MATH 133, section 1 – Calculus With Fundamentals 1
Exam 3 Practice Problems
November 4, 2015

I. Do not use the differentiation rules from Chapter 3 in this question.

A) State the limit definition of the derivative $f'(x)$.
Solution: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

B) Use the definition to compute the derivative function of $f(x) = \frac{1}{3x}$.
Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} \\ &= \lim_{h\to 0} \frac{3x - (3x+3h)}{9hx(x+h)} \\ &= \lim_{h\to 0} \frac{-3h}{9hx(x+h)} \\ &= \lim_{h\to 0} \frac{-3}{9x(x+h)} \\ &= \frac{-3}{9x^2} = \frac{-1}{3x^2}. \end{aligned}$$

(Note: this agrees with the result of applying the chain rule to $f(x) = (3x)^{-1}$.)

C) Find the equation of the line tangent to the graph $y = \frac{1}{3x}$ at $x = 2$.
Solution: At $x = 2$, $f(2) = \frac{1}{6}$ and $f'(2) = \frac{-1}{12}$. So the equation of the tangent line is $y - \frac{1}{6} = \left(\frac{-1}{12}\right)(x-2)$, or $y = \frac{-1}{12}x + \frac{1}{3}$.

II. Use the sum, product, quotient, and/or chain rules to compute the following derivatives. You may use any correct method, but must show work for full credit.

A)

$$\frac{d}{dx}\left(5x\sqrt{x} - \frac{2}{x^3} + 11x - 4\right)$$

Solution: The function can also be written as $5x^{3/2} - 2x^{-3} + 11x - 3$. In this form, we only need the power rule to differentiate:

$$y' = \frac{15}{2}x^{1/2} + 6x^{-4} + 11.$$

B)

$$\frac{d}{dt}\left(\frac{t^2e^{3t}}{t^4+1}\right)$$

Solution: By the quotient rule, product rule, and chain rule the derivative is:

$$\frac{(t^4+1)\cdot(3t^2e^{3t} + 2te^{3t}) - (t^2e^{3t})\cdot(4t^3)}{(t^4+1)^2}.$$

C)

$$\frac{d}{dz}\frac{z^2 - 2z + 4}{z^2 + 1}$$

Solution: Again using the quotient rule, the derivative is:

$$\frac{(z^2+1)(2z-2) - (z^2-2z+4)(2z)}{(z^2+1)^2} = \frac{2z^2 - 6z - 2}{(z^2+1)^2}.$$

D)

$$\frac{d}{dx}\left(\frac{\pi^2 + \tan(e^\pi) - 2x^e}{4}\right)$$

Solution: Most of this is constant; only the x^e from the last term on the top contributes anything nonzero to the derivative:

$$\frac{-e}{2}x^{e-1}.$$

E)

$$\frac{d}{dx}\left(\sin(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)\right)$$

Solution: Rewrite the fuction as $\sin(x)(x^7 - 4x^{-1/2})$. Then by the product rule the derivative is:

$$\sin(x)(7x^6 + 2x^{-3/2}) + (x^7 - 4x^{-1/2})\cos(x).$$

F) Find y' (note this is just another way of asking the same question!)

$$y = (e^{2x} + 2)^3$$

Solution: By the chain rule, the derivative is:

$$3(e^{2x}+2)^2(2e^{2x}) = 6e^{2x}(e^{2x}+2)^3.$$

G) Find y'

$$y = \frac{x+1}{3x^4 - 1}$$

Solution: By the quotient rule,

$$y' = \frac{(3x^4-1)(1) - (x+1)(12x^3)}{(3x^4-1)^2} = \frac{-9x^4 - 12x^3 - 1}{(3x^4-1)^2}.$$

H) Find y'

$$y = \frac{\sin(x)}{1+\cos(x)} + x^2\cos(x^3+3)$$

Solution: By the quotient, product, and chain rules:

$$\begin{aligned}
y' &= \frac{(1+\cos(x))\cos(x) - \sin(x)(-\sin(x))}{(1+\cos(x))^2} - x^2\sin(x^3+3)(3x^2) + 2x\cos(x^3+3) \\
&= \frac{1+\cos(x)}{(1+\cos(x))^2} - 3x^4\sin(x^3+3) + 2x\cos(x^3+3) \\
&= \frac{1}{1+\cos(x)} - 3x^4\sin(x^3+3) + 2x\cos(x^3+3).
\end{aligned}$$

III. The total cost (in \$) of repaying a car loan at interest rate of $r\%$ per year is $C = f(r)$.

A) What is the meaning of the statement $f(7) = 20000$?
Solution: At an interest rate of 7% per year, the cost of repaying the loan is 20000 dollars.

B) What is the meaning of the statement $f'(7) = 3000$? What are the units of $f'(7)$?
Solution: At an interest rate of 7% per year, the rate of change of the cost of repaying the loan is 3000 dollars per (% per year).

IV. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time t seconds for all $t \geq 0$. (Note: For a question like this, I could also give you the plot of the function and ask questions like those below. In this case you need to start from the formula and compute $Q'(t)$; if you were given the graph, you need to make the connection between slopes of tangent lines and signs of $Q'(t)$ visually.)

(a) Over which intervals with $t \geq 0$ is the amount increasing? (i.e. $Q'(t) > 0$) decreasing (i.e. $Q'(t) < 0$)?
Solution: $Q'(t) = 3t^2 - 6t + 1$. $Q'(t) = 0$ when

$$t = \frac{6 \pm \sqrt{36-12}}{6} = 1 \pm \frac{\sqrt{6}}{3} \doteq 1.816, .184.$$

Since this is a quadratic function with a positive t^2 coefficient, $Q'(t) > 0$ for $t > 1.816$ and $t < .184$. $Q'(t) < 0$ for $.184 < t < 1.816$ (t in seconds).

(b) Over which intervals is the rate of change of Q increasing? decreasing?
Solution: The rate of change of Q is increasing when $(Q')' > 0$ and decreasing when $(Q')' < 0$. The second derivative of Q is $Q''(t) = 6t - 6$. So $Q''(t) > 0$ for $t > 1$ and $Q''(t) < 0$ for $t < 1$ (t in seconds).

V. A spherical balloon is being inflated at 20 cubic inches per minute. When the radius is 6 inches, at what rate is the radius of the balloon increasing? At what rate is the surface area increasing? (The volume of a sphere of radius r is $V = \frac{4\pi r^3}{3}$ and the surface area is $A = 4\pi r^2$.)

Solution: By the chain rule, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. We are given $\frac{dV}{dt} = 20$ when $r = 6$, so

$$\frac{dr}{dt} = \frac{20}{4\pi(6)^2} = \frac{5}{36\pi}$$

inches per minute. The rate of change of the surface area is

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = \frac{48\pi \cdot 5}{36\pi} = \frac{20}{3}$$

square inches per minute.

VI. Review problems 2 and 3 from Quiz 5.