College of the Holy Cross, Fall Semester, 2015
MATH 133, section 1, Solutions for Midterm 2
Thursday, October 22

1. An object moves along a straight line path with position given by $x(t) = 2t^3 + t$, (t in seconds, x in feet).

   (a) (5) What is the average velocity of the object on the interval $[1, 4]$?

   Solution: The average velocity is

   $$v_{ave} = \frac{x(4) - x(1)}{4 - 1} = \frac{132 - 3}{4 - 1} = 43 \text{ ft/sec}$$

   (b) (10) Fill in the following table with average velocities computed over the indicated intervals. Using this information, estimate the *instantaneous velocity* at $t = 1$.

interval	$[1, 2]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
ave.vel.	15	7.62	7.0602	7.006002

   Estimated instantaneous velocity = probably 7 ft/sec.

2. Answer all parts of this question by referring to the graph $y = f(x)$ [on next page].

   (a) (6) $\lim_{x\to 0^-} f(x) = 3$ and $\lim_{x\to 0^+} f(x) = 1$.

   (b) (6) $f(x)$ has an infinite discontinuity at $x = 1$ (one-sided infinite discontinuities are possible).

   (c) (3) True/False: The limit $\lim_{x\to -1} f(x)$ does not exist. False – we have $\lim_{x\to -1} f(x) = 4$. The value $f(-1) = 3$ does not affect the limit.

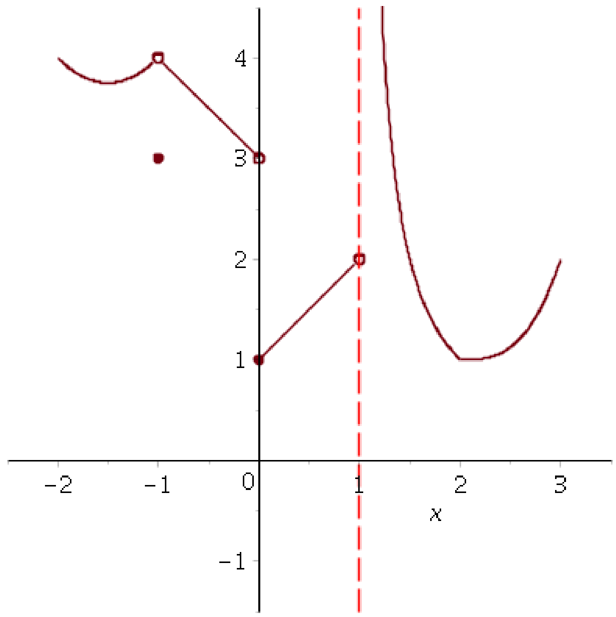
   (d) (9) $\lim_{x\to 2^-} f(x) = 1$ and $\lim_{x\to 2^+} f(x) = 1$. Given that $f(2) = 1$, what can we say about f at $x = 2$? f is *continuous* at $x = 2$.

   (e) (6) True/False: $f(x)$ has a removable discontinuity shown in this part of the graph. True. If so, where is it? $x = -1$. The limit exists, but is different from $f(-1)$. That means f has removable discontinuity there.

3. Compute *any four* of the following limits. (Only the best four will be counted for your total score.)

   (a) (10) This is zero/zero as $x \to 3$, so we try to factor and cancel, then compute the limit by continuity:

   $$\begin{aligned} \lim_{x\to 3} \frac{x^2 - 5x + 6}{3x^2 - 9x} &= \lim_{x\to 3} \frac{(x-2)(x-3)}{3x(x-3)} \\ &= \lim_{x\to 3} \frac{x-2}{3x} \\ &= \frac{1}{9}. \end{aligned}$$



(b) (10) This is a zero/zero form with a difference of square roots, so we use the conjugate radical:

$$\begin{aligned}\lim_{h\to 4}\frac{\sqrt{12+h}-\sqrt{16}}{h^2-16} &= \lim_{h\to 4}\frac{(\sqrt{12+h}-\sqrt{16})(\sqrt{12+h}+\sqrt{16})}{(h^2-16)(\sqrt{12+h}+\sqrt{16})}\\ &= \lim_{h\to 4}\frac{12+h-16}{(h-4)(h+4)(\sqrt{12+h}+\sqrt{16})}\\ &= \lim_{h\to 4}\frac{h-4}{(h-4)(h+4)(\sqrt{12+h}+\sqrt{16})}\\ &= \lim_{h\to 4}\frac{1}{(h+4)(\sqrt{12+h}+\sqrt{16})}\\ &= \frac{1}{64}.\end{aligned}$$

(c) (10) Use the Limit Product Law and $\lim_{t\to 0}\frac{\sin(t)}{t}=1$:

$$\lim_{t\to 0}\frac{(t^2+5)\sin(t)}{t}=\lim_{t\to 0}(t^2+5)\lim_{t\to 0}\frac{\sin(t)}{t}=5\cdot 1=5.$$

(d) (10) For the limit as $x \to \infty$, multiply the top and bottom by $\frac{1}{x^3}$:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+3x+1}{7x^3+x^2+4x} &= \lim_{x\to\infty} \frac{(x^3+3x+1)\frac{1}{x^3}}{(7x^3+x^2+4x)\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{1+\frac{3}{x^2}+\frac{1}{x^3}}{7+\frac{1}{x}+\frac{4}{x^2}} \\
&= \frac{1}{7}.
\end{aligned}$$

(e) (10) Put the two terms on top over a common denominator, simplify the fraction, then factor and cancel to take the limit:

$$\begin{aligned}
\lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} &= \lim_{x\to 2} \frac{4-x^2}{4x^2(x-2)} \\
&= \lim_{x\to 2} \frac{-(x+2)(x-2)}{4x^2(x-2)} \\
&= \lim_{x\to 2} \frac{-(x+2)}{4x^2} \\
&= \frac{-1}{4}.
\end{aligned}$$

4. Let $f(x) = x^2 + 4x + 1$.

(a) (5) What is the slope of the secant line to the graph through the points $(1, 6)$ and $(2, 13)$?

Solution: The slope is

$$\frac{13-6}{2-1} = 7.$$

(b) (5) Give a general formula for the slope of the secant line through the points $(1, 6)$ and $(1+h, (1+h)^2 + 4(1+h) + 1)$.

Solution: The general secant line slope is

$$m_{sec} = \frac{(1+h)^2+4(1+h)+1-6}{(1+h)-1} = \frac{1+2h+h^2+4h+4+1-6}{h} = \frac{6h+h^2}{h}.$$

(c) (5) Find the limit as $h \to 0$ of your slope from part (b).

Solution: Again a zero/zero form. We can cancel h's and then use continuity:

$$\lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0} 6+h = 6.$$