Mathematics 357 – Combinatorics
Discussion 3 – Inhomogeneous Linear Recurrences With Constant Coefficients
March 18, 2005

Background

Many of the most common recurrences that we need to solve in combinatorics are *linear recurrences with constant coefficients*, of the form:

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b(n) \tag{1}$$

We looked at the case $b(n) = 0$ (the homogeneous recurrences) last time. Today, we will investigate recurrences with $b(n) \neq 0$ in some detail.

Discussion Questions

A) The basis for the treatment of the inhomogeneous case is a fact that is parallel to patterns you have seen in Linear Algebra (and possibly Differential Equations too!). Show that if h_n, p_n are any two solutions of the same inhomogeneous recurrence (1):

$$\begin{aligned} h_n &= a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b(n) \\ p_n &= a_1 p_{n-1} + a_2 p_{n-2} + \cdots + a_k p_{n-k} + b(n) \end{aligned}$$

then the difference sequence $g_n = h_n - p_n$ is a solution of the homogeneous recurrence

$$g_n = a_1 g_{n-1} + a_2 g_{n-2} + \cdots + a_k g_{n-k} \tag{2}$$

associated to (1). (Note: the coefficients on the various terms from the sequence are the same, but the $b(n)$ term has been replaced by zero.)

B) Suppose we can find the *general solution* g_n of (2) by the methods we discussed in class last time, and that we know some "particular solution" of (1) like p_n above. What does that say about solving (1)? Explain.

To determine "particular solutions", we will use the following "rules of thumb" based on the form of the function $b(n)$:

- If $b(n) = b_\ell n^\ell + \cdots + b_1 n + b_0$, then try a particular solution of the same form

$$p_n = P_\ell n^\ell + \cdots + P_1 n + P_0$$

  but with *undetermined coefficients* P_k. (Be sure you understand the notation here: p_n is the sequence giving the particular solution; P_k with the capital letters are the undetermined coefficients that we want to solve for.)
- If $b(n) = q^n$, then try a particular solution $p_n = Pq^n$ (same q, of course!)

Then in both cases, substitute into (1) and try to determine the constants to make j_n a solution of (1) for all n.

C) For instance, find a particular solution of

$$h_n = 6h_{n-1} - 9h_{n-2} + 2n. \tag{3}$$

Then use that and the solution of the associated homogeneous recurrence to solve (3) with the initial conditions $h_0 = 1, h_1 = 0$.

D) The rules of thumb above do not always "work" (in particular they can fail if the proposed particular solution actually solves the homogeneous equation. Show that this happens, for instance for the recurrence

$$h_n = 5h_{n-1} - 6h_{n-2} + 2^n$$

Find a particular solution in this case of the form $p_n = P \cdot n2^n$ for some constant P. Then use that and the solution of the associated homogeneous recurrence to solve (3) with the initial conditions $h_0 = 1, h_1 = 0$.

Assignment

Group writeups due in class on Wednesday, March 23.