College of the Holy Cross, Fall Semester, 2015
MATH 133, section 1, Solutions for Midterm 3
Thursday, November 12

1. A) (5) State the limit definition of the derivative $f'(x)$ as an equation starting with $f'(x) =$.

   Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

   if the limit exists. (If so we say f is differentiable at x; if not we say f is not differentiable at x.)

   B) (10) Use the limit definition to compute $f'(x)$ for $f(x) = x^2 - 5x + 2$.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) + 2 - x^2 + 5x - 2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h + 2 - x^2 + 5x - 2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h} \\ &= \lim_{h \to 0} \frac{(2x - 5 + h)h}{h} \\ &= \lim_{h \to 0} (2x - 5 + h) = 2x - 5. \end{aligned}$$

   (This agrees with the shortcut rules, of course!)

   C) (5) Find the equation of the line tangent to the graph $y = x^2 - 5x + 2$ at $x = 2$.

   Solution: The y-coordinate is $f(2) = -4$ and the slope is $f'(2) = -1$ so the line is $y + 4 = (-1)(x - 2)$, or $y = -x - 2$.

2. Compute the indicated derivatives of the following functions. You may use any correct method. You do not need to simplify your answers unless specifically directed to do so. But you must show work for full credit.

   (A) (10) $f(x) = 6x^{4/3} + \dfrac{3}{x^5} - 4e^x$. Compute both $f'(x)$ and $f''(x)$.

   Solution: First, write the $\frac{3}{x^5}$ as $3x^{-5}$. Then

$$f'(x) = 8x^{1/3} - 15x^{-6} - 4e^x$$

   and

$$f''(x) = (f')'(x) = \frac{8}{3}x^{-2/3} + 90x^{-7} - 4e^x.$$

(B) (10) $g(x) = (x^4 + \cos(x))(x^2 - \sin(x))$. Compute $g'(x)$.

Solution: Using the product rule and the derivative rules for $\sin(x)$ and $\cos(x)$:

$$g'(x) = (x^4 + \cos(x))(2x - \cos(x)) + (x^2 - \sin(x))(4x^3 - \sin(x)).$$

The problem did not say to simplify, so we can leave it here(!)

(C) (10) $h(x) = \dfrac{x^3 - x}{x^2 + 1}$. Compute $h'(x)$ and simplify your answer.

Solution: By the quotient rule:

$$\begin{aligned} h'(x) &= \frac{(x^2+1)(3x^2-1) - (x^3 - x)(2x)}{(x^2+1)^2} \\ &= \frac{3x^4 + 2x^2 - 1 - 2x^4 + 2x^2}{(x^2+1)^2} \\ &= \frac{x^4 + 4x^2 - 1}{(x^2+1)^2}. \end{aligned}$$

(D) (10) $j(x) = \dfrac{1}{(x^5 + 3x + 1)^3}$. Compute $j'(x)$.

Solution: This is $(x^5 + 3x + 1)^{-3}$, so by the chain rule,

$$j'(x) = -3(x^5 + 3x + 1)^{-4}(5x^4 + 3) = \frac{-(15x^4 + 9)}{(x^5 + 3x + 1)^4}.$$

(E) (10) $k(x) = (x^2 + 3)e^{x^3 - 5\sin(2x)}$. Compute $k'(x)$.

Solution: By the product rule and the chain rule,

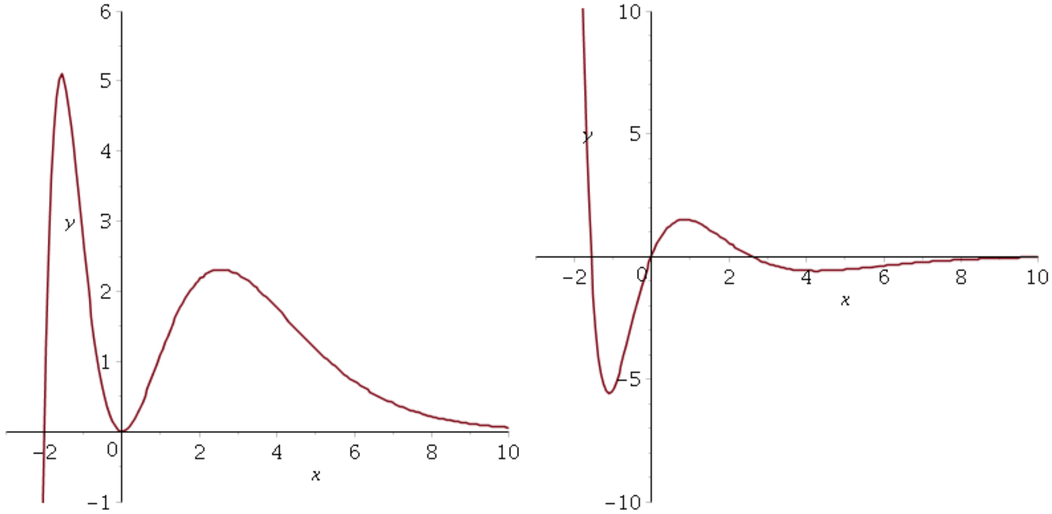
$$k'(x) = (x^2 + 3)e^{x^3 - 5\sin(2x)}(3x^2 - 10\cos(2x)) + e^{x^3 - 5\sin(2x)}(2x).$$

3. All parts of this question refer to the plots in Figure 1. Assume the whole domain of the functions is the interval $[-2, 10]$ shown.

(A) (3) Which has the larger slope: The secant line to $y = A(x)$ through $(-2, A(-2))$ and $(10, A(10))$ or the tangent line to $y = A(x)$ at $(4, A(4))$?
Answer: We see the secant line has a positive slope, but the tangent line at $x = 4$ has a negative slope. So the secant line has a larger slope.

(B) (3) Is $A'(6)$ positive or negative? Answer: $A'(6)$ is the slope of the tangent line to $y = A(x)$ at $x = 6$, which is negative.

(C) (3) At how many different points is $B'(x) = 0$? Estimate the x-values from the graph. Answer: There are three such points where the tangent line to $y = B(x)$ is horizontal – at approximately $x = -1.1$, $x = .85$, and $x = 4.25$.



(a) $y = A(x)$ (b) $y = B(x)$

Figure 1: Plots for Problem 3

(D) (3) On the x-interval $(2, 4)$, is $A''(x)$ positive or negative? Answer: We can see that the slopes of the tangent lines are starting positive, decreasing to 0 and then becoming negative. Since $A'' = (A')'$ and A' is decreasing, $A''(x)$ is negative on this interval.

(E) (3) On the x-interval $(1, 3)$ is $B'(x)$ positive or negative? Answer: The tangent lines have negative slopes, so $B'(x)$ is negative

(F) (3) One of the two functions $A(x)$ and $B(x)$ is the derivative of the other. Which is which? Answer: $B(x)$ is the derivative of $A(x)$. On intervals where $A(x)$ is increasing (resp. decreasing), $B(x)$ is positive (resp. negative).

(G) (2) Which plot is $y = e^{-x}(x^3 + 2x^2)$? Answer: That's $y = A(x)$. The easiest way to see that is that $A(x)$ takes only positive values for $x > -2$, but $B(x)$ has negative values for some of those x's. That says $B(x)$ cannot be $e^{-x}(x^3 + 2x^2) = e^{-x}x^2(x + 2)$, since that formula gives only positive values for $x > -2$.

4. (20) The bluebook value of a 2010 model car at time t years after 2010 is approximated by $P(t) = 20e^{(-.16)t}$ (in units of \$1000). What was the value in 2015, and what was the rate of change of the value then? (Give your answer with units.) Was the value increasing or decreasing?

Solution: Since t is the number of years after 2010, the year 2015 is $t = 5$, and the value in 2015 is $P(5) = 20e^{(-.16)(5)} \doteq 8.987$ in thousands of dollars, or \$8987. (Note: cars depreciate fast!!)

The rate of change of the value then is $P'(5)$. We compute $P'(t) = 20(-.16)e^{(-.16)t} = -3.2e^{(-.16)t}$ by the chain rule. So $P'(5) = -3.2e^{-.8} \doteq -1.438$ in thousands of dollars per year. This means the rate of change of the value is about -1438 dollars per year.