MATH 133 section 1 – Calculus with Fundamentals 1
Sample Questions for Exam 2 – Answers and Solutions
October 8, 2015

1. An object moves along a straight line path with position given by $x(t) = 4t^2 + t - 7$, (t in seconds, x in feet).

   (a) What is the average velocity of the object over the interval $[0, 5]$ of t-values?

   Solution: The average velocity is:

   $$v_{ave} = \frac{x(5) - x(0)}{5 - 0} = \frac{98 - (-7)}{5} = 21 \text{ ft/sec}$$

   (b) Fill in the following table with average velocities computed over the indicated intervals. Using this information, estimate the *instantaneous velocity* at $t = 2$.

   Solution:

interval	$[2, 2.5]$	$[2, 2.05]$	$[2, 2.005]$	$[2, 2.0005]$
ave.vel.	19.0	17.2	17.02	17.002

   It looks as though the average velocity is tending to 17 as the length of the interval goes to 0.

   (c) Construct a similar table for intervals *ending* at $t = 2$ and repeat the calculations in the previous part. If you estimate the instantaneous velocity at $t = 2$ using this new information, does your result agree with what you did before (it should!)

   Solution:

interval	$[1.5, 2]$	$[1.95, 2]$	$[1.995, 2]$	$[1.9995, 2]$
ave.vel.	15.0	16.8	16.98	16.998

   It looks as though the average velocity is tending to 17 as the length of the interval goes to 0 again.

2. (a) What is the slope of the secant line to the graph $y = x^3 + 1$ through the points with $x = 1$ and $x = 2$?

   Solution: The slope of the secant line is

   $$m_{sec} = \frac{(2^3 + 1) - (1^3 + 1)}{2 - 1} = \frac{9 - 2}{1} = 7.$$

   (b) What is the slope of the secant line to the graph $y = x^3 + 1$ through the points with $x = 1$ and $x = 1 + h$ for a general h?

Solution: The slope of the secant line is

$$m_{sec} = \frac{(1+h)^3 + 1 - (1^3+1)}{1+h-1} = \frac{3h+3h^2+h^3}{h}.$$

(c) The slope of the tangent line to $y = x^3 + 1$ at $x = 1$ would be obtained from what limit?

Solution: The slope of the tangent is

$$m_{tan} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h}.$$

(d) Estimate the limit in the previous part numerically (as in the first question).

Solution:

h	1	.5	.05	.005
$(3h+3h^2+h^3)/h$	7	4.75	3.1525	3.015025

It seems the limit as $h \to 0$ is about 3.

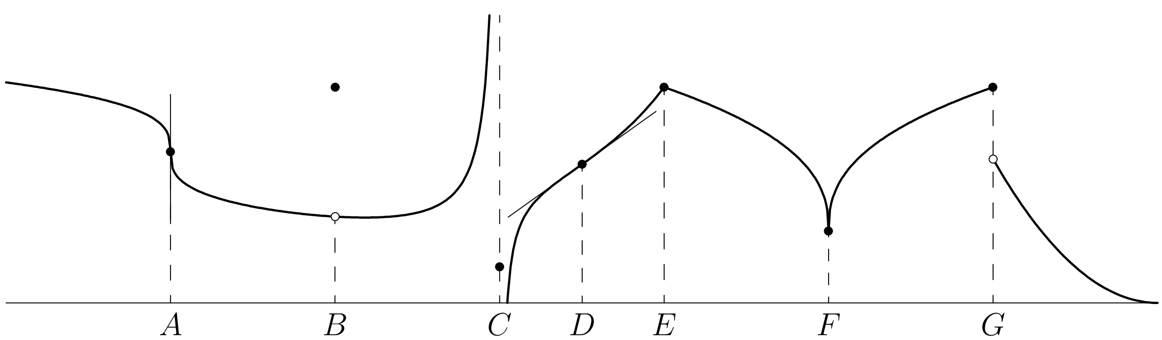
(e) Evaluate the limit exactly using our algebraic techniques.

Solution:

$$\lim_{h\to 0} \frac{3h+3h^2+h^3}{h} = \lim_{h\to 0} \frac{h(3+3h+h^2)}{h} = \lim_{h\to 0} 3+3h+h^2 = 3$$

(cancelling the h's top and bottom, then using the continuity of $3+3h+h^2$ as a function of h).

3. The graph of a function f is shown below with several points marked. Find all the marked points at which the following are true, and give explanations for your answers.



(a) f has an infinite discontinuity – *Answer:* C (from the left)

(b) f has a jump discontinuity (and: If you had a vertical axis with actual y-values to use, how would you find the one-sided limits at each such point?) – *Answer:* G (The other part of the question: To get the one sided limits, you would try to match the y-coordinates of the left and right sides of the graph with the axis scale. Then the limit $\lim_{x\to G^-} f(x)$ would be the limiting y-coordinate on the left and $\lim_{x\to G^+} f(x)$ would be the limiting value from the right.)

(c) f has a removable discontinuity – *Answer:* B, because $\lim_{x\to B} f(x)$ exists, but is different from $f(B)$.

(d) f is continuous – *Answer:* f is continuous at points A, D, E, F (and all the non-marked points too)

4. Compute the indicated limits. Show all work for full credit.

(a) $\displaystyle\lim_{x\to 1} \frac{3x^2-5x-2}{x^2-4x+4}$

Solution: This is not an indeterminate form, and the denominator is not zero at 1, so the answer can be found by continuity:

$$\lim_{x\to 1} \frac{3x^2-5x-2}{x^2-4x+4} = \frac{-4}{1} = -4.$$

(b) $\displaystyle\lim_{x\to 2} \frac{3x^2-5x-2}{x^2-4x+4}$

Answer: This is a 0/0 form. To try to evaluate, we aim to factor the bottom and cancel factors of $x-2$:

$$\lim_{x\to 2} \frac{3x^2-5x-2}{x^2-4x+4} = \lim_{x\to 2} \frac{(x-2)(3x+1)}{(x-2)(x-2)} = \lim_{x\to 2} \frac{3x+1}{x-2}.$$

This is not indeterminate any more, but the bottom is still 0 at $x = 2$. So this limit does not exist. (The function has an infinite discontinuity at $x = 2$.)

(c) $\displaystyle\lim_{x\to \infty} \frac{3x^2-5x-2}{x^2-4x+4}$

Solution: For this one, we multiply the top and bottom by $\frac{1}{x^2}$:

$$\begin{aligned} lim_{x\to\infty} \frac{3x^2-5x-2}{x^2-4x+4} &= \lim_{x\to\infty} \frac{(3x^2-5x-2)\frac{1}{x^2}}{(x^2-4x+4)\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{3-\frac{5}{x}-\frac{2}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}} \\ &= \frac{3-0-0}{1-0+0} = 3. \end{aligned}$$

(d) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \lim_{x\to 2}\frac{4-x^2}{4x^2(x-2)} = \lim_{x\to 2}\frac{(2-x)(2+x)}{4x^2(x-2)} = \lim_{x\to 2}\frac{-(2+x)}{4x^2} = \frac{-1}{4}.$$

(e) $\lim_{t\to 0}\frac{\sin(6t)}{\sin(7t)}$

Solution: For this one, we need to use the formula $\lim_{u\to 0}\frac{\sin(u)}{u}=1$ from Section 2.6. We have

$$\lim_{t\to 0}\frac{\sin(6t)}{\sin(7t)} = \lim_{t\to 0}\frac{6\cdot\sin(6t)/(6t)}{7\cdot\sin(7t)/(7t)} = 6/7.$$

(f) $\lim_{h\to 0}\frac{\sqrt{h+9}-\sqrt{9}}{h}$

Solution: Multiply top and bottom by the conjugate radical:

$$\lim_{h\to 0}\frac{\sqrt{h+9}-\sqrt{9}}{h} = \lim_{h\to 0}\frac{(\sqrt{h+9}-\sqrt{9})(\sqrt{h+9}+\sqrt{9})}{h(\sqrt{h+9}+\sqrt{9})} = \lim_{h\to 0}\frac{(h+9)-9}{h(\sqrt{h+9}+\sqrt{9})} = \lim_{h\to 0}\frac{h}{h(\sqrt{h+9}+\sqrt{9})} = \lim_{h\to 0}\frac{1}{(\sqrt{h+9}+\sqrt{9})} = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

5. Suppose you know each of the following conditions. What can you say about $\lim_{x\to c} f(x)$ for the indicated c? Why?

(a) $x^2+x\le f(x)\le x^3+3$ for all real x, at $c=0$.

Solution: $f(x)$ is not "squeezed" in this case because $l(x)=x^2+x$ has the value $l(0)=0$, which is strictly less than $u(0)=3$ for $u(x)=x^3+3$. The limit of $f(x)$ as $x\to 0$ could be any number between 0 and 3, or it might not exist at all.

(b) $-x^2 + 2x \leq f(x) \leq x^4 - 4x^3 + 6x^2 - 4x + 2$ for all real x, at $c = 1$

Solution: Now we have $l(1) = 1$ and $u(1) = 1$. So $f(x)$ is "squeezed" and the Squeeze Theorem says $\lim_{x\to 1} f(x) = 1$ also.

(c) $f(x) = x \sin\left(\frac{1}{x}\right)$ for all real $x \neq 0$, at $c = 0$.

Solution: This one is trickier because we are not given $l(x)$ and $u(x)$. However, note that for all $x \neq 0$, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$. Therefore, $-x \leq f(x) \leq x$ for all $x \neq 0$. Hence the Squeeze Theorem does apply and we can see $\lim_{x\to 0} f(x) = 0$. (This $f(x)$ has a graph that looks like the picture from Exercise 56 B on page 71.)