MATH 357 – Combinatorics
Solutions for Problem Set 7
March 31, 2017

6.2.6. From the recurrence $R_n = 2R_{n-1} + 35R_{n-2}$, with initial conditions $R_0 = 0$ and $R_1 = 1$, if $R(x) = \sum_{n=0}^{\infty} R_n x^n$, we have

$$\begin{aligned} R(x) &= R_0 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \cdots \\ 2xR(x) &= 2R_0 x + 2R_1 x^2 + 2R_2 x^3 + 2R_3 x^4 + \cdots \\ 35x^2 R(x) &= 35R_0 x^2 + 35R_1 x^3 + 35R_2 x^4 + \cdots \end{aligned}$$

so

$$(1 - 2x - 35x^2)R(x) = R_0 + (R_1 - 2R_0)x \Rightarrow R(x) = \frac{x}{1 - 2x - 35x^2}.$$

By partial fractions,

$$\frac{x}{1 - 2x - 35x^2} = \frac{A}{1 - 7x} + \frac{B}{1 + 5x}$$

with $x = A(1 + 5x) + B(1 - 7x)$, so $A = \frac{1}{12}$ and $B = \frac{-1}{12}$. From the geometric series expansions:

$$R_n = \frac{1}{12}(7^n - (-5)^n).$$

6.2.8. We follow the same method as in 6.2.6. However, now the recurrence is of 3rd order so we continue to $x^3 R(x)$ to get the combination that will cancel: From the recurrence $R_n = 5R_{n-1} + 29R_{n-2} - 105R_{n-3}$, with initial conditions $R_0 = 0$ and $R_1 = R_2 = 1$, if $R(x) = \sum_{n=0}^{\infty} R_n x^n$, we have

$$\begin{aligned} R(x) &= R_0 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \cdots \\ -5xR(x) &= -5R_0 x - 5R_1 x^2 - 5R_2 x^3 - 5R_3 x^4 + \cdots \\ -29x^2 R(x) &= -29R_0 x^2 - 29R_1 x^3 - 29R_2 x^4 + \cdots \\ 105x^3 R(x) &= 105R_0 x^3 + 105R_1 x^4 + \cdots \end{aligned}$$

so

$$(1 - 5x - 29x^2 + 105x^3)R(x) = R_0 + (R_1 - 5R_0)x + (R_2 - 5R_1 - 29R_0)x^2 \Rightarrow R(x) = \frac{x - 4x^2}{1 - 5x - 29x^2 + 105x^3}.$$

By partial fractions,

$$\frac{x - 4x^2}{1 - 5x - 29x^2 + 105x^3} = \frac{A}{1 - 3x} + \frac{B}{1 + 5x} + \frac{C}{1 - 7x}$$

with

$$x - 4x^2 = A(1 + 5x)(1 - 7x) + B(1 - 3x)(1 - 7x) + C(1 - 3x)(1 + 5x),$$

so $A = \frac{1}{32}$, $B = \frac{-3}{32}$, and $C = \frac{1}{16}$. From the geometric series expansions:

$$R_n = \frac{1}{32}(3^n - 3 \cdot (-5)^n + 2 \cdot 7^n).$$

6.3.9. The characteristic polynomial is $1 - x - x^2$, which factors as

$$1 - x - x^2 = \left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right).$$

Therefore the sequence will have the form

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants c_1, c_2. Note that $\alpha_1 = \frac{1+\sqrt{5}}{2} \doteq 1.618$ and $\alpha_2 = \frac{1-\sqrt{5}}{2} \doteq -.618$, hence $|\alpha_1| > |\alpha_2|$. Then $|\alpha_2/\alpha_1| < 1$, so $(\alpha_2/\alpha_1)^n \to 0$ as $n \to \infty$. It follows that as long as $c_1 \neq 0$ (that's the "genericity condition" you need for the statement given in the problem to be true),

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \alpha_1 = \frac{1+\sqrt{5}}{2}$$

like this: we have

$$\frac{F_n}{F_{n-1}} = \frac{c_1\alpha_1^n + c_2\alpha_2^n}{c_1\alpha_1^{n-1} + c_2\alpha_2^{n-1}}.$$

Multiply the top and bottom by $\frac{1}{\alpha_1^{n-1}}$:

$$\frac{F_n}{F_{n-1}} = \frac{c_1\alpha_1 + c_2\alpha_2 \cdot \left(\frac{\alpha_2}{\alpha_1}\right)^{n-1}}{c_1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{n-1}}.$$

Now let $n \to \infty$. The $(\alpha_2/\alpha_1)^{n-1}$ terms go to zero, leaving

$$\frac{c_1\alpha_1}{c_1} = \alpha_1.$$

However, note that if $c_1 = 0$, then the ratio F_n/F_{n-1} will equal α_2 for all n.

6.3.13. The characteristic polynomial is $1 - 7x - 5x^2 + 75x^3 = (1+3x)(1-5x)^2$. Hence the general solution is

$$R_n = A(-3)^n + (B + Cn)5^n$$

for some constants A, B, C. From the initial conditions

$$R_0 = 0 = A + B, R_1 = 0 = -3A + 5B + 5C, R_2 = 1 = 9A + 25B + 50C$$

Solving this system of linear equations, we find

$$A = \frac{1}{64}, B = \frac{-1}{64}, C = \frac{1}{40}.$$

Hence

$$R_n = \frac{1}{64} \cdot (-3)^n + \left(\frac{-1}{64} + \frac{1}{40} n \right) \cdot 5^n$$

6.5.18. The solutions of the second order homogeneous recurrence $R_n = 6R_{n-1} + 7R_{n-2}$ are determined by the characteristic polynomial, $1 - 6x - 7x^2 = (1+x)(1-7x)$, so they are of the form $R_n = A \cdot (-1)^n + B \cdot 7^n$. Because the inhomogeneous term is $g(n) = n^2 + 3n + 5$, a polynomial function of n, the table on page 187 indicates we should look for a particular solution of the form $Cn^2 + Dn + E$. So our "candidate solution" is

$$R_n = A \cdot (-1)^n + B \cdot 7^n + Cn^2 + Dn + E.$$

We have $R_0 = R_1 = 1$, so $R_2 = 28$, $R_3 = 198$, $R_4 = 1417$. Therefore

$$\begin{aligned} A + B + E &= 1 \\ -A + 7B + C + D + E &= 1 \\ A + 49B + 4C + 2D + E &= 28 \\ -A + 343B + 9C + 3D + E &= 198 \\ A + 2401B + 16C + 4D + E &= 1417. \end{aligned}$$

The solution has $A = \frac{47}{32}$, $B = \frac{511}{864}$, $C = \frac{-1}{12}$, $D = \frac{-19}{36}$, and $E = \frac{-229}{216}$.

6.5.20. The solutions of the second order homogeneous recurrence $R_n = 8R_{n-1} - 15R_{n-2}$ are determined by the characteristic polynomial, $1 - 8x + 15x^2 = (1-3x)(1-5x)$, so they are of the form $R_n = A \cdot 3^n + B \cdot 5^n$. Because the inhomogeneous term is $g(n) = 3^n$, an exponential in n *that appears in the homogeneous solution*, the table on page 187 indicates we should look for a particular solution of the form $Cn3^n$. So our "candidate solution" is

$$R_n = A \cdot 3^n + B \cdot 5^n + C \cdot n3^n.$$

We have $R_0 = R_1 = 1$, so $R_2 = -7 + 9 = 2$. Therefore

$$\begin{aligned} A + B &= 1 \\ 3A + 5B + 3C &= 1 \\ 9A + 25B + 18C &= 2 \end{aligned}$$

The solution has $A = \frac{-1}{4}$, $B = \frac{5}{4}$, $C = \frac{-3}{2}$.