College of the Holy Cross
MATH 133, section 1 – Calculus with Fundamentals
Solutions for Final Exam – December 15, 2015

I. A portion of the graph $y = f(x)$ is given in black (*on the top of the next page*). Match each equation with one of the red graphs identified with lower-case letters.

A) (5) $y = -f(x+5)$ is letter: **d** (shifted 5 units left and reflected across x-axis)

B) (5) $y = f(x+4) - 3.2$ is letter: **a** (shifted 4 units left and 3.2 units down)

C) (5) $y = 3f(x)$ is letter: **e** (stretched by a factor of 3 vertically)

D) (5) $y = f(x) + 2$ is letter: **c** (shifted 2 units up)

E) (5) $y = f(x-4)$ is letter: **b** (shifted 4 units right)

F) (5) The correct formula for $f(x)$ is $f(x) = xe^{-x}$, not $f(x) = \cos(x) - 1$. Note: $\cos(x) - 1$ would have a local maximum at $x = 0$ so that cannot be the correct formula.

II. The power delivered by a battery to an apparatus of resistance R (in ohms) is

$$P(R) = \frac{5R}{(R+0.5)^2}$$

(in watts).

A) (5) If $R = 10$ ohms, what is the power delivered by the battery?

$P(10) = \frac{5\cdot 10}{(10+0.5)^2} \doteq .45$ watts.

B) (5) A power of 2 watts can be obtained with two different values of the resistance R. What are they?

They are the solutions of $2 = \frac{5R}{(R+0.5)^2}$, which rearranges to $2R^2 - 3R + .5 = 0$. By the quadratic formula, the roots are

$$R = \frac{3 \pm \sqrt{9 - (4)(2)(.5)}}{4} = \frac{3 \pm \sqrt{5}}{4} \doteq .19, 1.3 \text{ ohms}.$$

C) (10) What is the *rate of change* of the power at $R = 10$ ohms?

This is the derivative $P'(10)$. By the quotient rule,

$$P'(R) = \frac{(R+.5)^2 \cdot 5 - 5R \cdot (2R+1)}{(R+.5)^4}$$

so $P'(10) \doteq -.041$ (units are: watts per ohm).

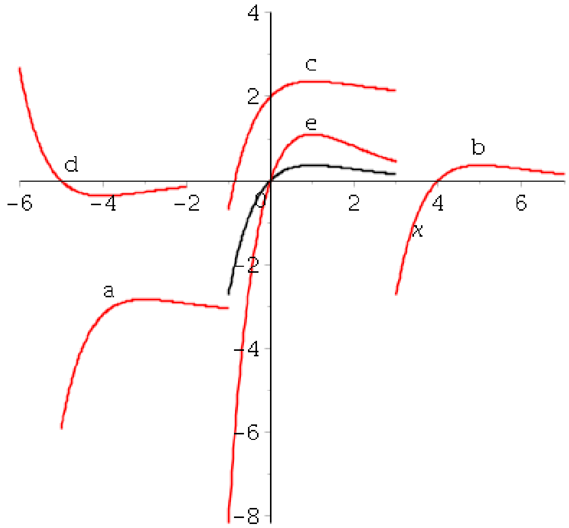

Figure 1: Figure for problem I

III. Compute the following limits. Any legal method is OK.

(A) (10) $\lim_{x\to 3} \dfrac{x^2-6x+9}{x^2-5x+6}$.

This is a 0/0 form. We can factor the top and bottom, cancel one $x-3$, then take the limit to obtain:

$$\begin{aligned}\lim_{x\to 3} \frac{x^2-6x+9}{x^2-5x+6} &= \lim_{x\to 3} \frac{(x-3)^2}{(x-2)(x-3)} \\ &= \lim_{x\to 3} \frac{(x-3)}{(x-2)} \\ &= 0.\end{aligned}$$

(B) (10) $\lim_{x\to 2} \dfrac{\sqrt{x+7}-3}{x-2}$.

Another 0/0 form. For this one, we multiply the top and bottom by the conjugate radical, simplify, then evaluate the resulting limit by continuity:

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2} &= \lim_{x\to 2}\frac{(\sqrt{x+7}-3)(\sqrt{x+7}+3)}{(x-2)(\sqrt{x+7}+3)} \\
&= \lim_{x\to 2}\frac{(x+7)-9}{(x-2)(\sqrt{x+7}+3)} \\
&= \lim_{x\to 2}\frac{(x-2)}{(x-2)(\sqrt{x+7}+3)} \\
&= \lim_{x\to 2}\frac{1}{(\sqrt{x+7}+3)} \\
&= \frac{1}{6}.
\end{aligned}$$

(C) (10) $\lim\limits_{x\to 0} x\cot(x)$

Rewrite $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and use $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, so $\lim_{x\to 0}\frac{x}{\sin(x)} = 1$ as well by the limit quotient rule. Then

$$\lim_{x\to 0} x\cot(x) = \lim_{x\to 0}\frac{x\cos(x)}{\sin(x)} = \lim_{x\to 0}\frac{x}{\sin(x)}\cdot\lim_{x\to 0}\cos(x) = 1\cdot 1 = 1.$$

IV.

A) (10) *Using the limit definition*, and showing all necessary steps to justify your answer, compute $f'(x)$ for $f(x) = \dfrac{1}{x-3}$.

We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{x+h-3}-\frac{1}{x-3}}{h} \\
&= \lim_{h\to 0}\frac{(x-3)-(x+h-3)}{h(x+h-3)(x-3)} \\
&= \lim_{h\to 0}\frac{-h}{h(x+h-3)(x-3)} \\
&= \lim_{h\to 0}\frac{-1}{(x+h-3)(x-3)} \\
&= \frac{-1}{(x-3)^2}.
\end{aligned}$$

(Note that this agrees with the result of applying the chain rule to the function $f(x) = (x-3)^{-1}$.)

Using appropriate derivative rules, compute the derivatives of the following functions. You do *not* need to simplify your answers.

B) (10) $g(x) = \pi x^e + \dfrac{3}{x^7} + \sqrt[3]{x} + 5$

$g'(x) = \pi e x^{e-1} - \frac{21}{x^8} + \frac{1}{3}x^{-2/3}$.

C) (10) $h(x) = \dfrac{x^2 - 4x + 1}{x^3 + 4}$

(quotient rule)

$$h'(x) = \frac{(x^3+4)(2x-4) - (x^2-4x+1)(3x^2)}{(x^3+4)^2}.$$

D) (10) $i(x) = x\ln(\cos(x)) + e^{\cos(x)}$

(product and chain rules)

$$i'(x) = x \cdot \left(\frac{-\sin(x)}{\cos(x)}\right) + \ln(\cos(x)) + e^{\cos(x)} \cdot (-\sin(x))$$

E) (10) $j(x) = \sin^{-1}(3x - 4)$

(derivative rule for inverse sine and chain rule)

$$j'(x) = \frac{1}{\sqrt{1-(3x-4)^2}} \cdot 3$$

V. The graph in Figure 2 shows the *derivative* $f'(x)$ for some function $f(x)$ defined on $-1.5 \le x \le 2.5$. In particular you should assume $f(1), f(2)$ are defined and finite. Note: This graph *is not* $y = f(x)$, *it is* $y = f'(x)$. Using the graph, answer these questions:

A) (5) What are the critical points of f in the interval $[-1.5, 2.5]$?

Answer: $x = 0, 1, 2$ are all critical points. The "outside ones" at $x = 0, 2$ have $f'(x) = 0$, the "middle one" $f'(1)$ does not exist because of the vertical asymptote to the graph $y = f'(x)$.

B) (5) Classify each of the points you found in part A) as a local maximum, local minimum, or neither.

$x = 0$ is a local minimum of f; $x = 1, 2$ are neither.

C) (5) Explain briefly how you know your answer in B) is correct.

Answer: From the given graph, $f'(x)$ changes sign from negative to positive at $x = 0$, so that is a local minimum by the First Derivative Test. At the other two critical points, f' does not change sign (it's positive on both sides of $x = 1, 2$). So they neither local maxima nor local minima.

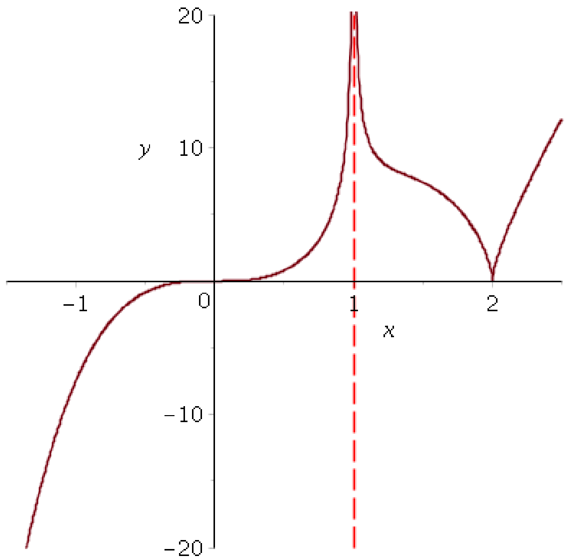

Figure 2: $y = f'(x)$ for problem V.

D) (5) Is $x = 0$ a point of inflection of f? Why or why not?

Answer: No – the concavity of $y = f(x)$ does not change at $x = 0$ because $f'(x)$ is increasing on both sides of 0.

E) (5) Does it appear that $f''(2)$ exists? (Yes/No):

Answer: No – $y = f'(x)$ appears to have a cusp (or maybe a "corner") at $x = 2$, so the derivative $f''(2) = (f')'(2)$ doesn't exist.

F) (5) Over which interval(s) contained in $[-1.5, 2.5]$ is the graph $y = f(x)$ concave down.

Answer: Where f' is decreasing, so $(1, 2)$ only.

VI. A Spanish factory can produce $P = 2LK^2$ million lightbulbs if labor costing L million euros is hired and equipment costing K million euros is obtained. If an order for 1.7 million lightbulbs is received, which combination of L and K will *minimize* the total cost of labor plus equipment to fill the order?

A) (5) Express the total cost of producing 1.7 million lightbulbs as a function of one of the two variables K, L. (Your choice, but choose wisely!)

Setting $P = 1.7$, we have

$$1.7 = 2LK^2 \Rightarrow L = \frac{.85}{K^2}.$$

So the total cost (labor, plus equipment) is

$$L + K = \frac{.85}{K^2} + K.$$

The function we want to minimize is

$$C(K) = \frac{.85}{K^2} + K.$$

(Note that you want to differentiate this, *not* the formula for P as a function of L and K.)

B) (10) Find a critical point of your function from part A which is a realistic solution of this problem; solve for the other variable.

We have

$$C'(K) = \frac{-1.7}{K^3} + 1.$$

This is 0 when

$$K = (1.7)^{1/3} \doteq 1.19 \text{ million euros.}$$

Then $L = \frac{.85}{(1.19)^2} \doteq .60$ million euros.

C) (5) How do you know that your solution *minimizes* the total cost?

One way to tell is to compute the second derivative of the total cost function (as a function of K):

$$C''(K) = \frac{5.1}{K^4} > 0$$

at $K = 1.19$ (in fact for all K.) This implies $K = 1.19$ is a local minimum by the Second Derivative Test.

VII. (15) A moth ball is evaporating and losing volume at the rate of .1 cm^3/week. It has the shape of a sphere at all times. How fast is the surface area of the moth ball shrinking when the radius of the ball is 1cm? (Note: A sphere of radius r has volume $V = \frac{4\pi r^3}{3}$ and surface area $A = 4\pi r^2$.)

Solution: We are given that the volume of the moth ball is decreasing at .1 cubic centimeters per week. This says

$$-0.1 = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Hence when $r = 1$,

$$\frac{dr}{dt} = \frac{-0.1}{4\pi} \doteq -.008$$

cm/week. Then

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = \frac{8\pi \cdot (-0.1)}{4\pi} = -0.2$$

(the units here are (square cm)/week).

VIII. (5) True/False: The graph obtained by shifting $y = \ln(x)$ vertically by 2 units can *also* be obtained from $y = \ln(x)$ by a horizontal compression. Explain your answer.

This is **True** because of properties of the logarithm function: Shifting $y = \ln(x)$ vertically by 2 units gives $y = \ln(x) + 2$. But

$$\ln(x) + 2 = \ln(x) + \ln(e^2) = \ln(e^2 \cdot x).$$

Multiplying x by a constant greater than 1 "inside" the function compresses the graph horizontally. Since $e^2 > 1$, the graph of this function is a horizontal compression of the graph $y = \ln(x)$.