Returning to the Roots of Mathematics: A Personal Journey

Faculty Scholarship Lunch

John B. Little

Department of Mathematics and Computer Science

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A true story

Soon after I started at Holy Cross, former Science Librarian Tony Stankus pointed out an extraordinary "flame war" carried out mid-1970's in *Arch. for Hist. of Exact Sci.* A sample:

" ... history of mathematics has been typically written by mathematicians ... who have either reached retirement age and ceased to be productive in their own specialties or become otherwise professionally sterile. ... The reader may judge for himself how wise a decision it is for a professional to start writing the history of his discipline when his only calling lies in professional senility."

I agreed(!) Wonderful irony and/or a complete confirmation of the author's point: I've recently gotten interested in the specific issues this particular article dealt with(!) More on that later ...

You can probably sympathize

- That awkward question ... "So, what do you do?"
- Because I'm an *algebraic geometer* by training, often start out by saying, "Well, I'm a mathematician – I study geometric objects defined by certain kinds of algebraic equations, ... "
- "For instance, the *conic sections* (ellipses, parabolas, hyperbolas) all defined by second degree polynomial equations in two variables:

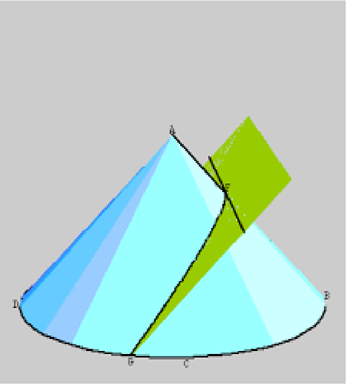
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

  for some coefficients A, B, C, D, E, F."
- "It all goes back to the ancient Greeks ... "

My own changing interests

- Of course, in this form, this uses key insights of *René Descartes* leading to *analytic* (i.e. coordinate) geometry – the Greeks did things differently.
- But at some point, I started to wonder: *Exactly how did* ancient Greek mathematicians such as Apollonius really think about the conic sections?
- Return to the roots! Goal: Read Apollonius (and Descartes) but not filtered through modern "interpretations," "explanations" and translations
- I had to learn some ancient Greek ... lucky I'm at Holy Cross with our world-class and wonderfully welcoming Classics department!

Context – previous work on conics



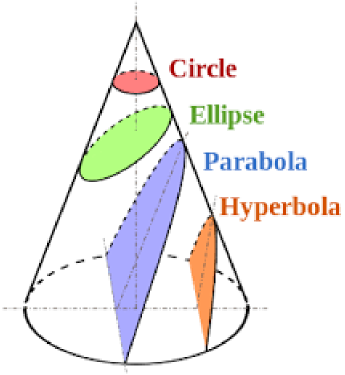
(Image by H. Mendel, Cal. State LA)

- Menaechmus (ca. 380 - 320 BCE; in Plato's circle) often credited with discovery
- Developed by Aristaeus (pre-Euclid) and Euclid (ca. 300 BCE)
- Known only through later commentaries and works by Archimedes; e.g. the parabola as a "section of a right-angled cone"

Apollonius of Perga, ca. 262–190 BCE

- Active roughly 75 - 100 years after time of Euclid (ca. 300 BCE); slightly younger than Archimedes (ca. 287–212 BCE); studied in Alexandria
- Have lists of his works from later commentaries, but most have not survived
- The *Conics*: Books I, II, III, IV – "Elements of conics" – survive in Greek versions.
- Book V, VI, VII – "Researches on conics" – only known in Arabic
- Book VIII – ? (lost – several attempts at "reconstruction" including one by E. Halley, 1710 CE)

Apollonius's framework



(Image from Wikipedia, "Conic Sections")

- Conic surfaces generated by lines through a vertex and points on a circle
- All conics are obtained by slicing any one conic surface by different planes.
- Get plane curves, but the construction inherently uses geometry in 3 dimensions(!)

Book I Proposition 11 – definition of the parabola (R.C.Taliaferro's translation)

If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if, further, the diameter of the section is parallel to one side of the axial triangle, and if any straight line is drawn from the section of the cone to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone's base, then this straight line to the diameter will equal in square the rectangle contained by (a) the straight line from the section's vertex to where the straight line to the diameter cuts it off and (b) another straight line which has the same ratio to the straight line between the angle of the cone and the vertex of the section as the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola.

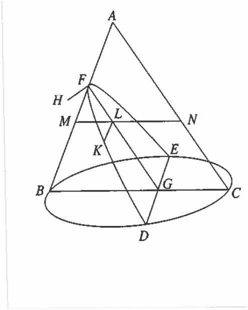
Comments

- Apollonius' prose is verbose, complicated in syntax, *and mathematically dense* – a "hard slog!"
- J. Kepler: "If anyone thinks that the obscurity of this presentation arises from the perplexity of my mind, ... I urge any such person to read the *Conics* of Apollonius. He will see that there are some matters which no mind, however gifted, can present in such a way as to be understood in a cursory reading. There is need of meditation, and a close thinking through of what is said."
- Descartes, from *La Géométrie*: " *... je vous prie de remarquer en passant que le scrupule que faisoient les anciens d'user des termes de l'arithmétique en la géométrie ... causoit beaucoup d'obscurité et d'embarras en la façon dont ils s'expliquoient ...* ""

Structure of a Euclidean or Apollonian proposition

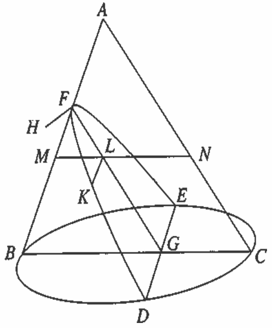
- Typically, there are 6 “parts” – *protasis*, [diagram and] *ekthesis*, *diorismos*, *kataskeuē*, *apodeixis*, *sumperasma*
- The supremely complex and convoluted first sentence above is the *protasis* of this proposition – the “statement”
- The *ekthesis* then “lays out” the statement by means of a figure and the sort of labeling of important points with letters familiar from school geometry (a *lifesaver* for the reader!)

Ekthesis of Proposition 11, beginning (my condensed translation)



Let A be the vertex, ΔABC the axial triangle, and let the other plane cut the plane of the base in DE perpendicular to BC so that FG is *parallel to AC*. The section is the curve DFE.

Conclusion of *ekthesis* and *diorismos* of Proposition 11



Let H be “contrived so that”

$$(*) \quad sq.BC : rect.BA, AC :: FH : FA$$

Finally let K be taken at random on the section and let KL be parallel to DE.

I say that $sq.KL = rect.HF, FL$.

A word on terminology and notation

- The notation here is Taliaferro's modern attempt to capture Apollonius in a (more) readable way
- The Greek is highly conventionalized and abbreviated
- Here $sq.XY$ means (the area of) the square with side XY (Apollonius just says *to apo XY*: literally "the from XY")
- $rect.XY, YZ$ stands for (the area of) the rectangle with sides XY and YZ (*to upo XYZ*: literally "the by XYZ")
- The : and :: are standard notation for comparing proportions that may be familiar from analogies

Historical comments

- This fact about parabolas was *certainly not discovered by Apollonius*
- Archimedes' *Quadrature of the Parabola*, for instance, states something very close, precisely – if KL and $K'L'$ are *two* such segments, then

$$sq.KL : sq.K'L' :: FL : FL',$$

  and it's given *without proof*. Usual interpretation (almost certainly correct): Archimedes is relying on a standard reference, probably the lost Euclid *Conics*
- What seems to be new in Apollonius is the use of this property to *define* a parabola, *and* the names we use for this curve and the other conic sections

The names “parabola,” “hyperbola” and “ellipse”

- Greek mathematical terminology often “borrowed” common words and gave them special meanings.
- Apollonius did this for the conics (following earlier work in a different context – “application of areas”)
- *parabolē* – noun: a “throwing alongside,” comparison, juxtaposition
- *hyperbolē* – noun: a “throwing beyond,” excess, superiority
- *elleipō* – verb: to fall short, be in want of, lack

What's in a word?

- The Greek phrase Apollonius uses for the segments *KL* before deserves special consideration: "*tetagmenōs* to the diameter" – literally means something like *"lined-up"* or *"in order"*, or perhaps even "drawn *in an orderly fashion*" (from the section to its diameter)
- OTOH, many standard English translations of Apollonius (e.g. Heath, Taliaferro, ...) say those parallels have been drawn *"ordinatewise"* to the diameter.
- Interestingly enough, the entry in the standard LSJ Greek lexicon for *tetagmenōs* gives the common meaning and then "ordinatewise" with a reference to Definition 4 in Book I of Apollonius(!) My guess: some mathematical historian *(maybe T.L.Heath?)* provided this citation to the compilers of the lexicon(!)

tetagmen$\bar{o}$s to “ordinatewise?”

- First Latin translation of Apollonius to circulate widely in Western Europe by Federigo Commandino (1509-1575 CE); then several others too, including one by E. Halley (1656-1742 CE).
- Commandino’s Latin rendering: *ordinatim applicatae* – pretty literal version of the *everyday Greek meaning of the word – “applied in an orderly fashion.”*
- Halley (also Latin) has something equivalent; later uses the word *abscissae* for distances along the segments “cut off” by the diameter and the section.
- Note slightly old-fashioned analytic geometry terminology: “abscissas and ordinates” are x and y coordinates(!) Was Apollonius was thinking in coordinate terms after all??

Apollonius' "equation of a parabola?"

- Recall the *diorismos* of Apollonius' Proposition 11: "I say that $sq.KL = rect.HF, FL$."
- Segments like KL are said to be drawn "ordinatewise" (mis?)reading Commandino
- If we write the "ordinates" $y = \overline{KL}$ and "abscissas" $x = \overline{FL}$, then noting that HF is a fixed segment of length c, say, we get the "sideways" parabola $y^2 = cx$ (and c corresponds to the length HF – called the *orthia pleura* or "upright side" in Apollonius – "latus rectum" later)
- Can get analogous statements for the hyperbola and ellipse as well!

Apollonius "interpreted" for modern mathematicians

- H. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, first ed. 1886
- T.L. Heath's version of Apollonius, first ed. 1896 – "*so entirely remodelled by the aid of accepted modern notation as to be thoroughly readable by any competent mathematician*" since it "*does not essentially differ from ... modern analytic geometry except that in Apollonius geometrical operations take the place of algebraical calculations*"
- C. Boyer (1906-1976) "*The work of Apollonius in many respects approaches so closely to the modern form of treatment that it not infrequently has been regarded as constituting analytic geometry*."

A contrary view

- S. Unguru (1931-present), “On the need to rewrite the history of Greek mathematics,” Archive for History of Exact Sciences 15 (1975/76), 67–114 (the article my introductory quotation came from!)
- Unguru later forcefully refutes prevailing mathematical historiography of the era of Zeuthen and Heath and the use of algebraic reformulations to explain Apollonius in a 2001 book with M. Fried
- Unguru’s main point: it’s geometry pure and simple; Greek mathematics did not have any of the apparatus of symbolic algebra or coordinates

Unguru's point, expanded

- "Explaining" Apollonius this way (and other similar "reconstructions" of Greek mathematics using modern concepts from the 19th and early 20th centuries) is *perniciously wrong* from the historical point of view
- A false description of a fundamentally different understanding of mathematics
- *Conceptual anachronism* or "Whig history" – presents the past as leading inevitably to the present
- Apollonius (following Euclid) *never* uses a numerical value as a measure of length or area – coordinate equations "don't compute" in this context

What can we learn from this?

- May seem like a minor thing
- But it points out a fundamental difference between doing mathematics and doing history of mathematics (as history)
- Recognizing logical connections between old and new work and making reinterpretations is a part of what mathematicians do.
- When apparently different things are logically the same, just expressed in different ways, mathematicians can and do treat them as the same(!) And we are always looking for those equivalences–finding them can represent an advance in our understanding!

And maybe Unguru had a (small) point?

- As Unguru insinuated in his own nasty way, Zeuthen, his “flame war opponents” van der Waerden, Freudenthal, Weil, etc. were certainly all primarily mathematicians who had eminent research records and then turned to writing history later in their professional careers
- Not surprising that they had the “habits of mind” and point of view of working mathematicians, not historians!
- In particular, to put words in their mouths: “if it’s logically equivalent to a coordinate equation of a parabola, but expressed in geometric terms, then it’s still essentially a coordinate equation”

The “take-home” message

- For intellectual historians, not so much logical equivalences that matter–it’s particular features, differences! Each culture, era, scientific school, etc. is a unique and separate thing
- Unguru: The mathematical historian’s first and most important job is to understand a body of mathematical work on its own terms, *not on our terms*
- A fundamentally different way of thinking
- The title of a recent article by K. Saito: “Mathematical Reconstructions Out, Textual Studies In” summarizes what’s up in mathematical historiography these days!

To be clear ...

- In a recent article, "Apollonius, Davidoff, Rorty, and Zeuthen: From A to Z, what else is there?" (Sudhoffs Archiv, 91 (2007), 1 - 19), Unguru and Fried make it clearer that their "issues" concern Zeuthen's work *qua history*, not *qua* mathematics
- and they contrast Zeuthen's well-intentioned and mathematically astute (mis)reading with a parodied "post-modern," deconstructionist view that would deny *any* intrinsic meaning in a text
- Make their point via a (hilarious, fictional) "sexual politics reading" of Apollonius. (Recall the *orthia pleura*? It's *phallic*; you get the idea!)

Conclusions

- The *Conics* is a masterwork (especially the later books), but *it's not* coordinate geometry (that didn't exist yet and he *never* uses coordinates on the whole plane!) But, of course, his work (together with summaries, commentaries) *was* read very carefully by Descartes and others and stimulated the development of analytic geometry(!)
- I think the standard histories of mathematics still seriously misrepresent a lot of this and that does a disservice to teachers and students; I want to try to participate in an active way and learn more about this past.
- Thanks for your attention and I'd be happy to take questions!