MATH 357 – Combinatorics
Solutions for Review Problems for Final Exam
May 1, 2017

1.3.13. Let H be the set of guests who have a hamburger and let D be the set who have a hot dog. The given information is $|H| = 25$, $|D| = 18$, and $|H \cap D| = 10$. Since it is given that everyone had either a hamburger or a hot dog, the set of guests is $H \cup D$ and by the Inclusion-Exclusion Principle,

$$|H \cup D| = |H| + |D| - |H \cap D| = 25 + 18 - 10 = 33.$$

1.4.8. If σ, τ are both bijections of X, then consider $\sigma \circ \tau$. If $(\sigma \circ \tau)(x) = (\sigma \circ \tau)(x')$, the $\sigma(\tau(x)) = \sigma(\tau(x'))$. Since σ is injective (one-to-one), this implies $\tau(x) = \tau(x')$. But then $x = x'$ since τ is also injective. This shows $\sigma \circ \tau$ is injective. Now, let $x \in X$ be arbitrary. Since σ is surjective (onto), there is some $y \in X$ such that $\sigma(y) = x$. But τ is also surjective, so there is some $z \in X$ such that $y = \tau(z)$. Substituting this into the previous equation gives $\sigma(\tau(z)) = (\sigma \circ \tau)(z) = x$. Since x was arbitrary, this shows $\sigma \circ \tau$ is surjective as well, hence bijective.

1.5.7. (This problem is very like Example 1.5.3 in the text.) Let $a_1, \ldots, a_{32}$ be the number of math problems done after day 1, day 2, ..., day 32. We have $a_{32} \leq \frac{9}{8} \cdot 32 = 36$. Now consider the "pigeons" to be the numbers

$$a_1, a_2, \ldots, a_{32}, a_1 + 27, a_2 + 27, \ldots, a_{32} + 27$$

and the "pigeonholes" to be the number values. We have $32 + 32 = 64$ numbers in the range 1 to $36 + 27 = 63$. Hence two of them must be the same, say

$$a_i = a_j + 27.$$

The a_i are strictly increasing with i, so we must have $i > j$ here, hence $a_i - a_j = 27$ and this represents the number of problems done between day $j + 1$ and day i, a consecutive string of days. This is what we wanted to show.

2.1.22. Note: I'm assuming the whole password consists of 8 characters.) (i) $26^6 \cdot 10^2$ ((ii) $25 \cdot 26^5 \cdot 10^2$. (iii) $26^6 \cdot 99$ (iv) $25 \cdot 26^5 \cdot 99$.

2.2.9. By the Inclusion-Exclusion Principle, the number is

$$26^5 + 25^6 - 25^5$$

(the last term is the number of words that begin with a and contain no b).

2.3.9. The number depends on whether n is even or odd. If n is even there are $n/2$ odds and $n/2$ evens in $[n]$. To make a permutation with evens and odds alternating, we can either begin with and even or an odd. The total number is $2 \cdot (n/2)!(n/2)!$. For instance, with

$n = 4$, there are 8 such permutations. In the "one-row" format (not cycle decomposition), they are:

$$(1234), (1432), (3214), (3412), (2143), (2341), (4123), (4321).$$

If n is odd, then there are $(n+1)/2$ odds and $(n-1)/2$ evens in $[n]$. To make one of the required permutations, we must start with an odd (i.e. 1 must map to an odd). Otherwise, we "run out" of evens before odds. There are $((n-1)/2)!((n+1)/2)!$ such permutations. For instance, with $n = 5$, there are 12 such permutations. (Write them down to make sure you understand!)

2.4.8. We use Corollary 2.4.2 and Inclusion-Exclusion. There are

$$\lfloor \frac{10000}{3} \rfloor = 476$$

numbers divisible by $21 = 3 \cdot 7$,

$$\lfloor \frac{10000}{33} \rfloor = 303$$

numbers divisible by $33 = 3 \cdot 11$, and

$$\lfloor \frac{10000}{231} \rfloor = 43$$

numbers divisible by $231 = 3 \cdot 7 \cdot 11$. Hence the number divisible by 21 or 33 is

$$476 + 303 - 43 = 736.$$

2.5.9. Let I, II, III, IV be the sets of elections where conditions (i),(ii),(iii),(iv) respectively are satisfied. Note that $|I| = P(19,3) = 19 \cdot 18 \cdot 17 = 5814$ (the other three positions are three other distinct people). We also have $|II| = P(19,3) = 19 \cdot 18 \cdot 17 = 5814$, but $|I \cap II| = P(18,2) = 18 \cdot 17 = 306$. Then $|III| = 18 \cdot 17 \cdot 2 = 612$ and $|IV| = P(18,4) = 18 \cdot 17 \cdot 16 \cdot 15 = 73440$. Now the set where at least one condition is satisfied is $I \cup II \cup III \cup IV$. Note that $(I \cup II) \cap III = \emptyset$, $(I \cup II) \cap IV = \emptyset$, $III \cap IV = \emptyset$. Hence by the Addition Principle, and Inclusion-Exclusion, we have

$$|(I \cup II) \cup III \cup IV| = |I \cup II| + |III| + |IV| = (5814 + 5814 - 306) + 612 + 73440 = 85347.$$

(There are $20 \cdot 19 \cdot 18 \cdot 17 = P(20,4) = 116280$ total different election outcomes.)

2.7.15. The number is

$$\left(\frac{P(25,8)}{8}\right)\left(\frac{P(17,8)}{8}\right)\left(\frac{P(9,6)}{6}\right)$$

(The last three numbers go into the three 1-cycles and there is only one way that can happen. You could also express the number of ways to pick them as

$$\frac{1}{3!} P(3,1) P(2,1) P(1,1) = 1.$$

3.1.15. There are $\binom{6}{3}$ ways to pick the digits that come from $\{2,3,5,9\}$. Since repeated digits are allowed, there are 4^3 ways to pick the digits themselves. The locations of the other digits from $\{1,4,6,7\}$ are then determined and there are 4^3 ways to pick them. The number of security codes is $\binom{6}{3}4^3 \cdot 4^3 = 81920$.

3.2.7. There are 13 denominations for the three cards and $\binom{4}{3} = 4$ possibilities for the suits. Then there are 12 possible denominations for the pair and $\binom{4}{2} = 6$ possibilities for the suits. This gives

$$13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

possible full house hands.

3.3.8. We construct such a bijection inductively. If $n = 1$, then the subsets of $[1]$ are $\emptyset$ (with an even cardinality $= 0$) and $[1]$ with odd cardinality. So clearly there are as many subsets of $[1]$ with even cardinality as with odd cardinality in this base case and the bijection just maps $\emptyset$ to $[1]$. Now, assume we have such a bijection for subsets of $[k-1]$ and consider subsets of $[k]$. Every such subset either contains k or it does not. If it does not then, it is a subset of $[k-1]$ and there is a bijection between the number of such subsets with even cardinality and those with odd cardinality by the induction hypothesis. If the subset $A \subseteq [k]$ does contain k, then it has the form $A = A' \cup \{k\}$ for some $A' \subseteq [k-1]$. Clearly $|A'|$ even $\Rightarrow |A|$ odd and vice versa. By the induction hypothesis again, there is a bijection between subsets of $[k]$ containing k with even cardinality and those with odd cardinality. This gives the required bijection for subsets of $[k]$ by patching the two bijections together (i.e. the one for subsets that do not contain k and the bijection for those that do) and we are done by induction.

3.5.6. This is slightly tricky but it also follows by the "dividers" method. The trick is that you now need to think of the k dominoes *as the dividers* and the $m - 2k$ unoccupied columns within the $m \times 1$ array as the *things between the dividers.* This gives a total number of placements equal to

$$\binom{(m-2k)+k}{k} = \binom{m-k}{k}.$$

3.6.20. Since the two Ns, the two Us and the two Os are indistinguishable, the number is

$$\binom{10}{2,2,2,1,1,1,1} = \frac{10!}{2!2!2!}.$$

See Example 3.6.9.

4.2.12. This is slightly ambiguous, depending on whether you interpret k-tuple as meaning an ordered list or an unordered list. If it is ordered, then this is the same as the number of ways to distribute n unlabeled balls to k labeled urns with no urn empty, or $\binom{n-1}{k-1}$ (dividers). If the list is unordered, then think unlabeled urns, so the number is $p(n,k)$ –

the number of partitions of n into exactly k positive parts. (Also note, it's the k-tuples that are "distinct" – this does not mean that the numbers in the k-tuple are distinct.)

4.2.19. The right side k^n is the number of ways of distributing k labeled balls into n labeled urns. The left side counts the same thing, but breaking it down into all the possible cases of how many balls go to each urn, and using the Addition Principle. The term $\binom{n}{\lambda_1,\ldots,\lambda_k}$ counts the number of ways to distribute the n labeled balls into the k labeled urns where urn 1 gets λ_1 balls, urn 2 gets λ_2 balls, etc. That equals the k^n since we are summing over all collections of $(\lambda_1,\ldots,\lambda_k)$ where $\lambda_i \geq 0$ for all i, and $\lambda_1 + \cdots + \lambda_k = n$.

4.3.14. The number is $p(11,5) = p(10,4) + p(6,5) = 9 + 1 = 10$. They can be represented like this (this is equivalent to counting partitions):

$$7+1+1+1+1, 6+2+1+1+1, 5+3+1+1+1, 5+2+2+1+1, 4+4+1+1+1,$$
$$4+3+2+1+1, 4+2+2+2+1, 3+3+3+1+1, 3+3+2+2+1, 3+2+2+2+2$$

4.3.15. (See Example 4.3.11 which is somewhat similar). Distribute the unlabeled red balls first. Suppose that out of the n_1, j_1 go into the unlabeled red urns and the other $n_1 - j_1$ of them go into the labeled white urns. There are

$$p(j_1,1) + \cdots + p(j_1,k_1) = \sum_{\ell=1}^{k_1} p(j_1,\ell)$$

ways to do the distribution to the red urns and then $\binom{n_1-j_1+k_2-1}{k_2-1}$ ways to distribute the others to the white urns. We then multiply and sum over j_1 to get the total number of ways to distribute the red balls:

$$(1) \quad \sum_{j_1=1}^{n_1} \left(\sum_{\ell=1}^{k_1} p(j_1,\ell) \right) \cdot \binom{n_1 - j_1 + k_2 - 1}{k_2 - 1}.$$

Now we do the labeled white balls. Suppose j_2 of them go into the unlabeled red urns and the other $n_2 - j_2$ go into the labeled white urns. There are

$$S(j_2,1) + \cdots + S(j_2,k_2) = \sum_{m=1}^{k_2} S(j_2,m)$$

ways to do the distribution to the red urns, and $k_2^{n_2-j_2}$ ways to the do the distribution to the white urns. Then the total number of ways to distribute the white balls is

$$(2) \quad \sum_{j_2=1}^{n_2} \left(\sum_{m=1}^{k_2} S(j_2,m) \right) \left(k_2^{n_2-j_2} \right).$$

Finally, by the Multiplication Principle, the total number of ways to distribute is the product of the sum in (1) times the sum in (2) – too complicated to write out on one line(!)

5.3.9 (just say how you would solve this using a generating function). This would be the coefficient of x^{100} in the Taylor expansion of

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{20}} \cdot \frac{1}{1-x^{50}} \cdot \frac{1}{1-x^{100}}.$$

5.4.9 (same directions). This would be the sum of the coefficients of all the terms $u^i v^j$ with $3 \le i \le 11$ and $5 \le j \le 13$ in the expansion of

$$\frac{1}{1-uv^2} \cdot \frac{1}{1-u^4v^3} \cdot \frac{1}{1-u^5v^5} \cdot \frac{1}{1-u^6v^7}.$$

6.1.20 (figure 6.2 is at the top of the page). Note that there are $R_0 = 1$ squares in s_0, $R_1 = 6$ squares in s_1, and $R_2 = 26$ squares in s_2. In going from s_{n-1} to s_n we place a copy of s_{n-1} into each quadrant of the larger graph, and then we also add the large outside square and the small central square. Therefore the total number of squares in s_n is 4 times the number of squares in s_{n-1}, plus 2:

$$R_n = 4R_{n-1} + 2$$

for all $n \ge 1$.

6.3.10 (solve with a generating function and by the "shortcut" method). The generating function $R(x)$ satisfies

$$(1 - x - 6x^2)R(x) = R_0 + (R_1 - R_0)x = x$$

so

$$R(x) = \frac{x}{1 - x - 6x^2}$$

Now, using partial fractions, we have

$$= \frac{x}{(1-3x)(1+2x)} = \frac{1/5}{1-3x} + \frac{-1/5}{1+2x}.$$

So expanding in geometric series, we see that

$$R_n = \frac{1}{5} \cdot (3^n - (-2)^n).$$

The shortcut method gets us to the same place somewhat more quickly. The characteristic polynomial is $1 - x - 6x^2 = (1-3x)(1+2x)$ so the general solution is

$$R_n = A \cdot 3^n + B \cdot (-2)^n$$

for some A, B. From $n = 0$ we get $0 = A + B$ and from $n = 1$, we get $1 = 3A - 2B$. Therefore $A = \frac{1}{5}$ and $B = \frac{-1}{5}$.

6.5.27. Because the characteristic polynomial is $1 - 8x + 16x^2 = (1 - 4x)^2$, we need to take a particular solution of the form $Cn^2 4^n$ using Table 6.1. The solution has the form $(A + Bn)4^n + Cn^2 4^n$ and we can solve for A, B, C from the given initial values $R_0 = 1$, $R_1 = 1$, $R_2 = 8 - 16 + 4^2 = 8$. Answer:

$$R_n = 4^n \left(1 - \frac{5n}{4} + \frac{n^2}{2}\right).$$

7.1.16 (just say how you would solve it using generating functions). We need to use generating functions and Inclusion-Exclusion for this since we are counting the number of elements of a union of four sets. Let A, B, C, D be the sets of solutions of the equation with each of the constraints separately. Then

$$\begin{aligned}
|A| &= \text{coeff. of } x^{35} \text{ in } \frac{1 + x + \cdots + x^{12}}{(1-x)^3} \\
|B| &= \text{coeff. of } x^{35} \text{ in } \frac{1 + x + \cdots + x^{10}}{(1-x)^3} \\
|C| &= \text{coeff. of } x^{35} \text{ in } \frac{1 + x + \cdots + x^{7}}{(1-x)^3} \\
|D| &= \text{coeff. of } x^{35} \text{ in } \frac{1 + x + \cdots + x^{4}}{(1-x)^3}
\end{aligned}$$

$$\begin{aligned}
|A \cap B| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{12})(1 + \cdots + x^{10})}{(1-x)^2} \\
|A \cap C| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{12})(1 + \cdots + x^{7})}{(1-x)^2} \\
|A \cap D| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{12})(1 + \cdots + x^{4})}{(1-x)^2} \\
|B \cap C| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{10})(1 + \cdots + x^{7})}{(1-x)^2} \\
|B \cap D| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{10})(1 + \cdots + x^{4})}{(1-x)^2} \\
|C \cap D| &= \text{coeff. of } x^{35} \text{ in } \frac{(1 + x + \cdots + x^{7})(1 + \cdots + x^{4})}{(1-x)^2}
\end{aligned}$$

$$|A \cap B \cap C| = \text{coeff. of } x^{35} \text{ in } \frac{(1+\cdots+x^{12})(1+\cdots+x^{10})(1+\cdots+x^{7})}{(1-x)}$$
$$|A \cap B \cap D| = \text{coeff. of } x^{35} \text{ in } \frac{(1+\cdots+x^{12})(1+\cdots+x^{10})(1+\cdots+x^{4})}{(1-x)}$$
$$|A \cap C \cap D| = \text{coeff. of } x^{35} \text{ in } \frac{(1+\cdots+x^{12})(1+\cdots+x^{7})(1+\cdots+x^{4})}{(1-x)}$$
$$|B \cap C \cap D| = \text{coeff. of } x^{35} \text{ in } \frac{(1+\cdots+x^{10})(1+\cdots+x^{7})(1+\cdots+x^{7})}{(1-x)}$$
$$|A \cap B \cap C \cap D| = \text{coeff. of } x^{35} \text{ in } (1+\cdots+x^{12})(1+\cdots+x^{10})(1+\cdots+x^{7})(1+\cdots+x^{4})$$

Then

$$|A \cup B \cup C \cup D| = \sum_{I \subseteq [4], I \neq \emptyset} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.$$

7.3.4. Recall that we know $D_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}$. Hence

$$\lim_{n\to\infty} \frac{D_n}{n!} = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{(-1)^m}{m!} = e^{-1}.$$

8.3.20. The 4×4 grid with squares labeled by $[16]$ is our X. The group acting is $C_4 = \{e, \rho, \rho^2, \rho^3\}$ acting by rotations about the center point of the grid. Let's label the squares $1, 2, 3, 4$ left to right across the top row, then $5, 6, 7, 8$ left to right across the second row, etc. We see

$$\begin{aligned} \operatorname{Inv}(e) &= [16] \\ \operatorname{Inv}(\rho) &= \emptyset \\ \operatorname{Inv}(\rho^2) &= \emptyset \\ \operatorname{Inv}(\rho^3) &= \emptyset \end{aligned}$$

By Burnside's Lemma, the number of orbits is equal to

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Inv}(g)| = \frac{1}{4}(16 + 0 + 0 + 0) = 4.$$

(The orbits are $\{1, 4, 13, 16\}$ (the corners), $\{2, 9, 15, 8\}$ and $\{3, 5, 14, 12\}$ (interior points of outside rows and columns), and $\{6, 7, 10, 11\}$ the 4 points in the "inside ring.")

8.5.12 (just say how you would solve it using the Polya Theorem). We begin by computing the cycle index polynomial for this action of C_4 on X. We have

$$\begin{aligned} \operatorname{cim}(e) &= x_1^{16} \\ \operatorname{cim}(\rho) &= x_4^{4} \\ \operatorname{cim}(\rho^2) &= x_2^{8} \\ \operatorname{cim}(\rho^3) &= x_4^{2} \end{aligned}$$

Hence

$$Z_{C_4}(x_1, x_2, x_4) = \frac{1}{4}(x_1^{16} + 2x_4^2 + x_2^8).$$

Then by the Polya Theorem, the number of distinct two-colorings of the 4×4 grid using the first color 10 times and the second 6 times is equal to the coefficient of $c_1^{10}c_2^6$ in the expansion of

$$Z_{C_4}(c_1 + c_2, c_1^2 + c_2^2, c_1^4 + c2^4).$$

(In case you are interested, this number is 2016.)