HISTORIA MATHEMATICA 3 (1976), 21-49

SHIFTING THE FOUNDATIONS: DESCARTES'S TRANSFORMATION OF ANCIENT GEOMETRY

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SUMMARIES

The aim of this paper is to analyse how the bases of Descartes's geometry differed from those of ancient geometry. Particular attention is paid to modes of specifying curves of which two types are distinguished - "Specification by genesis" and "Specification by property". For both Descartes and most of Greek geometry the former was fundamental, but Descartes diverged from ancient pure geometry by according an essential place to the imagination of mechanical instruments. As regards specification by property, Descartes's interpretation of the multiplication of (segments of) straight lines as giving rise to a straight line (segment), together with newer methods of articifical symbolism, led to more concise and suggestive modes of representation. Descartes's account of ancient procedures is historically very misleading, but it allowed him to introduce his own ideas more naturally.

Ce mémoire a pour but d'analyser comment les fondements de la géométrie de Descartes différaient de ceux de la géométrie antique. Une attention particulière est donnée aux modes de la spécification de courbes, dont deux genres sont distingués - "spécification par genèse" et "spécification par propriété". Pour Descartes et pour la plupart de la géométrie grecque, c'était le premier genre qui était fondamental, mais Descartes a divergé de la géométrie pure des anciens en accordant à l'imagination d'instruments mécaniques un rôle essentiel. En ce qui concerne la spécification par propriété, l'interprétation avancée par Descartes de la multiplication de lignes droites (segments de telles lignes) comme produisant (un segment d') une ligne droite, ainsi que des méthodes plus récentes de symbolisation artificielle, a abouti à des modes de représentation plus concis et plus suggestifs. La des-

cription de Descartes des procédures antiques est, historiquement, très trompeuse, mais elle lui a permis d'introduire ses idées propres d'une façon plus naturelle.

INTRODUCTION

The mathematical work of Descartes is rather an enigma. It was small in volume, but had great subsequent influence. This suggests that it contained something radically new. But to characterise this exactly has not been easy. A conventional view has been to say that Descartes was inventor (or co-inventor) of "analytical geometry". But, for various reasons, this is not satisfactory. One difficulty is that there has been a terminological change, and the use of algebra in geometry has come to usurp the term "analytical geometry". But even at the beginning of the last century John Leslie [1832] could speak favourably of the purely geometrical analysis of the ancients [on which see Mahoney 1968] as opposed to the algebraic analysis of the moderns. Moreover, although they were willing to quarrel over other matters [see e.g. Mahoney 1973, 57-60, 170-195], Descartes and Fermat, as Milhaud [1921, 136-141] sagely noted, did not see any need to contest priority over a new form of geometry; and indeed many historians [see e.g. Coolidge 1963, 117-122; Schramm 1965, 89-97; Zeuthen 1966, 192-215] have been able to emphasise how strong were the ancient roots of "analytical geometry". It may seem that these difficulties in characterisation could be eliminated by some tactic such as calling Descartes's achievement the "arithmetization of geometry", but, as Boyer [1959] has astutely argued, it can as appropriately be labelled the "geometrization of algebra". All this suggests the need to probe more deeply.

My aim in this paper is to contribute towards a clarification of the nature of Descartes's work by isolating certain fundamental differences between his geometry and that of Greek antiquity, and so I shall put a strong emphasis on assessing him on the basis of what had gone before. My particular focus will be on problems associated with the specification of geometric curves, and this demands a few general remarks. The geometer communicates with words and other artificial symbols. His diagrams are intended as no more than an aid to comprehension. Thus if a geometer wishes to speak of a particular curve, he must be able to characterise it by means of verbal or other symbols. I shall call a unique characterisation of a curve (which may or may not be treated as a definition) a specification of the curve. No more than a finite number of symbols may be used, and we meet a type of continuum problem, for not every curve imagined as if drawn with a "free movement of the hand" is susceptible of such specification [1]. The modes of specification may be continually extended and modified, but the process can reach no final completion. We must also remember that the existence of different modes of specification means that each mode determines its own range of specifiable curves.

We may compare the situation with ratios. The ratio between the diagonal of a square and its side is specifiable in those very terms (provided we have reason for regarding it as invariant). In modern mathematical language the same ratio is specifiable as $\sqrt{2}:1$. In late medieval language it was specifiable as *medietas duplae proportionis* (half the double ratio) [Molland, 1968, 117-119]. But it is not specifiable as the ratio between two natural numbers, and hence is called irrational. In Greek discussions of incommensurable quantities we meet the terms ἄλογος and ἄρρητος, both of which may be translated as "inexpressible", and in the thirteenth century both Campanus and Roger Bacon regarded the ratio between incommensurable quantities as known "neither to us nor to nature" [Molland, 1968, 116].

It is relatively easy to lay down criteria for different kinds of specification of ratios, but with curves the situation is more complicated. We shall find in both Descartes and the ancients a primary distinction between different modes of specification. We may speak of this as the distinction between *specification by property* and *specification by genesis*. Specification by property lays down a property (usually a quantitative property obeyed by all the points of the curve) which suffices to determine the curve. In Descartes this has characteristically the form of an equation. Specification by genesis determines a curve by saying how it is to be constructed. Specifications of this kind run up against the problem of what types of construction were regarded as acceptable at a given time. In what follows I shall consider the different types and roles of these two principal forms of specification in the geometry of Greek antiquity and in that of Descartes.

I. ANTIQUITY

We are faced with many difficulties in analysing the ancient Greek procedures. The number of writers involved is not small, and often because of the loss of their works we have to view them through the eyes of reporters, who may not always give an accurate presentation of individual nuances. We must therefore be on our guard against assuming a single monolithic view in all details, even if there is a basic invariant core running through all the Greek writings. Further, the issues in which we are interested are little analysed in extant Greek writings. We have therefore mainly to attempt to identify what was implicitly assumed rather than to isolate explicit statements. The explicit statements that were made often came from those whose primary interest was more philosophical than mathematical, and we should not be surprised to find differences between what the mathematicians actually did and what philosophers said was proper to their discipline. This fact has sometimes been obscured in the historiography of Greek mathematics, and its neglect is abetted by the fact that the early parts of Euclid's *Elements* conform more nearly to certain philosophical dicta than do its own later parts, or other geometrical works. It is as if care were taken to show how the most elementary and basic parts of geometry could be made philosophically acceptable while leaving greater latitude to the mathematician to follow his own intuitions in the higher reaches.

In our discussions of Greek geometry we shall have to be alive to the distinction between geometry and mechanics, and more particularly, as pertaining to constructions, the distinction between the geometrical (γεωμετρικός) and the instrumental (ὀργανικός). Contrary to what seems often to be assumed, references to instruments (including ruler and compasses) did not form part of Greek pure geometry. But constructions were certainly used, and indeed, as we shall see, formed the basis for the definitions of most curves. Thus the allowable modes of construction were a principal determining factor of what was admitted into Greek geometry. No explicit canonisation of admissible modes is extant, and so we shall have to try to identify the criteria from what was actually done. This will give us a fairly firm if not precisely delimited idea of what was allowable. We shall then consider some of the ways in which instrumental constructions infiltrated into geometrical contexts (although still not themselves being regarded as geometrical). We shall then return to pure geometry and focus on some of the roles of specification by property.

Before considering in detail geometrical construction, we must touch on one problem that has sometimes seemed serious, but which it is important not to exaggerate. This is the question of the place of motion in geometry. The *locus classicus* for the difficulty is a short passage from Plato's *Republic* [VII.9, 526C-527B], where Socrates, after admitting the incidental uses of geometry in warfare, insisted that its higher aim was the study of being rather than mere becoming. But this seemed to be belied by the geometers' talking as if they were doing something, such as squaring, applying or adding. This may be read as casting some doubt upon the propriety of constructions in geometry. But at that time, at least, constructions were necessary for geometry, and even if ontologically its objects were exempt from becoming, some element of becoming was necessary for geometric epistemology. Thus apparently Speusippus (Plato's nephew) and others resolved the dilemma by regarding constructions as processes of understanding "taking eternal things as if they were in the process of coming to be" [Proclus, 1873, 77-78; 1948, 69-70; cf. Aristotle, *De coelo* I.10, 279b32-280a12 and Becker 1927, 198] and much later in time Proclus [1873, 78-79;

1948, 70] spoke of human ideas shaping intelligible matter in the understanding [2].

Nevertheless the emphasis on the immovable nature of geometrical objects seems to have had some effect upon geometry, and in the early part of the *Elements* Euclid seems wary of using ideas of motion [Euclid 1956, 1, 224-228; Mugler 1948, 58-59]. This could square with Proclus's assertion that Euclid was a Platonist [Proclus 1873, 68; 1948, 61-62], but perhaps more plausibly it should be attributed to his borrowing from earlier sets of Elements. In the first book of Euclid's *Elements* the straight line and the circle are defined by property rather than by genesis [Euclid 1956, 1, 153]. "A straight line is a line which lies evenly with the points on itself." "A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another." But Euclid has to supplement these definitions by postulates laying down when straight lines and circles may be constructed, or, in Zeuthen's [1902, 98-100] interpretation, when they exist. "Let the following be postulated: To draw a straight line from any point to any point. To produce a finite straight line continuously in a straight line. To describe a circle with any centre and distance" [Euclid 1956, 1, 154]. These definitely seem to involve some kind of motion, and Proclus [1873, 185-187; 1948, 162-164] grasps the nettle firmly and grounds them in motions in the imagination. There were also in Antiquity definitions of the straight line and circle that explicitly appealed to motion. The circle was defined in terms of the rotation of a straight line about one of its extremities and the straight line as that which remained fixed on rotation when two points remained fixed [Euclid 1956, 1, 184, 168]. Hints of the latter definition can even be found in Plato's *Republic* [Mugler 1948, 26-27].

Despite the prominence of definition by property in the first book of Euclid's *Elements*, genetic definition was elsewhere the norm [3]. This applies even to Euclid, for in the eleventh book of the *Elements* he defines the sphere, cone and cylinder by rotations of, respectively, a semicircle, a triangle and a rectangle [Euclid 1956, 3, 261-262]. Outside Euclid's *Elements*, spires arose from the rotation of a circle about an axis in its plane but not passing through its centre [Proclus 1873, 119; 1948, 108]. Conic sections, as the name implies, were regularly defined as the sections of cones by planes, and Perseus investigated the figures arising from sections of spires [Proclus 1873, 111-112, 119; 1948, 101-102, 108]. Archimedes [1910-1915, 1, 246-255; n.d. 99-102] defined conoids and spheroids by rotating conic sections about their diameters. Archytas produced a "certain curve" from the inter-section of the circumference of a revolving semicircle with the surface of a half cylinder [Eutocius 1915, 85-89; Thomas 1957, 1, 284-287].

In the above instances the rotations involved were about an axis. Apollonius [1891-93, 1, 6-7; Thomas 1957, 2, 284-286; cf. Apollonius 1961, 1] gave a more general definition of the cone (to include oblique cones) in terms of the rotation of a straight line which passes through a fixed point and moves around the circumference of a circle (whose plane does not include the point). Serenus [1896, 1-5; 1969, 1-2] was careful by analogy to extend the definition of the cylinder to include oblique cylinders. He used two equal and parallel circles and had in each rotating parallel diameters. The line joining corresponding ends of the diameters produced the surface of the cylinder.

In a trivial sense Serenus's definition of the cylinder involved two simultaneous motions that had to be correlated in some way. Other definitions involved simultaneous motions in a non-trivial sense. Later, as we shall see, Descartes would reject such curves from geometry, but the ancients seem to have had no compunction about admitting them. The spiral was of this kind, being defined by Archimedes [1910-15, 2, 44-45; n.d. 165; Thomas 1957, 2, 182-183] in terms of the uniform motion of a point along a straight line, which itself was uniformly rotating. The definition of the cylindrical helix given by Proclus [1873, 105; 1948, 95] is from the uniform motion of a point along a straight line that is moving round the surface of a cylinder. Eudoxus's generation of the hippopede [Thomas 1957, 1, 412-415] had involved the uniform rotations of two spheres which had to complete their motions in the same time.

The quadratrix of Hippias has similar features, but at least for Pappus was the cause of some difficulties. The generation given by Pappus [1965, 252-253; 1933, 192; Thomas 1957, 1, 336-339] was of this kind. In a square *ABCD, B'C'* moves uniformly from *BC* to *AD*, while remaining parallel to *BC*. In the same time *AE* revolves uniformly about *A* from *AB* to *AD*. Both motions are completed in the same time. Then the intersection of *B'C'* and *AE* traces out the quadratrix. This curve had been applied to the squaring of the circle; but as Sporus [Pappus 1965, 252-255; 1933, 193-194; Thomas 1957, 1, 338-341] had pointed out, a *petitio principii* was involved, for how was the quadratrix to be constructed without knowledge of the ratio of the radius of the circle to a quarter of its circumference? Pappus expounded with approval Sporus's objections, but he himself [1965, 254-255; 258-259; 1933, 194, 197] had another difficulty with the curve, namely the extent to which its genesis was mechanical (he used the adjective μηχανικός rather than ὀργανικός). What he meant by this is not transparently clear, but he seemed happier when he had analysed the curve "by means of the loci on surfaces" in terms of first the cylindrical helix and then the Archimedean spiral [Pappus 1965, 258-265; 1933, 197-201].

The whole business of the "loci on surfaces" is rather obscure [cf. Euclid 1956, 1, 15-16], and the available evidence is scanty. We must therefore beware of drawing too many inferences from Pappus's procedures. It seems clear that Pappus regarded the spiral and the cylindrical helix as having a firmer claim to the status of being geometrical than the quadratrix, which could however receive authentication by being derived from them. The constructions used in the derivation must also have been regarded as having a fairly firm geometrical status [4]. The first derivation is from a cylindrical helix. A plectoidal [cf. Pappus 1933, 198, n.6] surface arises from the motion along the helix of a perpendicular from it to the axis of the cylinder. The section of this surface by a plane that can be determined by a property of the helix produces a curve, and the orthogonal projection of this curve onto the base of the cylinder produces the quadratrix [5]. The production of the quadratrix from the spiral is more complicated. A "cylindroidal surface" is formed perpendicular to the plane of the spiral and passing through the spiral. The intersection of this with a cone gives a curved line. A "plectoidal surface" is formed by the motion of a perpendicular from this line to the straight line through the origin of the spiral and perpendicular to its plane. The intersection of this surface with a plane (determined by the spiral and inclined to its plane at half a right angle) gives rise to another curved line, and the orthogonal projection of this line onto the plane of the spiral gives the quadratrix.

So far as I know, no ancient writer attempted to give a general account of what modes of construction were acceptable in geometry, and it would probably have been impossible to produce a universally agreed codification of the geometer's intuition. But clearly there had to be limits, since otherwise, for instance, a very simple construction could be given for the rectification of the circle. (The imagined motion could be a rolling of a circle.) Our analysis has suggested that restrictions were made to certain simple motions, and the dominant ideas seem to have been those of rotation and of constructing straight lines and planes. The passages from Pappus that we have just examined suggest that in the higher reaches of geometry there may have been standard procedures of constructing cylindroids (generalised "cylinders") from plane curves and "plectoids" from non-planar curves.

Pappus himself gave a rudimentary classification of curves, or more strictly of problems, in terms of the lines used for their solution. This classification, which Pappus attributes to much earlier geometers, is famous and was to be given a prominent place by Descartes. It is in terms of the geneses of the lines used, and the passage that Pappus annexes will serve to introduce us to the place of instrumental constructions in Greek geometrical works:

The ancients held that there were three genera of geometrical problems: some were called plane, some solid, and some linear.

Those that can be solved with straight lines and circumferences of circles are reasonably called plane, for the lines by which these problems are solved have their genesis in a plane. Problems that are solved by the use in their discovery of one or more sections of a cone are called solid, for in their construction it is necessary to use surfaces of solid figures, namely conic surfaces. There yet remains the third genus which is called linear, for lines other than those mentioned are used in the construction, which have a varied and more intricate genesis, such as the spirals, the quadratrixes, the conchoids and the cissoids, which have many marvellous properties.

As there is this difference between problems, the ancient geometers did not construct the aforementioned problem of the two straight lines, which is solid by nature, following geometrical reasoning, because it was not easy to draw the sections of the cone in a plane, but by using instruments they brought it to a manual construction and fit preparation, as is seen in the Mesolabe *of Eratosthenes and the* Mechanics *of Philo and Hero.* [Pappus 1965, 54-57; 1933, 38-39; cf. 1965, 270-271; 1933, 201-208].

From this passage it is clear that Pappus regarded instrumental solutions as being something of a concession to human weakness, or at least to human practical needs. Instrumental constructions were not properly geometrical, but they could indicate how a solution was physically to be performed. The imagination of idealised instruments can give constructions as exact as those of pure geometry, but they did not fit into the canons of Greek geometry, and were strictly regarded as part of mechanics.

In two famous passages of Plutarch we are shown Plato as fulminating against the use of instrumental constructions in geometry [*Quaestiones Conviviales* VIII. 2.1 in 1961, 120-123; *Vita Marcelli* XIV. 5-6 in 1914-26, 4, 470-473]. Although, as van der Waerden [n.d. 161-165] has suggested [6], these passages may derive not from Plato but from a dialogue by Eratosthenes in which Plato was a character, we may be sure that Plato would have wished a definite distinction between geometry and mechanics; and, from wherever they derive, the passages bear witness to the recognition of such a separation. Another explicit reference to the distinction, with hints of the practical man's need of instrumental constructions, may be found in a purported letter of Eratosthenes to Ptolemy Euergetes. The letter (like the other references we have so far considered) concerns the duplication of the cube, which was reduced to the problem of finding two mean proportionals between two straight lines. After reviewing the history of the problem, the writer continued [Eutocius 1915, 90-91; Thomas, 1957, 1, 260-261; cf. von Wilamowitz-Moellendorff 1894]: "It turned out that they all performed it demonstratively (ἀποδεικτικῶς), but they could not do it manually and turn it to use, except to a small extent Menaechmus, and that with difficulty. An easy instrumental solution was, however, found by us, by means of which we shall find, not only two means to the given straight lines, but as many as may be enjoined."

It is in the realm of solid, and to a certain extent linear, problems that the distinction is most apparent, for in these cases there was likely to be considerable divergence between the geometrical and instrumental procedures. In plane problems the distinction certainly existed, but for later historians it has sometimes been obscured by the tantalisingly close analogy existing between Euclid's first three postulates and the operations that can be performed with a straight edge and compasses. But even here, as de Morgan [1849, 6; cf. Euclid, 1956, 1, 246] emphasised, the analogy is not exact, for, in his words, the postulates "do not allow a circle to be drawn with a compass-carried distance; suppose the compasses to close of themselves the moment they cease to touch the paper." Euclid's propositions 2 and 3 of the first book are necessary to make the analogy complete.

We may glean further understanding of how even ruler-and-compass constructions were regarded as instrumental rather than properly geometrical from Book VIII of Pappus's *Collectio*. This is devoted to mechanics, and Pappus included a section on instrumental problems, which he clearly specified as belonging to mechanics. Some of these problems require only a straight edge and compasses, although Pappus does not explicitly specify the instruments. In association with two of them we have some rather enigmatic remarks on the status of instrumental problems:

The so-called instrumental problems in mechanics [are those which] *are deprived of geometrical authorities, such as those described by one interval and that of the cylinder with both bases multilated, which is put forward by the architects* [1965, 1072-75; 1933, 845].

The [problems] *among those which are especially called instrumental are also useful and most of all when, led to something easy by analysis, they can escape the proportionate trial* (πεῖρα) [1965, 1096-97; 1933, 860].

The first of these passages is followed by the problems of finding the diameter of a cylinder with two multilated bases, which Pappus reduces to that of constructing the minor axis of the ellipse that would pass through five given points. No instruments other than the straight edge and compasses are necessary, though the compasses have to be used on the surface of the cylinder. The second passage is followed by a problem in which it is demanded that seven regular hexagons be inscribed in a given circle. Pappus reduces this by analysis to the problem of constructing a certain triangle. He makes the problem more complicated than it need be, but no instruments other than straight edge and compasses are required.

It is not easy to infer much from Pappus's obscure general statements, and possibly he was a little confused himself.

However, it seems clear that he regarded instrumental solutions to geometrical problems as lacking in geometrical rigour, so that in the last resort they could only be justified on the basis of whether they worked in practice. They did not themselves fit into the strict deductive system of geometry, but often geometrical argumentation could produce conviction that they worked. Pappus's main difficulty in characterising their status may well have come from a problem of specifying in general terms the relations between geometry and mechanics.

We may derive even clearer evidence of how constructions with simple, as well as with complicated, instruments were regarded as mechanical from a passage of Book VIII of Pappus's *Collectio* that is only extant in Arabic. When reviewing Hultsch's edition of the *Collectio,* M. Cantor [1879; cf. 1894-1908, 1, 421] suggested that the phrase "those described by one interval" in the first of the passages that we have just quoted referred to problems in which the compasses could only be opened to one interval. This surmise found little favour with W. M. Kutta in his historical study of fixed compass problems [1898, 72-74], but the recently discovered Arabic version of Book VIII favours Cantor's view, for it includes a group of problems in which restrictions are placed on the use of the compasses [Jackson 1970, 63-73, A43-A51; cf. 1972]. There are reasonable grounds for attributing the passage to Pappus, or at least for assuming that the writer put the same interpretation on "described by one interval" as Pappus. The explicit restriction is that there "is a given distance which must not be exceeded when we draw circles", and the author later rephrased this in more definitely instrumental terms by saying that "we have only one small pair of compasses with which to work". But in fact, except for one problem, only one arbitrary opening of the compasses is required, and the exception would be easy to obviate. The author seems to have realised this, for at times he announced that only one distance had been used.

The broad distinction between geometrical and instrumental procedures is clear, but puzzles can arise in some cases, mainly it seems through lack of extant evidence. In particular we must consider the cases of the conchoid and the cissoid. It is clear that Pappus [1965, 54-55, 270-271; 1933, 38-39, 207-208] and Proclus [1873, 111, 128, 356; 1948, 103, 116, 304] had few qualms about accepting these as properly geometrical, and yet in both cases the accounts that survive are tinged with the instrumental. The conchoid was the invention of Nicomedes, but his original work is lost and we have to rely on reports by Pappus and by Eutocius. Pappus's account [1965, 242-245; 1933, 185-186; Thomas 1957, 1, 298-301] of its generation is in the following manner [7]. *AB* is a straight line and *E* a point not on it. Another straight line *CD* moves in such a way that *D* is always on *AB* (i.e. *CD* is constant) and *CD* produced always passes through *E*. The point *C* traces out the conchoid. Pappus [1965, 244-247; 1933, 187; Thomas 1957, 300-301] remarks that Nicomedes showed that the curve could be constructed instrumentally, and Eutocius [1915, 98-101] gives only an instrumental construction. This makes use of slotted rulers and pegs, and parallels exactly the genesis given by Pappus. Thus it seems that in this case Pappus may have regarded this analogue of an instrumental construction as geometrical even if it fitted rather loosely into the more usual criteria [8].

The case of the cissoid is less problematic, for Diocles' construction as reported by Eutocius [1915, 67-71; Thomas 1957, 1, 270-279] is solely instrumental and does not even effect a complete genesis of the curve [9]. Let *AB* and *CD* be perpendicular diameters of a circle. Mark off equal arcs *EB, BZ,* with *E* on the side of *C* and *Z* on the side of *D*. Drop the perpendicular *ZH* onto *CD*. The intersection of *ED* and *ZH* is a point of the required curve. "If in this way more parallels are drawn continually between *B, D,* and arcs equal to the arcs cut off between them and *B* are marked off from *B* in the direction of *C*, and straight lines are drawn from *D* to the points so obtained..., the parallels between *B* and *D* will be cut in certain points.... Joining these points with straight lines by applying a ruler we shall describe in the circle a certain curve." Both the reference to the ruler and the "construction" of the curve by joining points with straight lines make clear the instrumental orientation, which in any case would be expected in a work entitled, as Diocles's was, *On burning mirrors*. Diocles in fact has only given a method of constructing an arbitrary number of points on the curve and not a method of constructing the curve itself. We may suspect that a more acceptable geometrical construction (perhaps using two simultaneous motions) was discovered later, for Proclus [1873, 113; 1948, 103] reports that Geminus taught the genesis of cissoids as also of spirals and conchoids [10], but unfortunately we do not know his method.

Diocles's "construction" of the cissoid is easily translatable as laying down a property that each point of the curve must obey. To this extent we may regard it as leading to a specification by property of the cissoid. But, as we have seen, such specifications were not usually regarded as definitions, and would in any case need to be supplemented by an existence postulate or proof. Nevertheless specifications by property did play an important part in Greek geometrical methods, and we must now return definitely to the realms of pure geometry in order to explore some of their roles. We shall first see how certain essential properties were referred to curves, and then consider the class of locus problems and theorems.

When discussing parallel lines. Proclus referred to essential properties belonging to them as such (such as the equality of the alternate angles when the parallels were cut by a straight line). Such properties were unique to parallel lines and convertible with their definition. He added [1873, 356; 1948, 304]:

> *In this way also other mathematicians were accustomed to discourse on lines, giving the property (σύμπτωμα) of each species. For Apollonius showed for each of the conic lines what its property was, and Nicomedes likewise for the conchoids, Hippias for the quadratrices, and Perseus for the spirics. For after the genesis, the apprehension of the essential* [property] *belonging* [to it] *as such* [11] *differentiates the species constructed from all others.*

The example of conic sections may show us how important the establishment of such a property was, for it would clearly be very tedious to have to refer back to the original cone for each theorem. Thus early in his work Apollonius [1891-93, 1, 36-53; 1961, 8-12] produces particular planimetric properties for each of the three species of conic sections. The simplest case is the parabola. Suppose PM is a diameter (with P on the parabola) and PL the corresponding latus rectum. Then for any ordinate QV to the diameter PM, the square on QV is equal to the rectangle formed from PV and PL. Later Apollonius [1891-93, 1, 158-165; 1961, 42-43] will show essentially how any curve with such a property is a parabola. Thus this specification by property is unique to the parabola. We may see from the form of this planimetric property (and the corresponding ones for the ellipse and the hyperbola) how easy it is to read coordinate geometry back into ancient works, and this part of Descartes's method certainly had firm ancient roots.

Related to the establishment of such properties was the class of locus problems and theorems, although in this case the emphasis was on arguing from the property to the curve. Proclus [1873, 394; 1948, 337] defines a locus (τόπος) succinctly as "a position of a line or a surface producing one and the same property." We frequently meet with propositions of the form that, when certain properties of a point or line are given, that point or line is uniquely given in position [12]. But there were also problems without a unique solution of this kind, where, for instance, the required point could lie anywhere on a certain line, so that the line was the place or locus of the point, and it was to cases of this kind that the term "locus" was most frequently applied. Pappus, who had many relevant and no longer extant sources available to him, and is here drawing at least partially on Apollonius's *Plane loci,* divides loci into three classes [1965, 660-663; 1933, 495-496]. Some are ἐφεκτικοί, when a point is the locus of a point, a line of a line, or a surface of a surface. Other loci are διεξοδικοί, when a line is the locus of a point, or a surface of a line, or a solid of a surface. Finally others are ἀναστροφικοί, when a surface is the locus of a point, or a solid of a line. The most common form was διεξοδικοί loci of points. Pappus subdivided these into plane, solid and linear loci, in a way similar to that in which he had classified problems. Plane loci comprise straight lines and circles; solid loci comprise conic sections; all other loci are linear [13].

A complete locus theorem must include a proof that the point will lie on the given locus and also a proof that any point on the locus satisfies the given condition. We find both proofs in a proposition that Eutocius [1893, 180-185; cf. Heath 1949, 181-188] is apparently copying from Apollonius's *Plane loci*, to the effect that if two points are given and also the ratio between two unequal straight lines, then the locus of a point whose distances from the two given points is in the given ratio is a circle. Pappus [1965, 662-671; 1933, 496-501] gives an account of the contents of this work of Apollonius, and it is possible to see from this some rationale (besides that of their genesis) for treating the circle and straight line as forming a single class, for one general proposition in particular can be looked upon as dealing with transformations of circles or straight lines into circles or straight lines [Steele 1936, 358-360]. But in general we know very little of ancient locus procedures, and the reports on Aristaeus's *Solid loci* and Euclid's *Loci on surfaces* are even more sketchy than those on Apollonius's *Plane loci*. It is nevertheless clear that such inferences "from property to place" were an important part of Greek geometrical activity.

Let us now summarise. Many may have argued that geometrical definition should be by property rather than by genesis (for example, Eutocius [1893, 186-187] did not regard Apollonius's genetic definition of the cone as a definition), but in fact, despite the privileged position of definition by property in the first book of Euclid's *Elements*, genetic definitions of incomposite lines and surfaces tended to be the general rule. These, together with the constructions regularly used in geometry, demanded the imagination of certain simple motions. There seems to have been little explicit discussion of what motions were allowable, but rotations (understood in a broad sense) and the construction of straight lines and planes predominated. There was no ban on the use of two simultaneous motions.

Although genetic definitions were usually regarded as more basic, there was considerable interest in properties that could give a unique specification of a curve or surface, and usually these were in the form of some quantitative relation that had to be obeyed by all the points of the curve or surface. For the conic sections much use was made of fundamental planimetric properties, and the class of locus theorems dealt with what curves answered to what properties of points. Through all this, classification of curves was based on the mode of their genesis rather than on any properties that they possessed.

In pure theoretical geometry there was no mention of instruments. But practically-oriented solutions to geometrical problems could be given by specifying how certain instruments were to be employed. These were regarded as lacking some of the rigour of pure geometry, and in fact as belonging to mechanics rather than to geometry. Frequently both geometrical and instrumental solutions could be given to the same problem, and compilers often placed them next to each other. The distinction is usually quite clear, but a cursory reading could misconstrue it. And in fact it has often been blurred or misinterpreted, and in particular by Descartes.

II. DESCARTES

We may now confront Descartes's version of much of what we have treated. We may best start with an extended quotation from the beginning of Book II of the *Géométrie*, which is entitled *On the Nature of Curved Lines* [14]:

The ancients have well remarked that, among the problems of geometry, some are plane, others solid and others linear, that is to say, that some can be constructed by tracing only straight lines and circles, while others can only be constructed by using, at the least, some conic section, and finally others only by using some other more compounded line. But I am surprised that they did not beyond this distinguish different degrees among these more compounded lines, nor can I understand why they named them mechanical (mechaniques) *rather than geometrical. For in order to say that this was because there is need to use some machine for describing them, it would be necessary for the same reason to reject circles and straight lines, seeing that one only describes these on paper with compasses and a ruler, which one can also name machines. No more is it because the instruments used for tracing them, being more compounded than the ruler and compasses, cannot be so accurate* (si iustes); *for for that reason it would be necessary to reject them from mechanics, where the accuracy of works that issue from the hand is more desired than in geometry, where it is only accuracy of reasoning that is sought, which without doubt can be as perfect regarding these lines as regarding the others. Neither will I say that it was because they did not wish to increase the number of their demands, and were content to be granted that they could join two given points by a straight line and describe a circle with a given centre which would pass through a given point; for they made no scruple about further supposing, in order to treat the conic sections, that one could cut each given cone by a given plane. And, in order to trace all the curved lines that I intend to introduce here, there is only need to suppose that two or more lines can be moved, one by another* (l'une par l'autre) *and that their intersections mark out other* [lines], *and this does not seem to me at all more difficult. It is true that they did not entirely receive the conic sections into their geometry, and I do not wish to undertake to change names that have been approved by usage, but it is, it seems to me, very clear that, taking as one does for geometrical that which is precise and exact* (precis & exact) *and for mechanical that which is not, and considering geometry as a science that teaches generally how to know the measures of all bodies, one must no more exclude the more compounded lines than the more simple, provided that one can imagine them to be described by a continuous movement or by several which follow one another and of which the later are entirely determined* (entierement reglés) *by those which precede. For by these means one can always have an exact knowledge of their measure. But perhaps what prevented the ancient geometers from admitting* [curves] *that were more compounded than the conic sections was that the first that they considered happened to be (*lit. having by accident been*) the spiral, the quadratrix and such-like, which truly belong only to mechanics and are not of the number of those that I think must be received here, because one imagines them described by two separate movements, which have no ratio between them that one can measure exactly, although they* [the geometers] *afterwards examined the conchoid, the cissoid, and some few others which are among those* [to be received], *but because they perhaps did not sufficiently remark their properties, they made no more of them than the first. Or perhaps it was that, recognising that they still knew little regarding the conic sections, and that there even remained much they they did not know regarding what could be done with the ruler and compasses, they feared having to enter upon a more difficult matter.*

The fundamental error in this passage is the misconstrual of the ancient distinction between geometrical and instrumental constructions. This leads Descartes to hold that curves higher than the conic sections were regarded as mechanical rather than geometrical, and he compounds this error by suggesting (very oddly for someone who knew Apollonius's work) that even the conic sections were not fully accepted. Having produced this analysis, Descartes sets himself the task of making it intelligible, and produces further misleading statements. His first explanation suggests a ruler-and-compasses restriction in ancient works, and, although he rejects this as (in his own terms) misguided, the myth of this restriction has plagued much subsequent historiography [cf. Steele 1936]. He then tries to have Greek geometry determined by the postulates of the first book of Euclid's *Elements*, but here he has to admit that constructions of planes were used for the conic sections (even if he is grudging about admitting the place of these figures in Greek geometry). His third attempted explanation was that the spiral and quadratrix, which were not geometrical, were discovered first and only afterwards the acceptable conchoid and cissoid. But, as we have seen, there was no ancient compunction about admitting the spiral and little about the quadratrix, and there could well have been more doubt about the geneses of the conchoid and cissoid. His fourth attempt was to suggest that the ancient geometers wished to complete the lower reaches of geometry before moving on to the higher. But this does not in fact address itself to the same pseudo-problem, but only to the question of why more time was not spent on higher curves.

Descartes's historical errors are blatant, but his faulty exegeses allow him to introduce more naturally his own basis for geometry. This arises from the imagination of various articulated instruments in which the movements of all the parts are completely determined by the movement of one of them, and where the intersections of parts trace out curves [AT, 6, 391-395; 1954, 45-47]. The parts may all be straight lines or rulers, or one of them may be a figure already produced by a continuous movement, such as a parabola or a hyperbola. All such curves are to be received into geometry and Descartes [AT, 6, 392; 1954, 46-49] writes of those produced by one instrument that "I do not see what can prevent one from conceiving the description of the first [curve] as clearly and as distinctly as that of the circle or at least as that of the conic sections, nor what can prevent one from conceiving as well as one can the first the second, the third and all the others that one can describe, nor consequently why one should not receive them all in the same way for use in the speculations of geometry." On the other hand curves such as the spiral and the quadratrix, which arise from two simultaneous motions are to be rejected from geometry and regarded as only mechanical.

We shall have to examine later how Descartes relates this criterion for the acceptability of curves to the possibility of measure, but first we should note how the criterion goes right back to Descartes's earliest researches in mathematics. In the *Discours de la Méthode* he [AT, 6, 7; 1955, 1, 85] tells us how during his education he delighted most in mathematics, but at the time believed that "it was of service only in the mechanical arts". In the *Cogitationes Privatae* we see signs of his occupations with various practical problems of mathematics, including the invention of articulated instruments or compasses as they were generally called, for tracing curves and solving problems [AT, 10, 232-235, 238-242]. In a famous letter to Beeckman on March 26th, 1619 [AT, 10, 154-160] Descartes indicates how such generalised compasses were taking on a fundamental theoretical position in his mathematical thought. He speaks of how he has invented compasses for the division of angles into equal parts and for the solution of cubic equations. But such discoveries were suggesting to him a whole new programme for mathematical advance [AT, 10, 156-157]:

And certainly, to disclose openly to you what I am undertaking, I wish to propound not an ars brevis *of Lull but a completely new science, by which can be solved generally all questions that can be put forward in any genus of quantity, continuous as well as discrete. But each according to its nature.*

For, as in arithmetic some questions are solved by rational numbers, others only by surd numbers, and lastly others can be imagined but not solved, so I hope to demonstrate that in continuous quantity some problems can be solved with only straight lines and circular ones, others can only be solved by other curved lines, but such as arise from a single motion (sed quae ex unico motu oriuntur), *and so can be described* (duci possunt) *by new compasses, which I do not regard as less certain and geometrical than the common ones by which circles are described, and finally other* [problems] *can only be solved by curved lines generated from different motions which are not subordinated to one another, which lines are certainly only imaginary: such is the common enough* (satis vulgata) *quadratrix. And I judge that nothing can be imagined that cannot at least be solved by such lines. But I hope that I shall demonstrate which questions can be solved in this or that way and not the other, so that scarcely anything in geometry will remain to be discovered.*

In this way Descartes had formulated a programme for geometry from which he did not essentially diverge. But in the *Géométrie* we find it far more definitely related to specification by property. In the long passage that we quoted above we saw how Descartes regarded the distinguishing characteristic of geometry as opposed to mechanics as being that the geometrical was "precise and exact". Moreover, geometry was "a science that teaches generally how to know the measures of all bodies". Descartes also held in that passage that one could have an exact knowledge of the measure of curves that had been generated in accord with his criterion of one or more determined movements. Descartes's interpretation of what was meant by an exact knowledge of the measure of a curve may have undergone some development, but in the *Géométrie* he clearly explicates it in terms of equations "In order to understand together all those [curves] which are in nature and to distinguish them by order into certain genera I know nothing better than to say that all the points of those which one can name geometric, that is to say which fall under some precise and exact measure have necessarily some relation *(quelque rapport)* to all the points of a straight line, which can be expressed by some equation, and by one [equation] for them all" [AT, 6, 392; 1954, 49]. Curves are then classified according to the degree of their equations.

At the basis of Descartes's representation of curves by equations lies the close analogy that he makes between operations on straight lines (or, in later terminology, line segments) and operations on numbers. In particular, by the assumption of an arbitrarily chosen unit line, he is able to interpret the multiplication of two straight lines as given rise to a third straight line rather than to a rectangle. This step was of fundamental importance in making easier the representation of curves in algebraic terms. In setting up an equation for a curve Descartes [see e.g. AT. 6, 393-394, 1954, 49-54] chose a straight line, say *AB*, and a fixed point on it, say *A*. From an arbitrary point *C* of the curve a straight line *CB* was drawn to *AB* meeting it at a given angle. The lines *AB* and *BC* were then quantities that would determine the position of *C*, and they were called x and y. An equation in terms of x and y could determine the curve, by specifying a property that all its points had to obey. As we have noted above this has close similarities to such ancient procedures as Apollonius's establishment of *symptomata* for the conic sections. Descartes's innovations in this regard lie in his drawing heavily upon the traditions of algebra with the consequent assimilation of operations on straight lines to operations on number, and in his making this mode of representation standard and general.

Descartes held that the possibility of representing a curve by an equation (specification by property) was equivalent to its being constructible in terms of the determinate motion criterion (specification by genesis) [cf. Vuillemin 1960, 77-93]. We should note that the quasi-arithmetical operations which Descartes allows on straight lines mean that for him an equation is what we should call a polynomial equation: there was no place for what would later be called transcendental functions. This follows quite naturally from the four (or five if extraction of roots is counted) primary operations that Descartes allows upon straight lines. In some earlier writings we see him making use of compound ratios in moving towards his developed doctrine of multiplication of lines [15], and these may have played an explicit role in his original solution of the problem of Pappus. This problem was proposed to Descartes by Golius in 1631, and Descartes's occupation with it seems to have played a very important role in the development of his mature system of geometry. His original solution is lost, but the problem forms a central theme in his *Géométrie*, and an examination of it can give us much insight into Descartes's geometrical procedures.

The problem is as follows [16]. There are given n straight lines. From a point *C* lines are drawn making given angles with the given lines. If $n=3$ the ratio of the product of two of the lines from *C* to the square of the third is given. If n is even and greater than two the ratio of the product of $n/2$ of the lines from *C* to the product of the other $n/2$ lines is given. If n is odd and greater than three, the ratio of the product of $(n+1)/2$ of the lines to the product of the other $(n-1)/2$ lines together with a given line is given. It is required to find the locus of *C*. Pappus said that in the case of three or four lines, which had been investigated by Euclid and Apollonius [cf. Apollonius, 1891-93, 1, 4-5; 1896, lxxi], the locus was solid. In higher cases it was linear, but there had been virtually no study of those cases, and the curves had not in general been more fully identified [17].

Descartes [AT, 6, 382-384; 1954, 27-32] begins his attack on the problem with the use of his characteristic mode of algebraic representation. He takes one of the given lines *AB* as the reference line, and its point of intersection *A* with one of the other given lines as the fixed point. *CB* is the line from a possible position of *C* falling on *AB* at the given angle. He denotes *AB* and *CB* by x and y. He then shows how the lengths of the other lines from *C* to the given lines at the given angles can be expressed in terms of x and y and that these expressions are of the first degree in x and y. By multiplying these expressions an equation may be produced, whose degree depends upon the number of lines [AT, 6, 384-387, 396-399; 1954, 32-36, 57-65] so that, for instance, when there are four lines the equation will be of the second degree. Descartes holds that the locus of *C* is the curve represented by that equation. But an error has in fact crept in, owing to the inadequacies of Descartes's techniques for dealing with changes of sign. In each case there should be two equations of the given degree, so that in the four line problem the locus of *C* is not one conic section but two [18]. This error is not so fundamental as to vitiate the rest of Descartes's treatment, and interestingly enough he seems to have had some inkling of it himself, for, as we shall see, he recognises two curves in one case of the five line problem.

Descartes uses the degree of the equation as a basis for the classification of curves, although he hints at a finer classification on another principle by emphasizing that the circle is simpler than the ellipse, parabola or hyperbola [AT, 6, 392-393, 396; 1954, 49, 57]. The use of equations for classification is an important application of specification by property, but we should be wrong to think that the equation was for Descartes a substitute for genetic definition. The production of an equation did not solve Pappus's problem. The curve still had to be found.

Descartes gives most attention to the four-line locus [AT, 6, 397-407; 1954, 61-81]. From the form of the equation and with the help of a few simple constructions, he indicates very sketchily when the curve will be a parabola, hyperbola, ellipse or circle, and gives such parameters as the length of the *latus rectum*. In this he is drawing essentially on the principal planimetric properties of the various conic sections as established by Apollonius. But when he has achieved this he can also appeal to Apollonius for the geometric construction of the curves from particular sections of the cone. In this he is not being altogether consistent, for Apollonius's criteria for construction were different from his own.

But for the higher cases there was no Apollonius to appeal to, and Descartes was left to his own methods. He essayed no general treatment and concentrated most of his attention on a particular case of the five-line locus [AT, 6, 407-410; 1954, 81-86]. In this, four of the given lines are taken as parallel with equal intervals between them. The fifth line is perpendicular and all the given angles are right angles. The product of the perpendiculars from *C* to three of the parallel lines is equated with the product of a given line (equal to the interval between the parallel lines) and the perpendiculars from *C* to the other two lines. Descartes forms the equation, and solves the problem by showing that the same equation applies to a figure constructed by his method of determined motions. A parabola is constrained to move so that its axis always lies along one of the parallel lines. A straight line is constrained to move so that it always passes through one of the intersections of the given lines and also through a point on the axis of the parabola fixed relative to the parabola. The intersections of the parabola and the straight line trace out the curve, which is in fact a cubic curve with two branches. It was perhaps the symmetrical nature of this problem that made Descartes realise that the complete solution involved two cubic curves, the second being described by reversing the direction of the parabola.

Descartes's mathematical laziness is notorious [cf. Allard 1963, 158-160, section entitled "La lassitude de Descartes envers les Mathématiques"], and he himself frequently insisted upon it. Very early in his treatment of Pappus's problem in the *Géométrie* he claimed [AT, 6, 382; 1954, 27] that "it already wearies me to write so much about it," and he frequently said that he was only giving a sketchy treatment of particular issues in order that others could have the pleasures of discovery or at least realise how difficult the matter was. He had the type of mind that was happy in producing bold general conceptions, but became bored when it was a question of working out the detail, although he was quite capable of doing this. Also, after his early work in mathematics, he was far more interested in producing a natural philosophy with a strong mathematical basis than in working on actual problems of pure mathematics. It is thus not surprising that the *Géométrie* can read as if it ought to have been an early draft of itself. There are frequent obscurities and *lacunae* of reasoning, and diverse matters that had interested Descartes at different times of his life are sometimes thrown together with little attempt at unification. This seems particularly so in the case of the ovals, which Descartes saw as having application in optics, and when treating these his criteria for geometrical construction are more lax than his norm.

He prepared the ground for this at the end of his discussion of Pappus's problem, where he excused himself for not considering further higher cases. "I did not undertake to say everything, and having explained the manner of finding an infinity of points through which [the curves] pass I think that I have sufficiently given the means of describing them" [AT, 6, 411; 1954, 89]. Here Descartes's indolence seems to have led him to the brink of admitting definition by equation, but from what follows it is clear that he regarded this mode of description as subsidiary to genesis by determined motions [AT, 6, 411-412; 1954, 89-90]:

It is also to the point to remark that there is a great difference between this manner of finding several points in order to trace a curved line, and that which one uses for the spiral and similar [curves] [19]: *for by the latter one does not find indifferently all the points of the line that one is seeking, but only those which can be determined by some measure more simple than that which is required to compose it, and so strictly speaking one does not find one of its points, that is to say one of those which are so proper to the curve that they cannot be found without it. But there is no point in the lines that serve for the question at issue that cannot be met among those which are determined* (qui se determinent) *in the manner just explained. And because this manner of tracing a curved line by finding indifferently several of its points only extends to those which can also be described by a regular and continuous movement, one ought not to reject it entirely from geometry.*

Thus some point-wise descriptions are allowed, but with an inferior status. Descartes also feels it necessary to make a similar concession for certain constructions making use of strings [AT, 6, 412; 1954, 90-93]:

And one ought no more to reject that [manner of description] *in which one uses a thread or a doubled cord* (une chorde repliée) *to determine the equality or difference of two or more straight lines which can be drawn* (tirées) *from each point of the curve that one seeks to certain other points, or onto certain other lines at certain angles, as we did in the* Dioptrique *in order to explain the ellipse and the hyperbola. For, although one could not admit any lines which resembled cords, that is to say which became sometimes straight and sometimes curved, since the ratio between straight lines and curved ones, being unknown and even I believe being unable to be known by men, one could conclude nothing thence that was exact and ensured, yet because one only uses strings in these constructions to determine straight lines, whose lengths one knows perfectly, this ought not to make one reject them.*

Descartes points out that he has used constructions with strings for the ellipse and the hyperbola in his optical writings [see AT, 6, 165-178], and it is in an optical context that both pointwise descriptions and a string construction reappear in the *Géométrie*. Descartes introduces four ovals, all of which he claims are useful in optics [AT, 6, 424-429; 1954, 114-125; cf. AT, 10, 310-328]. In each case he gives a description by stating how an arbitrary point is to be constructed. For a special case of one of them he then asserts a construction using string. A rod is pivoted about a fixed point in the plane. A string travels from the free end of the rod, down the rod to the point where it is kept taut by a finger, from there to a fixed point in the plane and then back to the finger, and finally from the finger to another fixed point in the plane, where its other end is attached. As the rod is moved about the pivot, the finger traces out the oval. It is not immediately obvious that this construction serves Descartes's end, and he gives no justification for it, but he is in fact right. However, string constructions had a rather peripheral place in Descartes's foundations of geometry and we must regard them as being introduced as a second-rate substitute for constructions arising from articulated instruments. And certainly pointwise descriptions must be regarded as less than a full genesis, since they do not produce all the points of the curve. The norm remained the sequence of determined movements of rigid figures.

CONCLUSION

In John Aubrey's *Brief Lives* we have an anecdote of Descartes, which Aubrey claimed to have from one Alexander Cowper [Aubrey 1898, 1, 222]:

He was so eminently learned that all learned men made visits to him, and many of them would desire him to show them...his instruments (in those dayes mathematicall learning lay much in the knowledge of instruments, and, as Sir H[*enry*] *S*[*avile*] *sayd, in doeing of tricks), he would drawe out a little drawer under his table, and show them a paire of Compasses with one of the legges broken: and then, for his ruler, he used a sheet of paper folded double.*

This story, whether true or not, can help to highlight features of Descartes's approach to geometry. He sought for clear and distinct foundations of geometry, and found it easiest to achieve these by drawing on the traditions of contemporary practical geometry. This involved reflecting on the use of instruments, but it was by no means a case of actually using them. What had to be imagined were idealised instruments, and these could be conceived, when used in a certain way, to produce the acceptable curves of geometry. This involved a radical break with the ancient vision, but Descartes disguised the nature of the break by reading his own procedures into ancient writings and implying that they used criteria of his own kind, but in a more restricted way [20].

In Descartes as in the ancients we see clearly the distinction between specification by genesis and specification by property. For him the former was certainly fundamental, but the changes he made in the latter are more immediately apparent. These were grounded in his particular "algebra of straight lines" and in his use of a standard method for representing all acceptable curves by equations. His work within the theory of equations (particularly in Book 3 of the *Géométrie*) is also highly significant, but in this paper we have been concerned with how Descartes shifted the conceptual foundations of geometry. He was very conscious that he was making changes, but whereas he gave clear characterisations of the bases of his own procedures, his account of what he was changing from is often historically misleading. In this we may sympathise, for, as we have seen, the reconstruction of the ancient foundations is a matter of some complexity, and Descartes was not acting as historian. But a close examination of these foundations can show many unsuspected differences from Descartes. A fuller historical account of how Descartes came to make the changes that he did make would demand paying more detailed attention to his contemporary intellectual milieu than we have done here. But a historical explanation is not satisfactory without a clear analysis of what has to be explained, and it is to this end that this paper has been principally directed.

NOTES

1. For an interesting discussion of the analogous problems surrounding the "completely arbitrary" function see Becker [1927, 153-160].
2. The concept of intelligible matter in a mathematical context is found in Aristotle, *Metaphysica* Z. 10, 1036a9-12; Z. 11, 1037a1-5; H. 6, 1045a34, but there is some doubt about the status of the first two of these passages; see Aristotle [1957, 150, 152].

3. I speak here of what Geminus [Proclus 1873, 111; 1948, 100] would have called incomposite lines and surfaces, and do not consider the specification of such composite lines as the perimeters of polygons. For a useful account of curves in Antiquity see Tannery [1912].

4. My aim here is to identify the main thrust of Pappus's arguments. Particularly in the case of the first derivation, this has meant some reconstruction and deviation from his actual order of presentation.

5. Sporus [Pappus 1965, 254-255; 1933, 193-194; Thomas 1957, 1, 340-341] had pointed out that the original definition of the quadratrix did not produce its final point, but only the points before the motion was completed. There is thus a limiting process involved. The same applies to the construction of the quadratrix from the cylindrical helix and the Archimedean spiral.

6. For fragments of Eratosthenes' *Platonicus* see Hiller [1870]. It is not certain that the work was a dialogue.

7. We only consider the first conchoid, but Pappus remarks that there were others; cf. Heath [1921, 1, 240].

8. Nicomedes used the conchoid for the duplication of the cube, and Eutocius [1915, 98-99; Thomas 1957, 1, 296-299] reports that he derided Eratosthenes' discoveries as both "impracticable and

lacking in geometrical sense". Thus it may be that Nicomedes regarded the analogue of the instrumental construction as geometrical, or, less probably, that he had a completely different geometrical construction.

9. The cissoid is normally identified with the curve described by Diocles, but there is no absolute certainty; cf. Thomas [1957, 1, 270, n.a.]. It is possible, though unlikely, that the Arabic version of Diocles' work discovered by G. J. Toomer [see Toomer 1972, 190-191] contains a more geometrically acceptable genesis of the curve than that reported by Eutocius.

10. For Newton's "organic" (instrumental) construction of the cissoid, see Newton [1967-, 5, 464-466].

11. Proclus speaks of the property as τὸ καθ'αὑτὸ καὶ ᾗ αὐτὸ ὑπάρχον; cf. Aristotle, *Anal. Post.* I. 4, 73b26-32 and Ross's comment in Aristotle [1949, 522-523]. Pappus [1965, 234, 252; 1933, 178, 192] uses the phrase τὸ ἀρχικὸν σύμπτωμα, and gives this for the Archimedean spiral and the quadratrix. He does similarly for the conchoid, but in that case his phrase is simply τὸ σύμπτωμα [1965, 244; 1933, 186]. Apollonius [1891-93, 1, 4] spoke of the first book of his *Conics* as containing the γενέσεις and the ἀρχικά συμπτώματα of the conic sections.

12. Some of the propositions of Euclid's *Data* may be read in this way. See e.g. Euclid [1883-1916, 6, 46-55].

13. Pappus [1965, 662-663; 1933, 495-496]. Hultsch attributes this passage to an interpolator. Eutocius [1893, 184-185] also distinguishes plane and solid loci, but he sees the second group as comprising many more sections of solids than just the conic sections. He does not mention linear loci, but adds that there are also loci on surfaces. Proclus [1873, 394-395; 1948, 337-338] has the plane-solid distinction, but appears rather confused, and his whole discussion of loci seems to have been forced into a context that was not altogether appropriate. Cf. Euclid [1956, 1, 329-331]

14. Descartes [AT, 6, 388-390; 1954, 41-45]. The translations from Descartes are in general my own. I have striven for the literal, often at the expense of the elegant.

15. The steps by which Descartes struggled towards his mature form of algebraic representation cannot be considered in detail here. Among the rather scanty evidence, important pieces may be found in *Regulae ad directionem ingenii* 15-18 [AT, 10, 453-469; 1955, 65-77]. Compare with this Beeckman's notes from 1628-29 [AT, 10, 333-335], and see Milhaud's discussion [1921, 70-72, n.1]. In Descartes's extant letter to Golius on the problem of Pappus in

January 1632 [AT, 1, 232-235] there are puzzling references to the motion of construction being determined by simple relations, which are in turn explicated in terms of single ratios *(proportiones singulae)*.

16. Pappus [1965, 676-681; 1933, 506-510; Thomas 1957, 1, 488-489, 2, 600-603] had given a statement of the problem in his account of Apollonius's *Conics*. In the *Géométrie* Descartes [AT, 6, 377-379; 1954, 16-20] takes over this account from Commandinus's translation of the *Collectio*. In our exposition of the problem we shall, unlike Pappus, freely allow the product of straight lines to be a straight line, and we shall introduce the symbol n when speaking of the number of lines.

17. A textual obscurity in Pappus's account causes difficulties in assessing exactly what the ancients did in this regard, but according to him it was certainly little. Cf. Pappus [1933, 508, n.8].

18. In terms of modern "Cartesian coordinates" this may be seen as follows. Consider a line $ax+by+c = 0$ and a point $C(l,m)$ not on it. The perpendicular distance of C to the line is $(al+bm+c)/\sqrt{aa+bb}$. The length of a line from C to $ax+by+c = 0$ intersecting it at a given angle ϕ is $(\csc\phi)(al+bm+c)/\sqrt{aa+bb}$ Thus if we have four given lines distinguished by the subscripts 1,2,3,4, the conditions of the problem mean that $C(l,m)$ must satisfy:

$$(a_1l+b_1m+c_1)\ (a_2l+b_2m+c_2)/\ \sqrt{(a_1{}^2+b_1{}^2)\ (a_2{}^2+b_2{}^2)}$$

$$= \pm\ k(a_3l+b_3m+c_3)\ (a_4l+b_4m+c_4)/\ \sqrt{(a_3{}^2+b_3{}^2)\ (a_4{}^2+b_4{}^2)}$$

where k is a constant determined by the given angles. Thus C must lie on one of two conic sections, obtained respectively by taking the plus sign and the minus sign before k. Conversely any point on either curve satisfies the required relation of distances, and so the locus is two conic sections. Cf. [AT, 6, 724].

19. Christopher Clavius [1606, 320-322] had attempted such a description of the quadratrix.

20. We may see Descartes's interpretation of the ancient procedures as a natural progenitor of the great concern in the seventeenth and eighteenth centuries with organic (or instrumental) geneses of curves. For an account of such constructions from Antiquity to the eighteenth century see von Braunmühl [1892].

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ACKNOWLEDGMENTS

Some of the themes in this paper were treated in a lecture to the British Society for the History of Mathematics on December 15th 1972, and I am grateful for the comments then received. For incisive criticisms of an earlier draft and pertinent suggestions, I am much indebted to Professor J. E. McGuire. I should also like to thank: an anonymous referee and the editor for many helpful comments; Dr. D. E. P. Jackson for making his thesis available to me and for many useful discussions; Dr. D. Bloor for drawing my attention to Aubrey's anecdote. A welcome opportunity for much of the work on the paper was afforded by my tenure of an Andrew Mellon Postdoctoral Fellowship in the University of Pittsburgh.

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