Mathematics 357 – Combinatorics
Problem Set 8
*Due: April 22, 2005*

I. A *magic square* is an $n \times n$ integer matrix $M = (m_{ij})$ with the property that the sum of the entries in each row and in each column is the same. For instance, a famous $4 \times 4$ magic square appears in the engraving *Melencolia I* by Albrecht Dürer:

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

(see link on course homepage). The row and column sums in this array all equal 34. Although the extra condition that the $m_{ij}$ are the *distinct* integers $1, 2, \ldots, n^2$ (as in Dürer's magic square) is often included, we will *not* make that part of the definition. Also, many familiar examples of magic squares have equal diagonal sums and other interesting properties; we will not require that either. An interesting problem is this: *Given positive integers $s, n$, how many different $n \times n$ magic squares with row and column sum equal to $s$ and $m_{ij} \geq 0$ for all $i, j$ are there?*

A) In class, using Hall's Theorem, we showed that any $(0, 1)$-matrix with equal row and column sums $d$ is a sum of $d$ distinct permutation matrices. Adapt the proof to show that any integer $n \times n$ magic square is a linear combination of $n \times n$ permutation matrices with nonnegative integer coefficients.

From now on, we will restrict to the case $n = 3$ for simplicity. By part A, we know that any $3 \times 3$ non-negative integer magic square can be written as

$$M = aI + bS + cS^2 + dT_{12} + eT_{13} + fT_{23}$$

where $a, b, c, d, e, f$ are non-negative integers, $s = a + b + c + d + e + f$, and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

are the $3 \times 3$ permutation matrices.

B) Exhibit an explicit linear dependence relation between the 6 $3 \times 3$ permutation matrices and show that they only span a 5-dimensional subspace of the vector space of $3 \times 3$ matrices.

C) Taking the linear dependence from B into account, how many distinct integer $3 \times 3$ magic squares are there with $s = 2$, $s = 3$, $s = 4$?

D) Find and prove a general formula for the number of distinct non-negative integer $3 \times 3$ magic squares with row and column sum $s$.

*From the Text:* Chapter 11/22, 23, 24, 28, 29, 57.