HISTORIA MATHEMATICA 4 (1977), 141-151

DESCARTES AND THE BIRTH OF ANALYTIC GEOMETRY

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SUMMARIES

The traditional thesis that analytic geometry evolved from the concepts of axes of reference, co-ordinates, and loci, is rejected. The origins of this science are re-defined in terms of Egyptian, Greek, Babylonian, and Arabic influences merging in Vieta's Isagoge in artem analyticam *(1591) and culminating in a work of his pupil Ghetaldi published posthumously in 1630. Descartes'* Vera mathesis, *conceived over a decade earlier, served to revive and strengthen the important link with logic and thereby to extend the field of application of this analytic method to the corporeal and moral worlds.*

Die allgemein aufgestellte These, dass die analytische Geometrie, die aus den Begriffen Achse, Koordinate und Ort entfaltet wurde, wird abgelehnt. Diese mathematische Wissenschaft wird hier gedeutet durch ägyptische, griechische, babylonische sowie arabische Einflüsse, die in Vietas Isagoge in artem analyticam *(1591) vereinigt und 1630 in einem nachgelassenen Werk seines Schülers Ghetaldi umgestaltet werden. Die von Descartes über eine Dekade früher erfundene* Vera mathesis *diente dazu, das wichtige Bindeglied zur Logik wieder zu beleben und zu stärken und somit diese Methode auf physikalische und moralische Welt auszubreiten.*

As far as I am aware, the first person to challenge the belief that analytic geometry sprang like Athena from the head of René Descartes was the nineteenth-century German cartographer Sigmund Günther [1877]; according to whom there are three distinct conceptual stages which had to be progressively attained before that mathematical science came into existence:

(1) The specification of position on a surface with regard to two axes.
(2) The graphical representation of the relationship between the ordinates and the abscissae (i.e. between the dependent and independent variables).
(3) The discovery of the law, or algebraic equation, corresponding to that geometrical curve.

Matthias Schramm [1965] tells us that this is how common opinion still sees the situation--despite the fact that almost thirty years previously Julian Coolidge explicitly rejected Günther's point of view in favour of the thesis that "the essence of plane analytic geometry is the study of loci by means of their equations and...this was known to the Greeks and was the basis for their study of conic sections." [Coolidge 1936, 233]

Whether or not one is prepared to agree with Coolidge that the credit for this important discovery should go to Eudoxus's pupil Menaechmus, who is generally credited with having been the first to discover the conic sections, one must surely concede his point that the manner in which the Greeks treated the geometry of this class of curves is easily reducible to modern algebraic terminology.

It is consistent with Günther's interpretation to regard Apollonius of Perga (3rd century B.C.), who made use of coordinates and oblique axes in his *Conics*, as the 'father' of analytic geometry; and Descartes, who generalized those conics and reduced a hyperbola to an algebraic relationship between the section of the diameter and lines, as the 'midwife' who delivered the 'baby'. According to E. T. Bell, in *Men of Mathematics*, the date of birth was the 11 November 1619. This was supposedly when Descartes saw the Greek infant clearly for the first time, as a result of a dream. The 'delivery ward' was a stove-heated room somewhere in the south of Germany. Only after the 'child' had matured to the age of eighteen, did he allow it to make its 'debut' before the learned workd, in the form of an essay entitled simply "La Géométrie" appended to his first published work *Discours de la Méthode* (Amsterdam, 1637).

This homely analogy was implicitly accepted by Carl Boyer when he wrote his authoritative *History of Analytic Geometry* (1956). The present brief treatment of the early phases of such a complex story would naturally be inadequate as an attempt to re-examine the conceptual ramifications which are there so fully and ably discussed. Its value lies rather in its explicit rejection of Günther's thesis and reassessment of Descartes' achievement in association with an alternative framework for interpretation suggesting lines of research which may still be profitably explored.

Although, for reasons explained below, I am unwilling to accept Günther's evolutionary view of the birth of this subject, I would not wish to deny the fact that both axes of reference and co-ordinates were in widespread use in western Europe long before Descartes' own time. From the fourth century B.C. onwards, the ecliptic circle, or Sun's apparent annual path through the sky, was graduated from 0° to 360° and subdivided into 12 equal parts in order to serve as a calculating device by which a planet's celestial position could be expressed in terms of its angular distance relative to a bright star, or group of stars, in its neighbourhood. The origin of this single-axis reference system (or zodiacal circle) for obtaining celestial longitudes, was one of the two points at which the ecliptic intersects the projection of the terrestrial equator on the celestial sphere (viz. the Vernal Equinox). Hipparchus (2nd century B.C.) referred the positions of well over 800 bright stars to that same origin, at the same time introducing, as a second co-ordinate for uniquely specifying a star's position on the celestial sphere, its angular distance measured at right-angles north or south of the same fundamental reference plane (viz. celestial latitude). In the field of geometry, a very clear application of the coordinate principle is to be found in the first book of Apollonius's *Conics*. Hero of Alexandria used rectangular coordinates in geodetic measurements, and the Romans used them in their land surveys. The geographical maps of Ptolemy (2nd century A.D.) show terrestrial longitude and latitude differences.

In the Bavarian State Library in Munich there is a 10th century manuscript transcription of the Roman grammarian-philosopher Macrobius's commentary on Cicero's Dream, in which a graph is used to illustrate the inclinations of the planetary orbits as a function of time [Funkhouser 1936]. A late medieval example of the use of orthogonal axes to denote position in a plane is Nicolas Oresme's "latitude of forms", which Coolidge confesses to having studied hard without being able to understand its significance. It appears, however, that although the original purpose of Oresme's graphical representation of the notion of change was theological, it became widely known in scholastic circles during the 15th and 16th centuries through its application to the particular relationship between uniform and uniformly-accelerated motion. Mainly on this account, it has often been cited as a possible source of Descartes' own knowledge of the coordinate principle; yet no internal evidence in his mathematical writings has been found to support this belief. On the contrary, there is no reason to doubt the veracity of his statement that he acquired this insight while lying in bed watching a fly crawling across his bedroom ceiling!

Be that as it may, Schramm [1965] has explicitly dismissed as irrelevant the question of whether or not Descartes was fully aware of the coordinate principle, since in his view Greek geometry and the *Algebra* of Omar Khayyam are alone sufficient for interpreting the structure of *La Géomêtrie*. In the same article, Schramm puts another spoke into Günther's thesis by stressing that the concept of a function, or locus, was already implicit in the solar ephemerides of the Seleucid astronomers and in sequences with constant second-order differences which occur in the refraction table of Ptolemy's *Optics*. Thus he maintains that a training in *logistics*, meaning the technique of numerical calculation, was at the root of a tradition derived from the Babylonians and developed by Arabic scientists who also supplied the algebraic formulae necessary for the exposition of Greek geometrical methods.

An explanation of how logistics was linked to the theory of functions during the Alexandrian era of Greek culture has recently been given by Olaf Pedersen [1974]. After Plato, in Book 7 of his *Republic*, had advocated a separation between theory and practice, a formal distinction came to be made between *pure* mathematics (viz. arithmetic and geometry) and *applied* mathematics (viz. music and astronomy, geodesy, optics, mechanics and logistics). Despite the fact that no Greek exposition or manual of logistics has ever been found in Western Europe, Pedersen shows how the existence of this computational art can be established from a detailed study of Ptolemy's *Almagest*, in which a great number of practical methods for operating with functions of different kinds are presupposed. His analysis reveals that Hellenistic mathematicians carried logistics to a much higher degree of sophistication than has hitherto been suspected. They had methods for dealing with functions of one, two, and even three variables where 'function' in this context does not mean 'formula' but 'a general relation associating the elements of one set of numbers...with another set'; for example, the instants of time with some angular variable in planetary theory. Perhaps it was only the difficulty in understanding the concept of infinity which prevented the Greeks from developing an actual theory of functions.

Pedersen's discussion really refers to what Jacob Klein [1968] had christened earlier as 'theoretical logistics', or the theory of ratios and proportions such as was applied by Eudoxus to both incommensurable and commensurable magnitudes (see Euclid V) and to geometry (see Euclid VI). The traditional origins of these procedures, like those of geometry, were Egyptian; thus it is not surprising that one of the most outstanding examples of its subsequent development should be found in the *Arithmetic* of Diophantus of Alexandria (3rd century A.D.). The style of this treatise differs from that of books on modern algebra in not being organised around types of equations and methods of solution, but structured according to the types of relations that numbers can bear to one another. It is now recognised as representing a tradition stemming from early Greek (and perhaps Egyptian) sources--quite separate from the Babylonian-Arabic tradition of 'practical logistics' with which Schramm was primarily concerned, imported into Western Europe by Leonardo of Pisa at the beginning of the 13th century.

Diophantus's *Arithmetic*, and the 7th book of Pappus of Alexandria's *Collection*, were the two major sources of Vieta's *Isagoge in artem analyticam* (1591) which shows how, by reducing equations to the form of proportions, an algebraic equation can be treated in a geometric way. (e.g. $x^2 + bx = c^2$ may be otherwise written as $x/c = c/(x+b)$). In this respect, of course, Vieta was merely following the Greek geometrical tradition. There was no real novelty either in his introduction of a general algebraic symbolism transcending that of the earlier Arabic and Persian algebraists and the members of the so-called "cossic school" such as Stifel, Cardan, Tartaglia, Nuñez, and Clavius. The key to Vieta's fundamental transformation in the conceptual basis of this subject was his elaboration of its *methodological* foundations from the traditional two-fold analytic (or zetetic) art and synthetic (or poristic) art of classical Greek geometry, to a three-fold procedure introducing a computational (or exegetic) art.

In order to appreciate the full significance of this innovation, it is first necessary to recognise what was implied by the terms 'analysis' and 'synthesis'. The classical scholar H. D. P. Lee [1935] has argued convincingly that the first principles of geometry, along with its methodological procedures, can be identified with those laid down as being valid for the whole of science, in Aristotle's *Prior and Posterior Analytics*. In composing this logical treatise, Aristotle may have been borrowing from contemporary geometers, but it is impossible for us now to assess the extent of his own modifications of their work. Lee carefully examined the connections between the mathematical and philosophical principles, and summarised his conclusions as follows:

> *Euclid's Common Notions and Aristotle's Axioms, and the Definitions of both, are exactly parallel. The common notions and axioms are principles of reasoning whose scope extends further than that of a single science: the definitions are statements of the meaning of terms. To Aristotle's hypotheses answer Euclid's postulates. Both are a minimum of further assumptions necessary besides the axioms or common notions and the definitions. The hypotheses assume existence, the postulates the possibility of constructions, etc.* [Lee 1935, 117]

He is, however, careful to point out that Euclid need not have been influenced directly by Aristotle; the former was merely repeating and collating ideas which were in vogue at the time.

In Aristotle's account of the logical procedure of science, which he applies in his *Ethics*, there is first an intuitive movement of thought in which the mind grasps the requisite elementary principles (or 'archai'), followed by a deductive process in which the logical consequences of these principles are traced out. At the moral level, there is little difference between it and Plato's dialectic. At first sight, it would appear that *analysis*--the step which leads from the unknown to the known--is absent in Euclid's *Elements*; for this has the appearance of a purely *synthetic* treatise, proceeding from the known and simple to the unknown and more complex. From what is now known about the latent empiricism in Greek mathematics, one is nevertheless inclined to suspect that it played a part in the discovery of the proofs. Euclid's *reductio ad absurdum* proofs can themselves be regarded as involving analysis, since they begin with the reduction of the proposition that requires proof--assumed to be true--to something simpler that is immediately recognisable as true or false. The false case is the absurdity; the method which reveals it, analysis.

In Book 7 of Pappus's *Collection*, which was compiled almost six centuries later, the two procedures are defined explicitly as follows:

> Analysis *takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis; for in analysis we assume that which is sought as if it were already done, and we enquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until, by so retracing our steps, we come upon something already known or belonging to the class of first principles [viz. a corollary or porism] and such a method we call analysis as being solution backwards. But in* synthesis, *reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of that which was sought; and this we call* synthesis. [Heath 1963, 452]

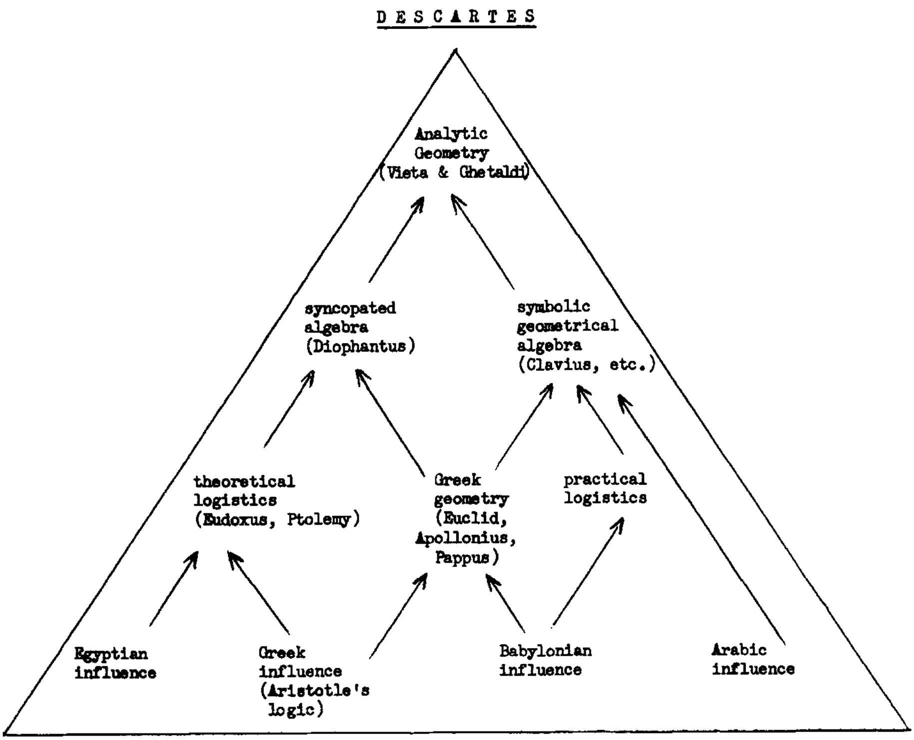
This was the very statement of method which Vieta was to read many centuries later, and adapt to his own purpose by introducing the following analogous definitions:

> *Zetesis:* *the procedure "through which the* equation *or the proportion is found which is to be* constructed *by the aid of the given magnitudes with a view to the magnitudes sought."*
>
> *Poristic:* *the procedure "through which by means of the* equation *or proportion the truth of the theorem set up [in them] is investigated."*
>
> [Klein 1968, Appendix]

Thus, as he remarks, theorems demonstrated by zetesis are then subjected to the law of synthesis since this was considered to be a more logical way of demonstrating; after which, the steps of the analysis were retraced. If, at some stage, one were to encounter an unexpected result requiring demonstration, this had to be done poristically. The loss of Euclid's three books of *Porisms* has obscured the precise meaning of this term, although the nearest synonym would appear to be 'corollary'--a type of proposition intermediate between a theorem and a problem, dealing with something already in existence yet which has to be found *by means of the construction of geometrical magnitudes.*

The novelty of the *exegetic art* "through which the magnitude sought was *itself produced out of* the equation or proportion set up", was that it introduced *the computation of arithmetical magnitudes*--effectively, by the solution of an algebraic equation--and thereby unified the Greek logistical procedure with the traditional method of analysis and synthesis. This is why Vieta believed he had created a new analytical art that would leave no problem unsolved. It would certainly appear that something more than geometrical algebra, the application of geometrical constructions to the solution of algebraic equations, is here involved; although that too unquestionably constitutes a necessary and indeed characteristic feature of Vieta's approach. The "something" is rather difficult to pin-point, but I believe that it may be connected with a class of porisms which form a kind of "missing link" between the procedures of Pappus and those contained in Diophantus's *Arithmetic*--namely, those concerned with the construction of loci from given geometrical conditions. Although our knowledge of these is scanty, one such porism, which has attained great historical importance on account of its central rôle in the formulation of Descartes *Géométrie*, is the famous Problem of Pappus. I believe that in this class of problem is to be found the earliest indication of the notion of an algebraic equation representing a geometrical curve. The lack of a well-developed symbolism in Pappus's and Diophantus's time, and historical accident, have conspired to hide this advanced Alexandrian development from posterity. The fact that Diophantus's *Arithmetic* also incorporated earlier Greek and Hellenistic sources, such as Plato's *Laws* and *Charmides*, Euclid's *Elements* (Books VII-IX), and Hero's *Metrika*, then later absorbed into the Arabic tradition by al-Būzjānī, Qusta ibn Lūqā, and possibly al-Haitham during the 10th century, unfortunately tends to obscure the precise nature of its influence upon Vieta himself. For the sake of simplicity, however, one might differentiate between his new analytical art and geometrical algebra by asserting that the former embodies the *direct* influence of the Graeco-Egyptian tradition of theoretical logistics, whereas the latter does not.

I do not feel inclined to identify Vieta's achievement with the birth of analytic geometry, although I am content to label his new art 'geometrical analysis' to distinguish it from 'geometrical algebra'. Still to be done to effect what in my view was the final stage in the embryonic evolution of analytic geometry was to transfer the purpose of Vieta's analysis away from geometrical constructions to the solution of algebraic equations, towards the application of the already well-known algebraic techniques to the solution of geometrical problems. This transformation was first effected by Vieta's contemporary and one-time pupil Marino Ghetaldi, in his posthumously-published *De resolutione et compositione mathematica* (Rome 1630). Here, the status of symbolic algebra was raised from a "means to an end" to a method in its own right, with a wider scope and application than it had hitherto possessed [Gelcich 1882]. Although it is therefore easy to appreciate why Ghetaldi has been hailed by various mathematicians as the father of analytic geometry, I feel that this title accords undue credit to what was primarily Vieta's achievement. Perhaps an acceptable compromise is to regard this science as having been conceived jointly by both of them. Vieta, as the originator of the indispensible unification of the various ancient and medieval traditions, would seem entitled to be designated as the 'father'; while Ghetaldi, as the person who nurtured the embryo of this new analytical art, to that of 'mother'. One might tentatively date the moment of conception as Ghetaldi's first meetings with Vieta in Paris in 1604, and the time of birth as shortly before his death in 1627 when he is known to have composed his treatise. Whatever one may think of this overworked and perhaps immoral analogy, two things follow automatically from the proposed interpretation of the birth of analytic geometry: one is that it did *not* require the adoption of the coordinate principle, as Günther claimed; and the other is that it occurred prior to the publication of Descartes' *Géométrie*. A visual aid for recalling the various mathematical developments described in the foregoing text is provided in the form of this diagrammatic summary.



The question which still demands an answer is: where *does* Descartes' work fit into this picture? Descartes' earliest known mathematical discoveries were made in March 1619, some four months after his memorable encounter with the Dutch mathematician Isaac Beeckman, when he envisaged four new compass constructions including one for the solution of the ancient Greek problem of trisecting an angle and another for solving a cubic equation. This preoccupation with compasses is usually taken to be an indication of his just having read Commandino's 1588 translation of Pappus's *Collection*, in which descriptions are given of constructions used by different Greek geometers for tracing complex curves like the conchoid of Nicomedes (3rd century B.C.). Other facts worth stressing in this connection are Descartes' conception of the magnitude x^3 as a *length*, not as a volume, his distinguishing of 13 different cases of a cubic equation (viz. $x^3 = \pm px \pm q$, $x^3 = \pm px^2 \pm q$, $x^3 = \pm px^2 + qx + r$ where $p,q,r > 0$), and his recognition that the method of construction used in solving a cubic equation seemed capable of being extended indefinitely to include higher powers of x.

It would also appear to be significant that, when writing down these equations, Descartes adopted the same "cossic" notation as that of Father Christopher Clavius in the second enlarged edition of his *Geometrica practica* (Mayence, 1611) which was a recommended text-book at the Jesuit school of La Flêche while Descartes was a pupil there. Clavius's book certainly contains the same geometrical constructions for representing the roots of algebraic equations as one finds at the beginning of Descartes' *Géométrie*, and an explanation of Cardan's and Vieta's method of solving the most general form of a cubic equation by eliminating the term in x^2. This latter procedure is exemplified for a particular case (viz. $x^3 = 6x^2 - 6x + 56$) in Descartes' *Cogitationes privatae* (1619). It has been suggested by Gaston Milhaud [1921] that Descartes might have followed up a clue in Clavius's book to Eutochius's *Commentary* on Euclid for his knowledge of how problems in solid geometry, such as those of Menaechmus, could be solved using conic sections. At any rate he had read this work, which is bound together with the first Latin translation of Archimedes' Greek text (Basle, 1544), and Apollonius's *Conics*, before the end of the year. Descartes' own treatise *De solidorum elementis* (1619)--containing the famous relationship $F + V = E + 2$ among the faces (F), vertices (V), and edges (E) of regular polyhedra, often erroneously attributed to Euler--could be interpreted as an attempt to algebrify solid geometry and hence the world of nature. If it were, it was a fruitless one.

Descartes' *Vera mathesis*, or *Universal mathematics*, conceived about this time, is no more distinguishable from his philosophic method than the principles and method of Euclidean geometry are from Aristotle's *Analytics*. It is founded upon the straight line as being the most elementary geometrical notion, coupled with the four fundamental algebraic operations (+, -, x, ÷) and the laws of ratio and proportion. These principles sufficed to define all curves in terms of a distance along each of two axes (which may, or may not, be orthogonal), and to express the relation between these distances symbolically (viz. by a formula). His recognition as early as March 1619 that such a symbolic logistic might be the methodological tool of a grandiose science that was "nothing less than a complete classification of all questions concerning the nature and resolution of quantity", stemmed from his knowledge that mathematical generalizations obtained through its use were capable of being extended beyond the realm of spatial relationships only, back into the broader realm of philosophy from whence they had sprung. The need for symbolism lay in the step of separating the logical results of geometrical deduction from the intuitive principles from which they had been logically derived. By representing these principles by symbols, one might divorce the geometrical properties from their spatial context and produce instead a self-consistent set of equations or symbolic logic. Then a symbol originally representing a length, area, or volume, could equally well be regarded as representing *any* type of dimension; for example, a dimension of motion (velocity) or a dimension of heaviness (weight). By this means, and by identifying spatial extension with the substance of a body, Universal Mathematics could be made to comprehend the corporeal world. This, in my view, was the essence of what Descartes claimed to have discovered on 11 November 1619, which several commentators have called the birth of analytic geometry but which I would prefer to call the birth of theoretical physics (or indeed, the birth of a new science of morals), since I feel that the realm of mathematics has been transcended. Only because analytic geometry has now grown to embody the formalistic expression of natural phenomena, and its logical symbolism to supplant the intuitively simpler but less flexible geometrical models or analogies, has this distinction become blurred.

We see, therefore, that one great intellectual leap carried Descartes over the restrictive boundaries of geometrical analysis into the realm of an "entirely new science" which it became his immediate mission in life to develop and propagate. The famous dream associated with this revelation should probably be regarded as a deliberately-chosen literary device for communicating in classical imagery the idea that his thinking was being divinely-inspired, and consequently embodied the truth about the world of nature. Moreover, it served to remind others of the fact that the door to true understanding was a form of intuition or natural enlightenment rather than syllogistic argumentation in which the conclusions could never transcend the precepts from which they had been derived. Thus intuition was the vitalizing element in Descartes' new philosophical method, just as the exegetic procedure had been in Vieta's new analytical art. His consciousness of what he was attempting to achieve is expressed in part 2 of his *Discourse on Method* (1637), where he refers to his early studies in logic, geometry, and algebra, and states: "I thought I must look for some other method which would combine the advantages of these three disciplines, and yet be exempt from their defects." It was this thought which led him to his four rules of method, the heart of his philosophy.

NOTE

1. This paper is a modified and reduced version of a lecture delivered by the author to the British Society for the History of Mathematics in London on 17 December 1974.

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