Mathematics 357 – Combinatorics
Corrections for Problem 32 in Chapter 6 of Brualdi, *version 2*
March 14, 2005

Unfortunately, there were several things wrong with problem 32 in Chapter 6 of the text (on Problem Set 5 for the course), *and also with the first corrected version* (!) (There must be something about this problem!) Here is a corrected version of the problem, incorporating some hints that should help you get started toward a solution.

A) *This is different from the original "corrected version"(!)* Show that if B_m is the board with m *unshaded* squares in locations

$$(1,1),(1,2),(2,2),(2,3),\ldots,$$

in a diagonal "staircase pattern" containing exactly m squares, then the rook polynomial of B_m is given by:

$$R(B_m,t)=1+\binom{m}{1}t+\binom{m-1}{2}t^2+\binom{m-2}{3}t^3+\cdots+\binom{m-k+1}{k}t^k+\cdots$$

(Note: B_{2n-1} is the "upper part" of the shaded squares in the board in the book problem – there are $m=n+n-1=2n-1$ X's. The cases B_m with $m=2n-1$ odd fit on $n\times n$ boards; the cases with $m=2n$ even don't quite fit on square boards if the top of the staircase occurs in row 1, column 1.)

B) Use part A and our theorems on rook polynomials to show that the rook polynomial of the complement $\overline{B}$ of the $n\times n$ board B given in the book problem is

$$R(\overline{B},t)=\sum_{k=0}^{n}\frac{2n}{2n-k}\binom{2n-k}{k}t^k$$

C) Use part B to find the number of ways of placing n non-attacking rooks on the *unshaded* squares in the book's $n\times n$ board B.

Comment: This problem is a version of the "hostess problem": In how many ways can n married couples be seated at a round table so that no husband sits next to his own wife. Do you see the connection? (Think of a round table, but ignore the circular symmetry!)