Greek Mathematics Recovered in Books 6 and 7 of Clavius' *Geometria Practica*

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- I've always been interested in the history of mathematics (in addition to my nominal specialty in algebraic geometry/computational methods/coding theory, etc.)
- Want to be able to engage with original texts on their own terms – you might recall the talks on Apollonius's *Conics* I gave at the last Clavius Group meeting at Holy Cross (two years ago)
- So, I've been taking Greek and Latin language courses in HC's Classics department
- The subject for today relates to a Latin-to-English translation project I have recently begun – working with the *Geometria Practica* of Christopher Clavius, S.J. (1538 - 1612, CE)

Overview

Clavius' Principal Mathematical Textbooks

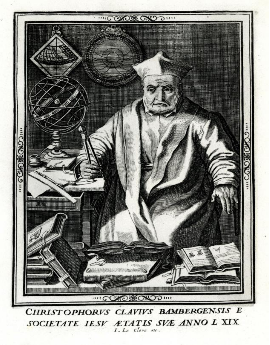

Figure: source: ricci.bc.edu/people/christopher-clavius.html

- *Euclidis Elementorum, Libri XV* (first ed. 1574)
- *Epitome artithmeticae practicae* (first ed. 1583)
- *Geometria practica* (first ed. 1604)
- *Algebra* (first ed. 1608)
- Also: Commentary on Theodosius, The Sine, Plane Triangles, Spherical Triangles

"Practical Geometry"

- Part of a long tradition from the late Medieval period (e.g. Leonardo Pisano – "Fibonacci") into the early 1800's;
- Showing influence of both Islamic and Greek traditions;
- Intended to be *useful* for questions in surveying, architecture, engineering, military science, etc.
- Quite stable tables of contents:
- Construction and use of actual geometrical instruments – astrolabe, quadrants, proportional compasses, etc.
- "Practical" methods for geometric constructions supplementing what Euclid gives
- *Numerical measures of lengths and angles used*; numerical examples given instead of, or in addition to, Euclidean-style proofs

"Practical Geometry," cont.

- Overtly "practical" problems: measuring heights of towers or mountains, depths of valleys, etc. via trigonometry
- Mensuration – area and volume formulas, often expressed in Archimedean terms: e.g. *The volume of a sphere equals the volume of the rectangular solid contained by the radius and a base with area the third part of the surface area of the sphere.*
- "Geodesy" – problems about subdividing plane areas or solid volumes into parts in given ratios

Contents of Clavius' *Geometria Practica*

- Book 1: Geometrical instruments, the quadrant, triangles (refers to his *Triangula Rectilinea* for proofs)
- Books 2 and 3: Measuring linear dimensions by the quadrant and pendulum (trigonometry)
- Book 4: Areas of plane figures (with Archimedes' approximation of π from *On the dimension of the circle* and a more accurate approx. to 19 decimal places(!))
- Book 5: Volumes and surface areas of solids (including Archimedes' results on the sphere and the cylinder)

Contents, cont.

- Book 6: "*In quo de Geodaesia, & de figuris augendis, minuendisque in data proportione: Item de duarum mediarum proportionalium inter duas rectas inuentione : Ac denique de radicum extractione agitur*"
- Section on "geodesy" – Clavius cites J. Dee and F. Commandino's translation of a book by "Machometus Bagdadinus" from Arabic, possibly partly derived from a Greek source – Euclid's lost *On Divisions of Figures*, and follow-up work by Simon Stevin
- Book 7: "*De figuris Isoperimetris disputans: cui Appendicis loco annectitur breuis de circulo per lineas quadrando tractatiuncula*"
- Book 8: Advanced constructions

Some initial observations

- Setting his work apart from other "Practical Geometries," Clavius had (what seems to me to rise to the level of) a *secondary goal* – to make known a number of potentially applicable results from Greek geometry *that did not "make it into Euclid"* (e.g. because they were developed later).
- The construction of two mean proportionals is one of these – context is the *duplication of the cube*
- One of the three problems that spurred development of Greek geometry; still at forefront in Clavius' lifetime due to Federigo Commandino (1509 - 1579) and others
- Latin translations from Greek, esp. the *Synagoge* of Pappus of Alexandria (ca. 290 - ca. 350 CE).

Relation of mean proportionals to duplication of the cube

- Given AB and GH, CD and EF are *two mean proportionals in continued proportion* if

$$AB : CD = CD : EF = EF : GH.$$

- Hippocrates of Chios (ca. 470–ca. 410 BCE): if $GH = 2AB$, then $CD^3 = 2AB^3$.
- In other words, if AB is the side of the original cube, then CD is the side of the cube with twice the volume.
- Geometric construction of the two mean proportionals still open, but all later work started from this reduction.

Clavius' Use of Eutocius' catalog

- Today, the main thing most mathematicians know about this problem is that it cannot be solved with straightedge and compass–the Greeks also seem to have guessed this; Descartes tried to prove it; first complete proof in 1800's using analytic geometry and algebra
- But the Greeks did not stop there and detailed accounts of other sorts of constructions have survived
- In Pappus and in a commentary on Archimedes' *On the sphere and cylinder* by Eutocius of Ascalon (ca. 480 – ca. 540 CE. Note: $\sim$ 750 years after Archimedes!)
- Clavius presented several solutions in this book because they apply to scaling solid objects up or down in a given proportion(!)

What Clavius says

Quo circa prius ... afferemus, quae antiqui Geometrae nobis hac de rescripta relinquerunt. Multorum enim ingenia res haec exercuit, atque torsit, quamuis nemo ad hanc vsque diem, verè, ac Geometricè duas medias proportionales inter duas rectas datas inuenerit. Praetermissis autem modis Eratosthenis; Platonis; Pappi Alexandrini; Spori; Menechmi tum beneficio Hyperbolae, ac parabolae, tum ope duarum parabolarum; & Architae Tarentini, quamuis acutissimis, subtilissimisque; solum quatuor ab Herone, Apollonio Pergaeo, Philone Bysantio, Philoppono, Diocle, & Nicomede traditos explicabimus, quos commodiores, facilioresque, & errori minus obnoxios iudicauimus. [my emphasis]

My translation

We will first report what the ancient geometers have left to us in their writings concerning this. For this problem drove and tormented the talents of many, although up to this day, no one will have truly and geometrically constructed two mean proportionals between two given lines. Although they are most acute and subtle, the solutions of Eratosthenes, Plato, Pappus of Alexandria, Sporus, Menaechmus by means of the hyperbola and parabola, then with the help of two parabolas, and Archytas of Tarentum will be omitted; we will explain only the four solutions from Heron, Apollonius of Perga, Philo of Byzantium, Philopponus, Diocles, and Nicomedes. We have judged these to be more convenient, easier, and less susceptible to error.

Hero's method – à la Clavius

Link

The construction – refer to Clavius' figure

- The given segments are AB, BC; with a right angle at B, the rectangle $ABCD$ is constructed, with diagonals intersecting at E. Perpendiculars are dropped from E to CD at H and AD at K.
- With the lines DA and DC extended to meet a line through B in F and G,
- The line is "moved," i.e. rotated, about B until a position is reached where $EG = EF$.

Claim and proof – in modern semi-algebraic language

- Claim: AF and CG are the mean proportionals, i.e.

$$\frac{AB}{AF} = \frac{AF}{CG} = \frac{CG}{BC}.$$

- Proposition II.6 of Euclid implies

$$AF \cdot DF + AK^2 = KF^2$$

- If EK^2 is added to both sides and we use the Pythagorean theorem (Proposition I.47),

$$AF \cdot DF + AE^2 = EF^2$$

- Similarly on the vertical side,

$$CG \cdot DG + CE^2 = EG^2$$

Proof, concluded

- $EF = EG$ by construction, and $AE = CE$ since the diagonals bisect each other.
- Hence $AF \cdot DF = CG \cdot DG$, or
$$\frac{AF}{CG} = \frac{DG}{DF}$$
- But
$$\frac{DG}{DF} = \frac{AB}{AF} = \frac{GC}{BC}$$
by similar triangles.
- This shows
$$\frac{AB}{AF} = \frac{AF}{CG} = \frac{CG}{BC},$$
which is what we wanted.

Observations

- Technically a *neusis* to find the line through B with $EF = EG$ – involves taking limits in modern language, and that's why Clavius qualifies this as "*inuenire prope verum*"
- Earlier in Book 1, Clavius refered to another work for proofs and just gave a procedure; here, it's clear he thinks the proof is important because it justifies the construction. So, he gives a complete argument,
- buttressed with marginal references to (his version of) Euclid to justify almost all of the steps.
- Clavius is often described as more of teacher than a researcher, and this discussion actually shows a typical example of how he reworked texts for greater clarity.

Observations, cont.

- In Eutocius and Pappus, the methods of Heron and Apollonius are presented separately with the connection claimed – Clavius makes that clear by putting the diagrams together.
- Similar to the approach in his version of Euclid – he's always rethinking, making connections, ...
- After this discussion and propositions about rescaling geometric figures, Clavius concludes Book 6 of the *Geometria practica* with an extensive discussion of numerical algorithms for extracting approximate square and cube roots(!)

"Squaring the circle"

- Recall the following statement: "The volume of a sphere equals the volume of the rectangular solid contained by the radius and a base equal to the third part of the surface area of the sphere. "
- Related: "The area of a circle is equal to the area of the rectangle contained by the radius and the semiperimeter."
- Question: How could you construct such a rectangular solid or rectangle?
- Question: Suppose you wanted the base of the rectangular solid to be a *square* with area equal to the area of the given circle?

Book 7 – Main topic

- Begins with a discussion of *isoperimetric problems*, following the approach of Zenodorus (ca. 200 - ca. 140 BCE), as preserved by Pappus and by Theon of Alexandria, 335 - 405 CE, father of Hypatia
- Key fact: both circles and regular polygons satisfy the fact mentioned before:

  $$\text{Area} = \text{apothem} \cdot \text{semiperimeter}$$

  (an apothem is a line dropped perpendicular from the center to one of the sides $\leftrightarrow$ a radius of a circle)
- Hence, if perimeter is fixed, more sides $\Rightarrow$ greater area, and circle with that perimeter has greater area than all the polygons.

The connection

- Note that this gives a way to describe a rectangle with the same area as a given regular polygon or circle:
- Take one side as an apothem, the other as the semiperimeter(!)
- But how do you get a straight line equal to the semicircumference of a circle?
- And if we want a square with this area, we need a *mean proportional* between the apothem and the semiperimeter, and this reduces to finding $\sqrt{\pi}$.
- Hence this cannot be done by straightedge and compass either. But the Greeks were happy to ask: What other tools, auxiliary curves, etc. would yield a construction?

Clavius' discussion

- As in the discussion in Book 6, he starts off with a summary of previous work
- Dismisses some approximate modern quadratures by "Arabs," Durer, as only approximate and Campano *scathingly* as "not worthy of geometry and plainly ridiculous"
- Turns to Greek work, especially an attempt by Hippocrates (same figure we have seen before in this talk in connection with the duplication of the cube)

A partial success(?)

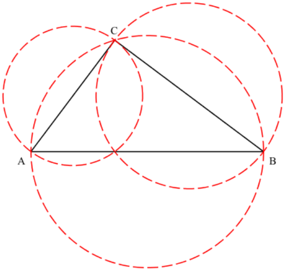

Figure: Lunes of Hippocrates

- ΔABC has right angle at C; with circumscribed circle and circles on the legs.
- The *lunes* are the two regions between the circles.
- Claim is: *The area of the lunes together equals the area of the triangle*

A partial success(?)

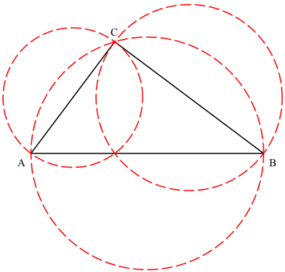

Figure: Lunes of Hippocrates

- Let a, b, c be the sides in the triangle, labeled as usual.
- We have from semicircles

$$\text{area}(ABC) + \frac{\pi a^2}{8} + \frac{\pi b^2}{8} = \text{area(lunes)} + \frac{\pi c^2}{8}$$

- But also $a^2 + b^2 = c^2$, so the claim follows.

The quadratrix

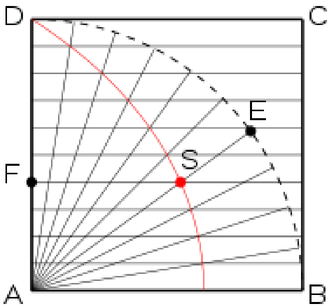

Figure: source: en.wikipedia.org

- Attributed to Dinostratus (died ca. 320 BCE), and a Hippias (probably *not* the sophist by same name)
- Mechanical description: *DC* drops at a constant rate, while *AD* rotates
- The red curve is the quadratrix – locus of intersections *S*
- Used to trisect angles *and* to square circles(!)

Squaring a circle with the quadratrix

- Assuming the point S tends to some S' along the line AB (this was controversial even in Clavius' period – it really requires understanding of limits):
- Clavius gives proof to show $AS' : AB = AB : \text{arc } DEB$
- Since there is a well-known construction for a line segment in this proportion to AS', AB too, we can find a straight line GH equal to the arc of the circle in this quadrant
- Area of quadrant is equal to one half the rectangle contained by AB and GH (by the fact we noted above in connection with the isoperimetric problem)
- To find a square with same area find *mean proportional* between AB and GH.

Clavius' point of view here

- Taken from an earlier pamphlet on the quadratrix, from 1589, included also in some editions of his Euclid
- Clavius *claimed at first*: the quadratrix can be constructed by Euclidean geometry, so it gives an "orthodox" quadrature of a circle
- His idea: you could construct arbitrarily many points by bisection of the angle and the length AD, *ad infinitum*.
- In modern language – gives a dense set of points on the curve, not the whole curve; *not an exact construction of the line* AS' – just a way to get approximations
- By 1604, he had modified his claim to recognize this.

Overall conclusions

- Clavius' textbooks were developed to support his teaching activities at the *Collegio Romano*, and then used all over the world in Jesuit schools, and other institutions
- *Prolegomena* to his Euclid – Clavius justified the study of mathematics as *both noble* – an abstract way of discovering universal truths (hence preparation for philosophy and theology) – *and useful* as a tool for answering concrete questions about the physical world, especially for future leaders in society who were (still are) a "target audience" for the Jesuit schools
- *Geometria practica* a perfect illustration of this view, and of
- his abiding interest in the mathematics of "the ancients"

References

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