MATH 133 – Calculus with Fundamentals 1
Second Derivative and Concavity
November 30, 2015

Background

We say f (or the graph $y = f(x)$) is *concave up* on an interval if f' is increasing on that interval, and similarly, f or its graph is *concave down* of f' is decreasing on that interval. Combined with our results from last time, this says:

- If $f''(x) > 0$ on an interval, then f or its graph is concave up on that interval
- If $f''(x) < 0$ on an interval, then f or its graph is concave down on that interval
- A point $(c, f(c))$ on the graph of f where the concavity changes is called a *point of inflection* of f.

The notion of concavity can also be used to state a second method for determining whether critical points are local maxima or local minima, called the Second Derivative Test:

Theorem 1 *(Second Derivative Test) Let f be differentiable on some open interval containing a critical point c. In addition, assume $f''(c)$ exists.*

(a) If $f''(c) > 0$, then $f(c)$ is a local minimum

(b) If $f''(c) < 0$, then $f(c)$ is a local maximum

(c) If $f''(c) = 0$, there is no conclusion.

In the last case here, f could have either a local maximum or a local minimum, or neither, so no conclusion is possible. *Technical Comment*: In the other cases, the intuition is that f' should be increasing or decreasing on an interval containing c depending on the sign of $f''(c) = (f')'(c)$, so that (a) corresponds to a case where the graph is concave up at c and (b) corresponds to a case where the graph is concave down at c. This would follow, for instance, if we knew (in addition) that f'' was continuous on some interval containing c. But the conclusion of the Theorem is valid even without that extra continuity hypothesis, as is shown in Exercise 67 in Section 4.4.

Questions

1. Consider the graph $f(x) = x^3 - 3x^2 + 2x$ on the interval $[-1, 3]$ from last time (the plot is also on the back of this sheet). Find the intervals where f is concave up and the intervals where f is concave down. How many points of inflection are there on this graph and where are they located?

2. Consider $f(x) = x^2e^{-x}$.

   (a) Compute $f'(x)$ and find all critical points.

   (b) Determine the sign of $f'(x)$ on each interval between successive critical points, and use that to classify the critical points as local maxima or local minima by the First Derivative Test.

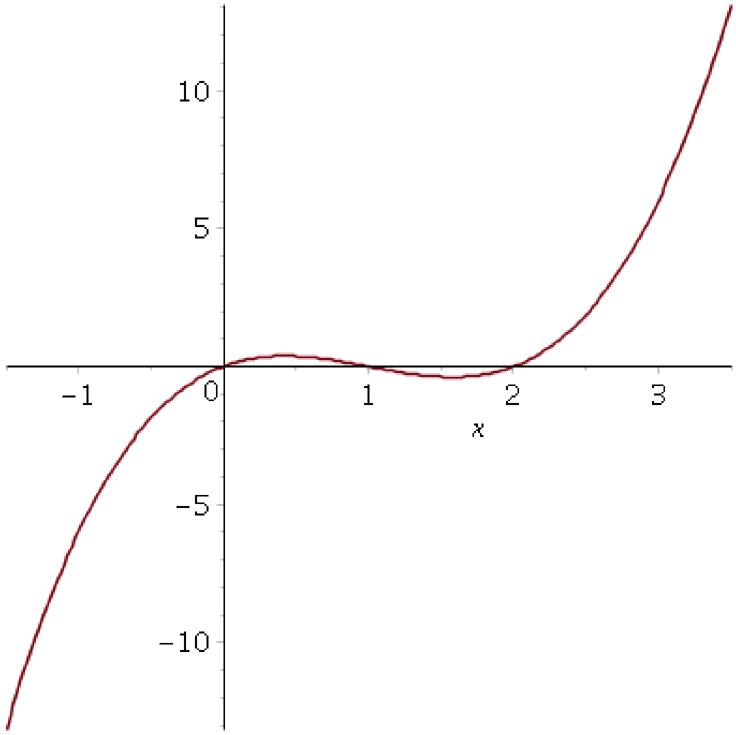

Figure 1: Plot for question 1

(c) Now compute $f''(x)$ and check your answers in (b) by using the Second Derivative Test.

(d) Determine all points of inflection of f.

3. Repeat question 2 for $f(x) = 2x^4 - 3x^2 + 2$.