MATH 133 – Calculus with Fundamentals 1
Exam 4 Solutions for Sample/Practice Problems
December 4, 2015

I. Find y'; don't simplify.

(A)

$$y = \ln(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)$$

Solution: Rewrite the function as $\ln(x)(x^7 - 4x^{-1/2})$. Then by the product rule and the derivative formula for the natural logarithm:

$$y' = \ln(x)(7x^6 + 2x^{-3/2}) + (x^7 - 4x^{-1/2}) \cdot \frac{1}{x}.$$

(B)

$$y = \sin^{-1}(e^{2x} + 2)$$

Solution: By the derivative formula for the inverse sine and the chain rule:

$$y' = \frac{1}{\sqrt{1 - (e^{2x} + 2)^2}} \cdot 2e^{2x}.$$

(C)

$$y = \frac{\ln(x+1)}{3x^4 - 1}$$

Solution: By the quotient rule and the derivative formula for the natural logarithm:

$$y' = \frac{(3x^4 - 1) \cdot \frac{1}{x+1} - \ln(x+1) \cdot 12x^3}{(3x^4 - 1)^2}.$$

(D)

$$y = \frac{\sin(x)}{1 + \cos(x)}$$

Solution: By the quotient rule:

$$y' = \frac{(1 + \cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} = \frac{1}{1 + \cos(x)}$$

(if you use a trig indentity to simplify).

(E)

$$y = \tan^{-1}(x^2 + x)$$

Solution: By the derivative formula for the inverse tangent and the chain rule:

$$y' = \frac{1}{1 + (x^2 + x)^2} \cdot (2x + 1).$$

(F) Using implicit differentiation:

$$xy^2 - 3y^3 + 2x^4 - 4xy = 2$$

Solution: By the implicit differentiation method (treating y as an implicitly defined function of x) we get:

$$2xyy' + y^2 - 9y^2y' + 8x^3 - 4xy' - 4y = 0.$$

Solving for y', we get

$$y' = \frac{-y^2 - 8x^3 + 4y}{2xy - 9y^2 - 4x}$$

(G) Find the equation of the line tangent to the curve from (F) at $(x, y) = (1, 0)$

Solution: At $(x, y) = (1, 0)$, we have $y' = -8/-4 = 2$. So the tangent line is $y = 2(x - 1)$ or $y = 2x - 2$.

II. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time $t \geq 0$ seconds.

(A) Over which intervals with $t \geq 0$ is the amount increasing? decreasing?

Solution: We need $Q'(t) = 3t^2 - 6t + 1$ to see this. Note that $Q'(t) = 0$ when $t = \frac{6 \pm \sqrt{36-12}}{6} = \frac{3 \pm \sqrt{6}}{3} \doteq .18, 1.82$. $Q'(t) < 0$ between these two values and positive outside that interval. So for $t \geq 0$, $Q(t)$ is increasing on $[0, .18)$ and $(1.82, +\infty)$. It is decreasing on $(.18, 1.82)$.

(B) Over which intervals is the rate of change of Q increasing? decreasing?

Solution: The rate of change of Q is $Q'(t)$. This is increasing on intervals where $(Q')'(t) = Q''(t)$ is positive and negative on intervals where $Q''(t)$ is negative. Since $Q''(t) = 6t - 6$, we see that the rate of change of Q is increasing on $(1, +\infty)$ and decreasing on $[0, 1)$ (intervals of t-values).

III. A spherical balloon is being inflated at 20 cubic inches per minute. (The volume of a sphere of radius r is $V = \frac{4\pi r^3}{3}$ and the surface area is $V = 4\pi r^2$.)

(A) When the radius is 6 inches, at what rate is the radius of the balloon increasing?

Solution: From $V = \frac{4\pi r^3}{3}$, if we differentiate both sides with respect to t (time), we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

The given value 20 cubic inches per minute is $\frac{dV}{dt}$. When $r = 6$, we have

$$20 = 4\pi 6^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{20}{144\pi} = \frac{5}{36\pi}$$

inches per minute.

(B) When the radius is 6 inches, at what rate is the surface area increasing?

Solution: From $A = 4\pi r^2$, if we differentiate both sides with respect to t (time), we get that the rate of change of the surface area is

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}.$$

Using the value $\frac{dr}{dt} = \frac{5}{36\pi}$ from part (A), we have

$$\frac{dA}{dt} = \frac{240\pi}{36\pi} = \frac{20}{3}$$

square inches per minute.

IV. A baseball diamond is a square with side of length 90 feet. After hitting the ball, a player leaves home plate and runs toward first base at 15 ft/sec. (Assume the runner is running straight along the base path – this is a bit unrealistic, of course, but let's keep it simple for the purposes of the problem!) How fast is the runner's distance from second base changing when he is half way to first base?

Solution: Let x be the distance traveled by the runner along the basepath. From a diagram of the diamond, we can see that the runner's position, first base, and second base form a right triangle (with right angle at first base) at all times up until the runner reaches first base. The two legs of that triangle are 90 and $90 - x$, so by the Pythagorean theorem, the distance from the runner to second base is

$$\ell = \sqrt{90^2 + (90 - x)^2} = \sqrt{16200 - 180x + x^2}.$$

Take derivatives with respect to t (time) everywhere. Then

$$\frac{d\ell}{dt} = \frac{1}{2}(16200 - 180x + x^2)^{-1/2}(2x - 180)\frac{dx}{dt}$$

The given value 15 ft/sec is the $\frac{dx}{dt}$, and we want the instant when $x = 45$. So at that time,

$$\frac{d\ell}{dt} = \frac{1}{2}(16200 - 180 \cdot 45 + 45^2)^{-1/2}(-90)(15) = -\frac{15}{\sqrt{5}}$$

ft/sec. (This is negative, so the distance from the runner to second base is *decreasing.*)

V. All parts of this question refer to $f(x) = 4x^3 - x^4$.

(A) Find and classify all the critical points of f using the First Derivative Test.

Solution: $f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x)$. This is defined for all x and equal to zero at $x = 0$ and $x = 3$. Note that $4x^2 \geq 0$ for all x. So the sign of $f'(x)$ comes from the $3 - x$ factor. That is negative for $x > 3$ and positive for $x < 3$. Hence f' changes sign from positive to negative at $x = 3$ and the First Derivative Test says f has a local local maximum at $x = 3$. On the other hand, $f'(x)$ does not change sign at $x = 0$, so that critical point is neither a local maximum nor a local minimum.

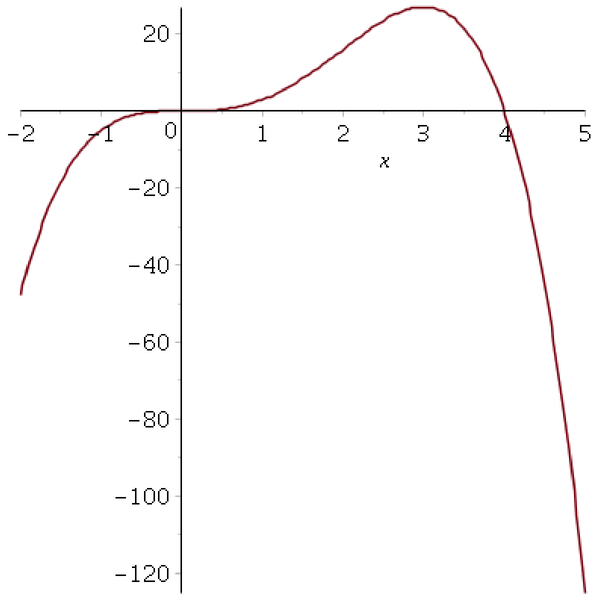

Figure 1: Plot of $y = f(x)$ for Problem V

(B) Over which intervals is the graph $y = f(x)$ concave up? concave down?

Solution: $f''(x) = 24x - 12x^2 = 12x(2-x)$, which is zero at $x = 0$ and $x = 2$. Then $f''(x) > 0$ and the graph $y = f(x)$ is concave up on $(0, 2)$ and $f''(x) < 0$ and the graph $y = f(x)$ is concave down on $(-\infty, 0)$ and $(2, \infty)$.

(C) Sketch the graph $y = f(x)$.

Solution: See Figure for problem V.

(D) Find the absolute maximum and minimum of $f(x)$ on the interval $[1, 4]$.

Solution: Only the critical point $x = 3$ is in this interval. $f(1) = 8$, $f(3) = 27$ and $f(4) = 0$. So $f(3) = 27$ is the maximum value and $f(4) = 0$ is the minimum value on the interval $[1, 4]$.

VI. All three parts of this question refer to the function $f(x)$ *whose derivative is plotted in the Figure for problem VI. NOTE: This is the graph* $y = f'(x)$ *not* $y = f(x)$.

(A) Give approximate values for all the critical points of $f(x)$ in the interval shown, and say whether f has a local maximum, a local minimum, or neither at each.

Solution: By inspection of the plot in Figure 1, we see that $f'(x) = 0$ at approximately $x = -5.2, -2$, and 1.2. Since f' changes sign from $+$ to $-$ at $x = -5.2$ and $x = 1.2$, those are local maxima for f. Since f' changes sign from negative to positive at $x = -2$, that is a local minimum for f.

(B) Find approximate values for all the inflection points of $f(x)$.

Solution: $y = f(x)$ has inflection points where f' changes from increasing to decreasing. That happens here at roughly $x = -3.9$ and $x = -0.8$.

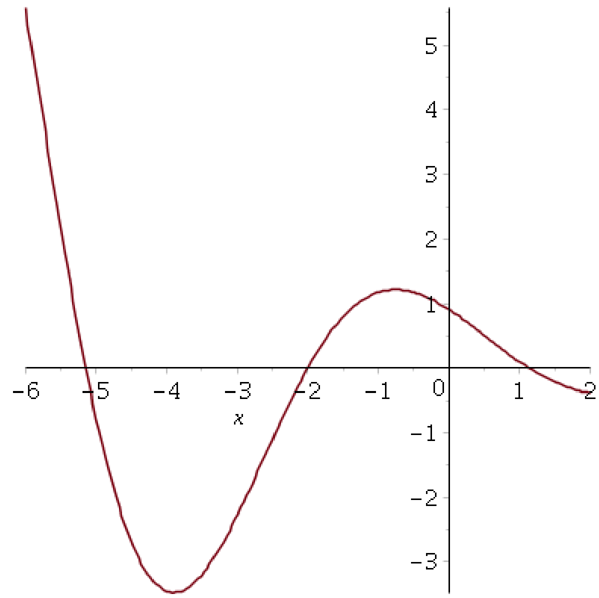

Figure 2: Plot of $y = f'(x)$ for Problem VI

(C) Over which intervals is $y = f(x)$ concave up? concave down?

Solution: Following on from (B), $y = f(x)$ will be concave down on intervals where $f'(x)$ is decreasing – roughly $(-6, -3.9)$ and $(-0.8, 2)$. $y = f(x)$ will be concave up on intervals where $f'(x)$ is increasing – roughly $(-3.9, -0.8)$.

VII. Optimization problems.

(A) A rectangular box with no top is created out of a rectangular piece of cardboard by cutting equal squares out of the four corners and folding up the sides. If the original piece of cardboard was 20 inches by 15 inches, what is the largest volume possible for the resulting box? (Hint: Let x be the side of the four squares cut out of the corners. The volume is length times width times height.)

Solution: As suggested, let x be the side of the small squares cut out of the rectangular sheet of cardboard. Then, after folding up the sides, the rectangular box will have length $20 - 2x$, width $15 - 2x$ and height x. (Note that this says x is restricted to the interval $[0, 15/2] = [0, 7.5]$ since $15 - 2x \geq 0$ is necessary to be able to fold up the sides to get a box!) The volume is

$$V(x) = (20 - 2x)(15 - 2x)x = 300x - 70x^2 + 4x^3.$$

Taking the derivative with respect to x and set equal to zero:

$$V' = 300 - 140x + 12x^2 = 0.$$

The solutions are $x \doteq 2.83, 8.84$ (quadratic formula). Only $x \doteq 2.83$ is in the interval $[0, 7.5]$ noted before. So that is our candidate for the side of the small squares. This is a local maximum since $V'' = -140 + 24x < 0$ at $x \doteq 2.83$. This x also gives the global maximum of the volume since it is the only critical point on the interval of feasible x-values. The actual answer to the problem is the volume

$$V(2.83) = (20 - 5.66)(15 - 5.66)(2.83) \doteq 379$$

cubic inches.

(B) A rectangular poster is to be created with 400 square inches of printed material surrounded by 2 inch margins on the top and bottom and the left and right edges. What should the dimensions of the poster be to minimize the total area (printed material plus margins)?

Solution: Let the overall dimensions of the poster be x, y. Then the inner printed area is a rectangle with area $400 = (x-4)(y-4)$. This lets us solve for y in terms of x: $y = \frac{400}{x-4} + 4$. The total area of the poster is

$$A = xy = x\left(\frac{400}{x-4} + 4\right) = \frac{400x}{x-4} + 4x$$

Then by the quotient rule,

$$A'(x) = \frac{(x-4) \cdot 400 - 400x}{(x-4)^2} + 4 = \frac{-1600}{(x-4)^2} + 4$$

For a critical point, we set this equal to zero and solve for x:

$$\frac{-1600}{(x-4)^2} + 4 = 0 \Rightarrow (x-4)^2 = 400 \Rightarrow x = 24,$$

and then from $(x-4)(y-4) = 400$, we get $y = 24$ as well. This corresponds to a square poster. Is this a minimum? Well, differentiating again,

$$A''(x) = \frac{3200}{(x-4)^3} \Rightarrow A''(24) = \frac{3200}{20^3} > 0.$$

By the Second Derivative Test, this says we have a local and global minimum (again, it has to be global minimum since it's the only critical point and a local minimum).

(C) A billboard 20 feet tall is mounted 10 feet above eye level on the wall of a building. How far should a person stand from the wall in order to maximize the angle θ subtended by the billboard at the person's eye. (Hint: Draw a diagram first; see the top diagram in Figure 30 on page 248 of the text if you cannot figure out what this means.)

Solution: Let x be the distance from the wall that the person stands. Referring to the diagram we see

$$\theta = \tan^{-1}(30/x) - \tan^{-1}(10/x)$$

We want to maximize this angle as a function of x. Taking derivatives, we get

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{1}{1 + \left(\frac{30}{x}\right)^2} \cdot \frac{-30}{x^2} - \frac{1}{1 + \left(\frac{10}{x}\right)^2} \cdot \frac{-10}{x^2} \\ &= \frac{-30}{x^2 + 30^2} + \frac{10}{x^2 + 10^2}. \end{aligned}$$

This equals zero (i.e. we have a critical point) when

$$\frac{30}{x^2+30^2} = \frac{10}{x^2+10^2}$$

which implies

$$30(x^2+100) = 10(x^2+900)$$

or

$$30x^2+3000 = 10x^2+9000$$

or $20x^2 = 6000$, so $x = \sqrt{300} = 17.32$ feet. The best way to see that this is a local and global maximum is to use the first derivative test. $\frac{d\theta}{dx} > 0$ for small x and $\frac{d\theta}{dx} < 0$ for large x.

(D) A window has the shape of a rectangle surmounted by a semicircle (see Figure 10 on page 245 of our textbook if you don't understand what this means). The total perimeter of the window is 600 cm. What should the dimensions be to make the area of the window be as large as possible (so that it will admit the most light possible)?

Solution: Let the dimensions of the rectangle be x, y and say x is the horizontal side, so the radius of the semicircle is $x/2$. Then the perimeter is

$$600 = x + 2y + \frac{\pi x}{2}.$$

This lets us solve for y in terms of x:

$$y = 300 - \frac{x}{2} - \frac{\pi x}{4} = 300 - \frac{2+\pi}{4}x.$$

The total area is the area of the rectangle plus the area of the semicircle:

$$A = xy + \frac{\pi(x/2)^2}{2} = xy + \frac{\pi x^2}{8} = x\left(300 - \frac{2+\pi}{4}x\right) + \frac{\pi x^2}{8} = 300x - \frac{4+\pi}{8}x^2.$$

Taking the derivative with respect to x we get

$$A'(x) = 300 - \frac{4+\pi}{4}x.$$

Setting this equal to zero and solving for x,

$$x = \frac{1200}{4+\pi} \doteq 168 \text{ cm}.$$

Substituting this into the formula for y above, we get $y \doteq 84$ cm. These dimensions maximize the area because $A'' = -\frac{4+\pi}{4} < 0$, so our critical point is a local and global maximum.