MATH 357 – Combinatorics
Discussion 3 – Polya's Theorem
April 24 and 28, 2017

Background

Last Friday, we stated the Polya Counting Theorem and did a first example using it. Here's the statement again:

Polya's Theorem. Let a finite group G act on a set S and hence on the set X of m-colorings of S. The number of distinguishable colorings using color i the number ℓ_i times (where $\sum_{i=1}^{m} \ell_i = |S|$) is given by the coefficient of $c_1^{\ell_1} \cdots c_m^{\ell_m}$ in

$$Z_G(c_1 + \cdots + c_m, c_1^2 + \cdots + c_m^2, \ldots, c_1^{|S|} + \cdots c_m^{|S|}).$$

Here $Z_G(x_1, \ldots, x_{|S|})$ is the *cycle index polynomial* of G acting on S:

$$Z_G(x_1, \ldots, x_{|S|}) = \frac{1}{|G|} \sum_{g \in G} \operatorname{cim}(g)$$

and $\operatorname{cim}(g)$ is the *cycle index monomial*

$$\operatorname{cim}(g) = \prod_{i=1} x_i^{k_i(g)}.$$

The variables are $x_1, x_2, \ldots, x_{|S|}$ and $k_i(g)$ = the number of cycles of length i in the permutation representation of g acting on S (using some labeling of the elements of $|S|$ by the integers in $[|S|]$). Usually there will not be cycles of every possible length $1, 2, 3, \ldots, |S|$. If so, we usually just ignore those variables.

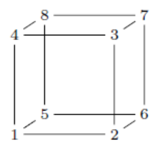
Recall the idea – colorings are not distinguishable if there is some group element that takes one to the other. They are distinguishable if and only if they are in different orbits under the action of G on the set X of colorings.

To finish off the semester before the final project presentations, we will work through an interesting and complicated example using this! All of the calculations will be do-able by hand (although somewhat tedious). You may also use Maple to compute the relevant cycle index polynomial and apply the Polya Theorem if you prefer.

Questions

(A) The group acting on the nodes (= vertices) of a cube is a group of order 24. As permutations of the vertices, the elements are listed in the table on the back of this sheet. For each element, find $\operatorname{Inv}(g)$, $\operatorname{cyc}(g)$ and $\operatorname{cim}(g)$.

The group of symmetries of the nodes of a cube



is

$G = \{(1)(2)(3)(4)(5)(6)(7)(8),$ no rotation
$(3)(5)(1,8,6)(2,4,7), (3)(5)(1,6,8)(2,7,4),$ rotations around axis 3-5
$(2)(8)(1,3,6)(4,7,5), (2)(8)(1,6,3)(4,5,7),$ rotations around axis 2-8
$(1)(7)(4,5,2)(3,8,6), (1)(7)(4,2,5)(3,6,8),$ rotations around axis 1-7
$(4)(6)(1,3,8)(2,7,5), (4)(6)(1,8,3)(2,5,7),$ rotations around axis 4-6
$(1,2,3,4)(5,6,7,8), (1,3)(2,4)(5,7)(6,8), (4,3,2,1)(8,7,6,5),$
$(1,4,8,5)(2,3,7,6), (1,8)(4,5)(2,7)(3,6), (5,8,4,1)(6,7,3,2),$
$(3,4,8,7)(2,1,5,6), (3,8)(4,7)(2,5)(1,6), (7,8,4,3)(6,5,1,2),$
$(3,4)(5,6)(1,7)(2,8)$ flip 1-8,7-2
$(4,8)(2,6)(1,7)(3,5)$ flip 1-3,7-5
$(8,7)(1,2)(3,5)(4,6)$ flip 3-6,5-4
$(1,5)(3,7)(2,8)(4,6)$ flip 3-7,1-5
$(1,4)(6,7)(2,8)(3,5)$ flip 5-2,3-8
$(2,3)(5,8)(4,6)(1,7)$ flip 1-6,7-4
$\}$

Figure 1: The symmetries of the nodes (= vertices) of a cube

(B) How many distinct colorings of the vertices are there if we have $m = 4$ colorings to use?

(C) How many distinguishable colorings are there if each of the 4 colors is used exactly twice?

(D) How many if three of the colors are used once each and the other is used five times?

(E) **Extra Credit** The group you are working with here is isomorphic to S_4. However, it shouldn't be immediately obvious what four things are being permuted to produce the isomorphism. Find such a set of four things (related to the geometry of the cube), such that the group G is acting as the set of all permutations of those four objects.