)(14)(23)}

8.2.13. S_4 contains all the $4! = 24$ permutations of [4], which have disjoint cycle decompositions like this: the identity, then 6 4-cycles

$$(1)(2)(3)(4), (1234), (1432), (1243), (1342), (1324), (1423)$$

then 8 products of a 1-cycle and a 3-cycle and 3 products of two 2-cycles

$$(1)(234), (1)(243), (2)(134), (2)(143), (3)(124), (3)(142), (4)(123), (4)(132), (12)(34), (13)(24), (14)(23),$$

and finally 6 products of a 2-cycle and two 1-cycles:

$$(12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2).$$

The other information asked for is as follows. Instead of listing all the elements here, I have included just one of each cycle type:

g	$Inv(g)$	cyc(g)
(1)(2)(3)(4)	[4]	4
(1234)	$\emptyset$	1
(1)(234)	{1}	2
(12)(34)	$\emptyset$	2
(12)(3)(4)	{3, 4}	3

The stabilizers of each $x \in [4]$ look like a copy of S_3 sitting inside S_4. For instance:

x	$st(x)$
4	$\{(1)(2)(3)(4), (123)(4), (132)(4), (12)(3)(4), (13)(2)(4), (23)(1)(4)\}$

(If you strip the (4) from the end of each of these permutations, you have the list of 3! elements of S_3.)

8.3.13. We use Burnside's Lemma:

$$\begin{aligned}
\text{Inv}(e) &= [6] \\
\text{Inv}(\pi_1) &= \{4,5,6\} \\
\text{Inv}(\pi_2) &= \{4,5,6\} \\
\text{Inv}(\pi_3) &= \{1,2,3\} \\
\text{Inv}(\pi_4) &= \emptyset \\
\text{Inv}(\pi_5) &= \emptyset \\
\text{Inv}(\pi_6) &= \{1,2,3\} \\
\text{Inv}(\pi_7) &= \emptyset \\
\text{Inv}(\pi_8) &= \emptyset
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\{\text{orbits}\}| &= \frac{1}{|G|}\sum_{g \in G} |\text{Inv}(g)| \\
&= \frac{1}{9}(6+3+3+3+0+0+3+0+0) \\
&= 2.
\end{aligned}$$

By examining the form of the cycle decompositions of the elements of G, it is clear that the two orbits are $\{1,2,3\}$ and $\{4,5,6\}$.