MATH 133 – Calculus with Fundamentals 1
Differentiability – Graphical and Symbolic Approaches
October 27, 2015

Background

We say f is *differentiable* at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. This is not always true, though, and today we want to understand, from graphical and symbolic approaches, some of the most common ways differentiability can fail. We already know some of these:

(A) From the video for today, we know that if f is differentiable at x, then f is also continuous at x. Turning that around, if f is not continuous at x, then f is not differentiable there either. You don't need to do anything for this part except to understand the statements above(!)

(B) Another pattern – a graph like $y = x^{2/3}$ has a *cusp point* at $x = 0$ and the function $f(x) = x^{2/3}$ is not differentiable at the cusp.

Questions

(1) First consider $f(x) = x^{1/3}$ at $x = 0$.

  (a) What happens if we apply the limit definition to compute $f'(0)$ for this function?

  (b) The graph $y = x^{1/3}$ can be obtained by reflecting $y = x^3$ across the line $y = x$ (why?) Use this to sketch the graph $y = x^{1/3}$. What's happening at $x = 0$? Hint: The graph $y = x^{1/3}$ *does have a tangent line at* $x = 0$, but what can you say about that line? Why isn't $f'(0)$ defined?

(2) Now consider

$$f(x) = x|x-2| = \begin{cases} 2x - x^2 & \text{if } x \le 2 \\ x^2 - 2x & \text{if } x > 2 \end{cases}$$

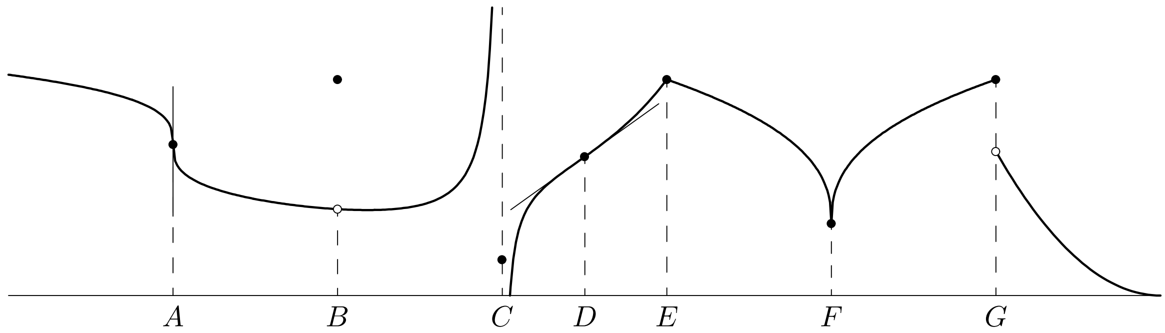
  (a) Sketch the graph $y = f(x)$ for $0 \le x \le 4$. What happens at $x = 2$? Why might we expect that there would be a problem defining $f'(2)$? (Does this graph seem to have a reasonable tangent line at $x = 2$?)

  (b) Determine the two one-sided limits

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} \quad \text{and} \quad \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$$

  This should reinforce what you were saying in part (a)

(3) At which of the marked points on the graph on the back is the function $f(x)$ *not differentiable.* Explain.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