MATH 357 – Combinatorics
Solutions for Problem Set 3
February 17, 2017

3.1.13. There are $\binom{5}{2}$ ways to choose the senior members and $\binom{100}{3}$ to choose the junior members. By the multiplication principle, the number is

$$\binom{5}{2} \cdot \binom{100}{3} = 10 \cdot 161,700 = 1,617,000.$$

3.1.14. There are 5 choices for the bun, 8 choices for the patty, 5 choices for the cooking method,

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = 1 + 10 + 45 + 120 = 176$$

different choices for the cheese(s), $2^8 = 256$ different combinations of vegetables, and

$$\binom{13}{0} + \binom{13}{1} + \binom{13}{2} + \binom{13}{3} + \binom{13}{4} = 1 + 13 + 78 + 286 + 715 = 1093$$

combinations of up to 4 condiments. By the Multiplication Principle, the total number of different burgers you can order is

$$5 \cdot 8 \cdot 5 \cdot 176 \cdot 256 \cdot 1093 = 9,849,241,600$$

different burgers. ("Burgero Magnifico" indeed, but one wonders how many of those different possibilities have ever been ordered!)

3.1.17. (See Proposition 3.8) In the 8 distinct numbers, there are $\binom{8}{2} = 28$ different pairs, but some of the pairs contain the same element. As in the proof given in the text, we'll say two pairs a, b and a, c overlap at a if $b - a = a - c$, or $b + c = 2a$. If we have additional pairs a, b' and a, c' overlapping at the same a and b, c, b', c' are all distinct, then $b + c = 2a = b' + c'$ and we have found what is asked for. So without loss of generality, we may assume there are at most two overlapping a, b and a, c pairs at each a. At each a in the set where overlapping pairs exist, we *remove one of the pairs*. Since we had 28 pairs to begin with and there are at most 8 such a, there are at least $28 - 8 = 20$ remaining pairs. We subtract the smaller number in the pair from the larger number in each case. Since the numbers are coming from [20], they elements of a pair differ by at most 19. This gives at most 19 possibilities for the difference. By the Pigeonhole Principle, two different pairs must have the same difference, so there are pairs $a > d$ and $c > b$ among the 8 numbers such that a, b, c, d are distinct and $a - d = c - b$. But then $a + b = c + d$.

3.1.19. (Extra Credit) As stated, the problem seems to be defective. First, we must clearly have $n \geq 15$ in order to have at least 15 distinct elements in $[n]$. If $n = 15$ then there are also clearly distinct a, b, c, d in [15] itself for which $a + b = c + d$. Take $a = 1, b = 15$, $c = 2, d = 14$ for instance. So the *smallest* n for which this is true is $n = 15$, but this seems like a silly thing to be asking about. I have the feeling that the author meant "largest" instead of "smallest." There are $\binom{15}{2} = \frac{15(15-1)}{2} = 105$ pairs. As in the solution for 3.1.17 we are done if our set contains a and b, c, b', c' distinct such that $b-a = a-c$ and $b'-a = a-c'$. So we may assume that there is at most one such overlap for each a in the set and we remove one of the pairs, leaving at least $105-15 = 90$ pairs. If we subtract the smaller from the larger, we get at least 90 differences. We want to be able to apply the Pigeonhole Principle as in the last problem, so we want $90 > n-1$, so $n < 91$. $n = 90$ is the largest n for which this argument will work. It is not hard to construct subsets of $[n]$ of size 15 for which $a + b \neq c + d$ whenever a, b, c, d are distinct and n is large enough. Do you see how?

3.2.6. The denominations of the two pairs can be any two distinct denominations out of the 13 in the deck so there are $\binom{13}{2}$ ways to pick those. Within the denominations, the pairs can be from any two of the suits, so there are $\binom{4}{2}$ ways to choose the suits of the first pair and $\binom{4}{2}$ ways to choose the suits in the second pair. Then the fifth card in the hand must be a card in one of the 11 other denominations *not* represented in the two pairs; there are $11 \cdot 44$ ways to choose that last card. So by the Multiplication Principle the total number is

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123,552.$$

3.3.6.

(i) $\binom{6}{3}y^3 = 20y^3$.

(ii) $\binom{7}{3} \cdot 3^4 = 2835$.

(iii) $\binom{4}{3} \cdot 2^3 \cdot 5 = 160$.

(iv) Think of expanding like this:

$$(2x + 3y + 5)^7 = \sum_{k=0}^{7} \binom{7}{k} (2x)^k (3y + 5)^{7-k}$$

So the coefficient of x^3 is

$$\binom{7}{3} \cdot 2^3 \cdot (3y + 5)^4 = 280(3y + 5)^4.$$

3.3.7. From the binomial theorem we know

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiate both sides m times with respect to x. On the left we get

$$n(n-1)\cdots(n-m+1)(1+x)^{n-m} = P(n,m)(1+x)^{n-m}.$$

on the right, the mth derivative of x^k is just

$$\begin{cases} k(k-1)\cdots(k-m+1)x^{k-m} = P(k,m)x^{k-m} & \text{if } k \geq m \\ 0 & \text{if } k < m. \end{cases}$$

Therefore,

$$P(n,m)(1+x)^{n-m} = \sum_{k=0}^{n} \binom{k}{n} P(k,m)x^{k-m}.$$

The desired equality then follows by setting $x = 1$:

$$P(n,m) \cdot 2^{n-m} = \sum_{k=0}^{n} \binom{k}{n} P(k,m).$$

Comment: If $m > n$, then $P(n,m) = 0$ and $P(k,m) = 0$ for all k with $0 \leq k \leq n$. So the formula is true in those cases without any additional argument.

3.4.6. Think of the process of choosing a committee of k people from a pool of n possible members, if one of the members is also singled out as a chairperson. We can count the number of ways to do this in more than one way. First, there are n choices for the committee chair, and then $\binom{n-1}{k-1}$ ways to choose the other $k-1$ members of the committee Hence by the Multiplication Principle, we would have $n \cdot \binom{n-1}{k-1}$ to make the selection by that process. Second, we could think of choosing the committee first, with $\binom{n}{k}$ possible choices, and then electing the chair from the membership, so k choices. Since these are two different ways of counting the same collection of things, we have

$$n \cdot \binom{n-1}{k-1} = \binom{n}{k} \cdot k.$$

3.4.7. Continuing from the previous problem, consider the problem of counting a committee with a chairperson chosen from a pool of n candidates. The committee can have any number of members 1 up to n. So if we add the right hand sides in the last formula from 3.4.6 we get the total number of committees with a chairperson counted one way:

$$\sum_{k=1}^{n} k \cdot \binom{n}{k}, \text{ which equals } \sum_{k=0}^{n} k \cdot \binom{n}{k}$$

since the term with $k = 0$ is 0. Now on the other hand, to choose the committee we could also pick the chairperson first, then any combination of the other 2^{n-1} people to fill out the membership. By the Multiplication Principle the number of choices is $n \cdot 2^{n-1}$. Again since we are counting the same collection of things in two different ways, we have

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k}.$$

3.5.5. There are $\binom{n}{k}$ such strings because there are that many ways to place the k ones among the n bits in the string and the other entries must then be zeroes. (Note: This *is not* a case to apply the "dividers" or "stars and bars" method. Our author is being slightly devious to test your understanding and review previously covered material!)