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Conjugate diameters: Apollonius of Perga and Eutocius of Ascalon

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CONJUGATE DIAMETERS: APOLLONIUS OF PERGA AND EUTOCIUS OF ASCALON

by

Colin Bryan Powell McKinney

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

July 2010

Thesis Supervisor: Professor Daniel D. Anderson

ABSTRACT

The *Conics* of Apollonius remains a central work of Greek mathematics to this day. Despite this, much recent scholarship has neglected the *Conics* in favor of works of Archimedes. While these are no less important in their own right, a full understanding of the Greek mathematical corpus cannot be bereft of systematic studies of the *Conics*. However, recent scholarship on Archimedes has revealed that the role of secondary commentaries is also important. In this thesis, I provide a translation of Eutocius' commentary on the *Conics*, demonstrating the interplay between the two works and their authors as what I call *conjugate*. I also give a treatment on the duplication problem and on compound ratios, topics which are tightly linked to the *Conics* and the rest of the Greek mathematical corpus. My discussion of the duplication problem also includes two computer programs useful for visualizing Archytas' and Eratosthenes' solutions.

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Thesis Supervisor

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Title and Department

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the July 2010 graduation.

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Οὐκοῦν τοῦτο ἔτι διομολογητέον; Τὸ ποῖον; Ὡς τοῦ ἀεὶ ὄντος γνώσεως, ἀλλ᾽ οὐ τοῦ ποτέ τι γιγνομένου καὶ ἀπολλυμένου. Εὐομολόγητον, ἔφη· τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν. Ὁκλὸν ἄρα, ὦ γενναῖε, ψυχῆς πρὸς ἀλήθειαν εἴη ἂν καὶ ἀπεργαστκὸν φιλοσόφου διανοίας πρὸς τὸ ἄνω σχεῖν ἃ νυν κάτω οὐ δέον ἔχομεν. Ὡς οἷόν τε μάλιστα, ἔφη.

ΠΛΑΤΩΝ, ΠΟΛΙΤΕΙΑ Ζ΄

ABSTRACT

The *Conics* of Apollonius remains a central work of Greek mathematics to this day. Despite this, much recent scholarship has neglected the *Conics* in favor of works of Archimedes. While these are no less important in their own right, a full understanding of the Greek mathematical corpus cannot be bereft of systematic studies of the *Conics*. However, recent scholarship on Archimedes has revealed that the role of secondary commentaries is also important. In this thesis, I provide a translation of Eutocius' commentary on the *Conics*, demonstrating the interplay between the two works and their authors as what I call *conjugate*. I also give a treatment on the duplication problem and on compound ratios, topics which are tightly linked to the *Conics* and the rest of the Greek mathematical corpus. My discussion of the duplication problem also includes two computer programs useful for visualizing Archytas' and Eratosthenes' solutions.

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CHAPTER 1
INTRODUCTION AND HISTORICAL OVERVIEW

1.1 A Brief Historical Overview

This thesis approaches several periods within the history of Greek mathematics. They are all mathematically related, of course; but I focus on them as, say, mathematical vignettes.

The Greek mathematical period ranges a considerable quantity of time, from Thales of Miletus (7th century BCE), through Pythagoras of Samos (5th century BCE) and moving to Eudoxus of Cnidus, Archytas of Tarentum, and others who were associates of Plato. After the end of the Athenian hegemony and the conquest of Alexander, our story moves to the Hellenistic period, where the center of mathematical activity was the newly founded city of Alexandria on the coast of the now-Ptolemaic realm of Egypt. Euclid and Apollonius are two key figures here; we know that both lived and worked primarily in Alexandria. And while Archimedes mostly kept to himself in Syracuse, in Sicily, he was in contact with mathematicians in Alexandria such as Eratosthenes of Cyrene, near the western extent of Ptolemaic Egypt.

With the Roman conquests of mainland Greece, the cession of the Attalid empire in Anatolia to Rome, and the eventual defeat of Marcus Antonius and Cleopatra, the so-called Hellenistic era came to a close. Several notable Greek mathematicians emerge, though, after the founding of the Roman Empire: chief amongst them, of course, is Claudius Ptolemaeus, whose *Syntaxis* formed the mathematical and physical explanation for the geocentric Greco-Roman cosmos.

Another important class of authors appears during the Roman (and, after the division of the Empire, the Byzantine) period. These are the great commentators, who aimed to organize and expound upon the mathematical wisdoms of their ancestors. Notable here are Pappus of Alexandria, Theon of Alexandria and his daughter Hypatia, Proclus Lycaeus (also called Proclus Diadochus, *the Successor*), and lastly, Eutocius of Ascalon. These mathematicians wrote commentaries on works such as the *Elements*, the Archimedean corpus, and the *Conics*; in addition, they also edited and compiled new editions of these texts.

Eutocius specifically edited the *Conics* and several of Archimedes' works; his edition and commentaries on the *Conics* form the basis for our story.

1.2 Eutocius, Apollonius, and Archimedes

In the current literature, Archimedes is by far a more popular author to study. This in itself is not surprising, for several reasons. First, Archimedean works are smaller and self-contained; each approaching a specific mathematical concept with distinct goals in mind. Second, of course, is the re-discovery of the Archimedes Palimpsest. After its purchase at auction by an anonymous buyer, the Palimpsest came to reside at the Walters Art Museum in Baltimore, where, under the careful study of the Museum's staff, the text has been thoroughly analyzed. The rediscovery of the Palimpsest has allowed Reviel Netz to publish a new English edition of *On the Sphere and the Cylinder*, together with Eutocius' commentaries thereon ([16, 24]); future works are forthcoming.

With the exception of Michael N. Fried and Sabetai Unguru's recent book ([3]), and a few papers by Ken Saito, Apollonius' *Conics* has largely been neglected in recent times: the last major work on the subject was that of H.G. Zeuthen (in [28, 29]). It was not even until Fried's dissertation that Book IV of *Conics* was available in English (now published in [22]; Books I-III had been translated from the Greek by R. Catesby Taliaferro at the request of St. John's College in Annapolis, Maryland. Books V-VII, which are extant only from the Arabic tradition, were translated by Gerald Toomer ([19, 20]). This left the very awkward situation of Book IV being the lone untranslated extant book from the Greek.

The lack of attention to Apollonius, though, is not altogether surprising. As Saito quite eloquently puts it (in [27], his review of [3]):

> If we define a classic as “something that everybody wants to have read and nobody wants to read” (Mark Twain), then there is no doubt that Apollonius’

> *Conica* is the classic of the classics in Greek mathematics. There have been few people who have gone through this long and difficult text during the 20th century; one-third of those who have gone through Archimedes may be too optimistic an estimate. The Archimedean Corpus, the only extant document comparable to Apollonius' *Conica* in the level of its mathematical content, is more readable by far, consisting of several single works of medium length (up to about 50 propositions), each of which accomplishes some definite aim. The *Conica*, in contrast, is a massive heap. Though it is divided into seven books (eight, if the lost Book VIII is taken into account), its one and only aim is to investigate the properties of conic sections—practically an endless task. There are certainly admirable propositions here and there, but their significance is often very unclear, and one can get far less satisfaction from reading Apollonius than from reading Archimedes, who at least rewards the reader's patience by arriving at some significant result at the end of each work. Thus a reader of the *Conica* is always left with a sense of being lost in a labyrinth, the intention of whose creator is obscure. No wonder this work has stimulated the intellectual appetite of modern historians far less than the Archimedean Corpus.

Netz' edition of *On the Sphere and the Cylinder* adds a new layer to this, by considering not just the Archimedean text, but also the commentary written by Eutocius. However, there is currently no complete translation of Eutocius' commentaries on the *Conics*, a situation which this thesis partially aims to rectify[1]. My translation, together with some analysis of my own, is presented in Chapter 3.

1.3 Conjugate Diameters

Throughout this thesis, I make considerable use and reference to the secondary commentaries and scholia. In a large sense, I believe these commentaries are permanently linked to the texts on which they comment. For example, Eutocius himself tells us, in his commentary on Apollonius, that his aim was twofold: to produce *a new edition of the* Conics *itself*, and to add commentary and alternate proofs as scholia. From his commentary:

> But since there are so many editions, as he himself [Apollonius] says in his letter, I considered it better to bring them together, from those available, placing the more manifest things in the text side by side for the benefit of beginning students, and to note outside in the adjacent scholia the different courses of the proofs, as seemed reasonable.

[1] A French translation, together with a new critical edition of the Greek text, is underway by Roshdi Rashed, Micheline Decorps-Foulquier, and Michel Federspiel. The text is to be published in the near future by de Gruyter.

The letter to which Eutocius is referring is Apollonius' introductory letter to *Conics* I:

> ...and how on arranging them in eight books we immediately communicated them in great haste because of his [Naukrates'] near departure, not revisiting them but putting down whatever came to us with the intention of a final going over. And so finding now the occasion of correcting them, one book after another, we publish them. And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision, don't be surprised if you come upon them in a different form.

It is in this sense that I call Eutocius' commentary and the *Conics* itself *conjugate*, and why I place such importance on understanding these secondary commentaries. Further, given that Eutocius comments on Books I-IV of the *Conics*, and that these are precisely the books extant in Greek, one cannot help but agree with Heath (in [4, 5] in his assertion that the commentary aided the survival and transmission of the text.

1.4 Editions, Texts, Copies, Commentaries, and Scholia

It is important to make a distinction between the following terms: edition, text, copy, commentary, and scholia, and to understand what role they had to Eutocius. By edition, Eutocius refers to the different editions of the original *Conics*. We know that different editions existed even in Apollonius' time, as confirmed by his introductory letter to *Conics* I. The idea of different editions of a work holds even today, and the existence of the concept in antiquity is not surprising.

A major difference, however, is that of *copies*. In modern publishing, two copies of a given edition are largely identical. Even when not identical (such as when an edition is reprinted) the differences are usually extremely minor and in some cases these reprints are facsimiles of the source edition. These differences are more extreme with ancient texts, which were hand copied over time, often from other previously made copies. The result is that different copies disagree from each other and common ancestor source copies. The job of sorting these differences out is left to the textual critic; recent scholarship of Netz et al. has added a focus on diagrams to mathematical texts.

Eutocius essentially was a textual critic. He had at his disposal a number of source copies, perhaps representing many different lineages from Apollonius' two editions. The differences in these copies forms the basis for many parts of edition of the *Conics*: he wishes to use the best versions of proofs in order to make the overall text more clear and more understandable to students and other non-specialists. Alternate versions of proofs found from the various copies are given in his commentary when appropriate. He also adds in the scholia found in his source copies.

A word on Eutocius' audience: in some sense there are three. First and foremost, as Eutocius himself tells us, he is concerned with students and how they will be using the text. He wants the proofs to be as clear and concise as possible for the students. Second, he is writing directly for Anthemius, who he addresses in the introduction. And third, he is writing for future mathematicians, who will use his commentary to see alternative proofs and glimpses of his own source copies.

1.5 Notation and The Structure of Greek Mathematical Arguments

In general I follow the notational style of Heath's edition of the *Elements*.[2] Other works use a similar format, specifically the Green Lion editions of the *Conics* ([21, 22]). A summary of notational elements is given in Table 1.1.

There are six traditional parts of a Greek mathematical proof. Extended discussions of them are given in [4, 5, 10]. A short description is given below:

1. The *protasis* (πρότασις): The enunciation of the proposition, which states the hypotheses and thing to be proved without reference to any particular diagram.

2. The *ekthesis* (ἔκθεσις): A restatement of part of the protasis, in which the hypotheses are given in terms of a constructed diagram. In Greek, ekthesis literally means "setting out." A common Latinate rendering is "exposition."

3. The *diorismos* (διορισμός): The claim about what is to be proved. In Greek this generally starts with the words "λέγω ὅτι," or, "I say that..."

4. The *kataskeuē* (κατασκευή): Completion of the diagram, constructing any additional auxiliary parts not already constructed in the *ekthesis*.

[2] The text of this is the same as in the Greek Lion Edition ([12]), however, that edition dispenses with the mathematical notation in favor of keeping things in word form (as in the Greek). The technical nature of the *Conics*, however, I feel justifies using this notation.

Table 1.1: Summary of Notation

tri.($AB\Gamma$)	The triangle with vertices A, B, Γ.
sq.(AB)	The square with side AB.
rect.($AB, \Gamma\Delta$)	The rectangle with sides $AB, \Gamma\Delta$.
rect.($A\Gamma$)	The rectangle about the diagonal $A\Gamma$.
quad.($A\Gamma$)	the quadrilateral about the diagonal $A\Gamma$.
pllg.($AB, \Gamma\Delta$)	The parallelogram with sides $AB, \Gamma\Delta$.
pllg.($A\Gamma$)	The parallelogram about the diagonal $A\Gamma$.
pllpd.($AB, \Gamma\Delta, EZ$)	The parallelepipedal solid with sides $AB, \Gamma\Delta, EZ$.
pllpd.($A\Delta$)	The parallelepipedal solid about the diagonal $A\Delta$.
$A : B = \Gamma : \Delta$	The ratio of A to B is equal to that between Γ and Δ.
dup.($A : B$)	The duplicate ratio of $A : B$.
trip.($A : B$)	The triplicate ratio of $A : B$.
($A : B$) comp. ($\Gamma : \Delta$)	The ratio compounded from the ratios $A : B$ and $\Gamma : \Delta$.
arc($AB\Gamma$)	The arc through points A, B, Γ.
section($AB\Gamma$)	The conic section through points A, B, Γ.

5. The *apodeixis* (ἀπόδειξις): This is the actual proof. The word here is from the verb ἀποδείκνυμι.
6. The *symperasma* (συμπέρασμα): The conclusion, in which the claim of the diorismos is restated as now having been proved. Apollonius occasionally adds a definition in this part, such as "...and let such a section be called a parabola."

1.6 On Translation

The style and structure of mathematics in Greek differs considerably from that of modern mathematics. The following is a short discussion of some of the differences and how they affect translators and their translations.

To start, the text is written continuously; the practice of putting "equations" on their own lines is much more modern. Greek also has the ability to leave out many more words than English due to its inflected nature. If a word is to be the direct object of a verb, for example, a pronoun will suffice when this word is mentioned again. This presents a difficulty in English, where typically a pronoun is taken to refer to the last explicitly mentioned noun. In translating Greek to English, a translator often must resupply the appropriate noun, lest the passage turn into an impenetrable pile of confusing pronouns. Since there are only archaic vestiges of inflection left in English, the only other way to keep pronouns organized would be to color them, and make the red pronoun "it" and the blue pronoun "it" different[3].

For a translator, this presents a few difficulties. First, of course, is that it makes total fidelity to the Greek impossible. But it is important to recognize that Greek and English are totally different languages, with different syntax, word order restrictions, *et cetera*, so it should not be totally surprising that issues such as these come up. The second issue is more one of utility and readability. Reading a translation with resupplied nouns, for example, may falsely lead the reader into thinking that this is how it appeared in Greek. Making note of this issue, then, is absolutely necessary. The question is how and when these notes are best made, and where on the continuum between total fidelity to the Greek

[3]Interestingly, this can work in a more pictorial setting: Oliver Byrne's edition of the *Elements* illustrates this beautifully.

and ease of use for a modern English reader the work should be.

Further, words in mathematical Greek are sometimes completely omitted. For example, in Eutocius, the phrase "the square on AB" never actually mentions the word square. In Greek, it appears as "τὸ ἀπὸ AB", literally, "the from AB." Another example is the differences between "angles" and "rectangles": in Greek, they differ only by gender. The first appears as the feminine "ἡ ὑπὸ ABΓ", whereas the second appears as the neuter "τὸ ὑπὸ ABΓ". These phrases are essentially abbreviations; they come up often enough in mathematical works for the omission of "square" or "angle" or "rectangle" to be practical. But how do we make note of this as translators? One way would be to always reinsert the appropriate noun, such as "square," but to somehow note that it does not actually appear in the Greek. Netz takes this approach; for "τὸ ἀπὸ AB", he writes "the <square> on AB." By doing so, Netz makes very clear what actually appears in the Greek and what are his insertions. However, it does make for some laborious reading, and the presence of so many pointed brackets is, frankly, an eyesore. I prefer to take a more selective approach; even though "square" may not appear in the Greek, I prefer to put it in the English without any brackets or other distracting notation except in special situations. While this may be less transparent than Netz' approach, I feel the benefits to the reader justify my preferred practice.

The matter of vocabulary has also been a significant challenge. In most cases, I try to use the same translations of technical vocabulary as exist in the current English literature, especially the Heath, Taliaferro, and Netz translations. Heath made many contributions to mathematical uses listed in the Liddell, Scott, and Jones dictionary ([8, 9]), which has been my primary source for matters of translation. Also quite helpful has been Mugler's *Dictionnaire Historique de la Terminologie Géométrique des Grecs* ([11]).

1.7 Mathematical Vignettes

The structure of my thesis is divided as follows: in Chapter 2, I give a mathematical overview of the state of conic sections both before and after Apollonius. Chapter

3 presents my translation of Book I of Eutocius' commentary, statements of the theorems from Apollonius', and my own discussion and analysis where appropriate. For this I have translated Heiberg's edition of the Greek, from [14] and the digital version of it on Thesaurus Linguae Graecae. Chapter 4 explores the problem of duplicating a cube, a problem whose many varied solutions come to us primarily through Eutocius. Chapter 5 deals exclusively with the topic of compound ratios, as it appears in major mathematical authors and in Eutocius' work. Appendix A contains Heiberg's Greek text of Eutocius' commentary on the *Conics*, which is rather hard to find in print due to the scarcity of the century-old Teubner editions (and the unpublished status of the forthcoming Rashed, Decorps-Foulquier, and Federspiel edition). Though this text is also available on Thesaurus Linguae Graecae, I have included it here for the convenience of interested readers or those unable to access the TLG. Appendix B is included for the same reason, but presents the Greek text for various discussions of compound ratios subordinate to Chapter 5. The Greek texts for this appendix comes from [15, 14, 13], with English translations by Netz (from [16]), myself (from Chapter 3), and Knorr (from [7]).

CHAPTER 2
INTRODUCTION TO CONIC SECTIONS

2.1 Pre-Apollonian Notions

Eutocius himself gives an account of cones and conics before Apollonius. Quite before him, however, Euclid defines a (pre-Apollonian) cone in *Elements* XI, though it is in his lost *Elements of Conics* that the conics themselves are defined and examined. We know, for example, that Archimedes was familiar with this work, himself referring to it in his *Quadrature of the Parabola* [23].

Before Apollonius, a cone was constructed by means of a right triangle, say, ΑΒΓ. Keeping one leg about the right angle fixed, the triangle was revolved, until it returned to its original position. The surface so swept out was defined as a cone; the vertex was the endpoint of the hypotenuse which remained fixed, and the base was the circle generated by the rotating leg. The axis of the cone is the fixed leg, which thus is perpendicular to the base circle and connects the vertex and the center of the base circle. A cone was said to be acute, right, or obtuse, according to whether the angle subtending the rotating leg was less than, equal to, or greater than one-half a right angle, respectively. Such is the definition in the *Elements*, and is confirmed on the authority of Geminus as seen in Eutocius' commentary on Apollonius.

Each conic, then, was defined by a particular intersection of a plane with one of the cones. In all instances, the plane intersects the cone at right angles, in the sense that it is perpendicular to the generating hypotenuse. The common section of the cutting plane and the plane of the generating triangle was called the diameter of the section. If the original cone is right, then the diameter is parallel to a generator, and the resulting section was called *the section of a right-angled cone*. Although we would call such a section a

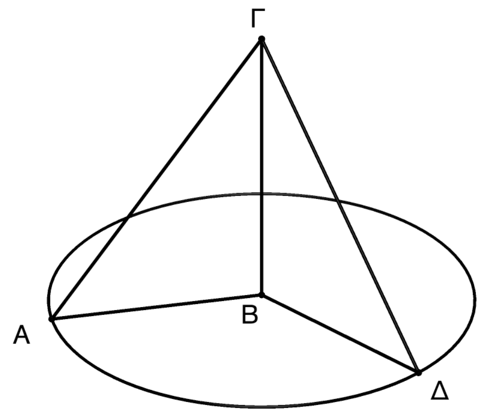

Figure 2.1: A pre-Apollonian Cone

parabola, it was Apollonius himself who first coined this term. Similarly, then, a hyperbola is the section of an obtuse angled-cone; the ellipse the section of an acute angled-cone. Even Archimedes, a mere twenty-five year elder to Apollonius, uses these "old" names for the sections.

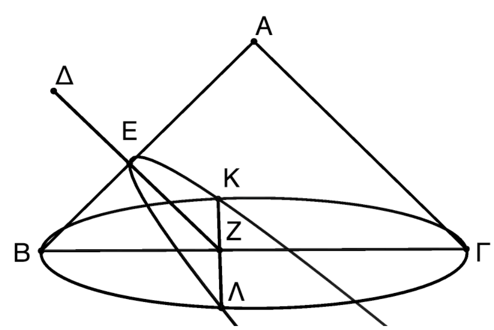

Figure 2.2: A pre-Apollonian Section of a Cone

Heath reports that the conics were first investigated by Menaechmus (a student of Eudoxus and Archytas); indeed, Eutocius' quotation of Eratosthenes' solution to the duplication of the cube calls him "three-conic cutting Menaechmus" ([16]) Why it is that Menaechmus first conceived of the conics remains a mystery: whether it was as a specific tool to solve the duplication problem or as a curious mathematical object, we cannot know. But it should be stated that his solutions to the duplication problem are of such simplicity, assuming the principle properties of conics, that it would not be entirely unlikely that he first investigated cones expressly for this purpose. One of his teachers, Archytas (an associate of Plato), gives a wonderfully grand construction that involves intersections of rotating figures and a cylinder, yielding non-planar curved lines. Details of these constructions are given in the chapter on the duplication problem. If his teacher was capable of imagining such a bold construction, perhaps it is not so unlikely that Menaechmus thus imagined the conics.

2.2 Apollonian Notions

Apollonius, however, defines his cones and conics differently. Instead of beginning with a triangle, as did his predecessors, Apollonius considers a circle and a point not on the plane of that circle. Taking a point on the circle, and joining the two points, this line is extended both directions indefinitely. As the point on the circle is rotated about the circle, until it returns to its starting position, the infinite line sweeps a surface, really *two* surfaces, which Apollonius calls collectively the *conic surface*. The vertex is the original point not on the plane of the circle, the base is the given circle, and the portion of the surface between he defines as a cone. The conic surface, then, may be imagined as a double-napped cone, much as in the modern conception, though he often concentrates on one cone instead of the conic surface. In fact, Fried and Unguru (in [3]) believe that he tries to avoid the conic surface entirely, due to its awkward singular/dual existence. One surface or two?

Apollonius differentiates two types of cones: those which are right, and those which are not. The criteria for each depends on whether the axis of the cone meet the base at right angles. In pre-Apollonian cones, of course, this is always the case; therefore all cones before Apollonius, even the so-called acute angled or obtuse angled cones, are right as far as Apollonius is concerned. The second type of cone is the oblique[1]. In essence, then,

[1]We tend to use oblique rather than scalene, despite the Greek word in question being σκαληνός.

Apollonius' work will capture everything about the works of his predecessors, but doing so in sufficient generality so as to demonstrate new and extended results.

Apollonius defines next the *axial triangle*, which serves to give him a scaffolding with which to construct the other sections. There is a unique diameter of the (base) circle which, when a plane containing it is set up perpendicular to the base, the plane also contains the vertex. The common section of this plane and the cone is called the axial triangle[2]; manifestly, then, it contains the axis. The cutting plane that will yield the sections is established so that it intersects one side of the axial triangle and cuts the aforementioned diameter at right angles. Depending on whether or not it cuts the other side of the axial triangle, and where, the section is called either a parabola, an hyperbola, or an ellipse.

It is also interesting to note here the intrinsic use of the "true" form of Euclid's fifth postulate, often misquoted, as it appears in the *Elements*. For completeness, I state it here, using Heath's translation in [12]. I also give a diagram as an example.

> ***Elements* I, Postulate 5:** That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

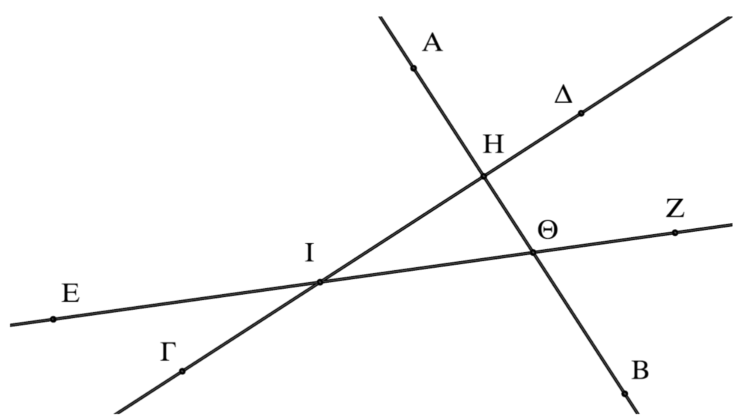

Figure 2.3: Euclid's Fifth Postulate

[2]The cone, properly speaking, consists of the base circle too. So the axial triangle is really a triangle, with one side the diameter of the circle, and the other two on the surface of the cone and terminating at the vertex.

In the figure, take the straight line as AB, and the two straight lines as ΓΔ, EZ. The interior angles in question here are IHΘ and IΘH, which are together less than two right angles. Accordingly, the lines HΔ, ΘZ, when produced, meet on the same side of AB as these two angles. In the figure, this happens at I.

But back to the conic sections. Eutocius himself plays a bit of a word game with the names: for in a parabola, the cutting plane (and thus the diameter of the section, being the common section of the cutting plane and the axial triangle) is parallel to the other side of the axial triangle; in the ellipse it meets on the same side of the vertex as the cone, and in the hyperbola it meets beyond the vertex. Considering certain angles in the figures, he describes that the hyperbola occurs when these angles exceed (ὑπερβάλλον) two right angles, the ellipse when they are deficient (ἐλλεῖπον), or when equal, the diameter is thus parallel. In the case of the parabola in Figure 2, for example, the angles in question are ZEA and EAΓ.

I say word games because this is not precisely the reason that Apollonius chooses these names, for he himself is more concerned with the excess/equality/deficiency upon the application of areas to certain lines. It is certainly possible, though, that Eutocius intended a double word play: contrasting the parallelism (arising from the perpendicular application) of the cutting planes in pre-Apollonian sections with the non-parallel cutting planes in Apollonius.

This method of cutting the cone thus differs from his predecessors in a complementary or dual way: his predecessors kept the sectioning procedure fixed for all cones, creating the varying sections by varying the type of cone which is sectioned. Apollonius, however, shows that for a fixed cone of any type whatsoever, it is possible, by means of a different application of the cutting plane, to cut any of the sections. The result of this is that we might include the circle amongst Apollonius' conic sections. Though it certainly would be clear to his predecessors that a plane cutting the cone parallel to the base will generate a circle, the sectioning procedure is not that of the other conics. Apollonius, on the other hand, is free to section as he likes, and indeed, the fact that one may cut such a circle is the fourth proposition in the *Conics*. What is perhaps more unusual, however, is the existence of a second way to section a circle: this by means of a plane cutting subcontrariwise[3]. On account of this similarity, it should not be surprising that a plane that cuts the cone with this third side as a diameter should be a circle; indeed Apollonius proves this fact. It should be noted that this occurs only in the oblique cone: for in a right cone, the subcontrary plane is itself parallel to the base on account of the axial triangle being isosceles.

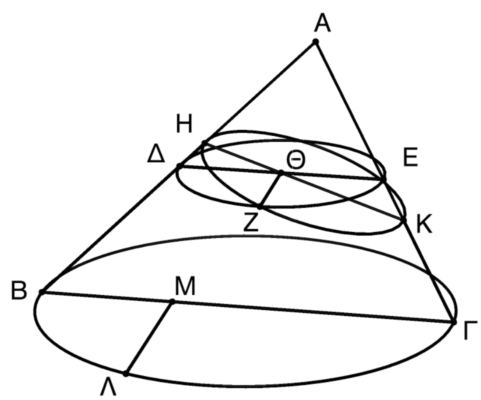

Figure 2.4: A Subcontrary Circle

2.3 The *Symptomata*

The following *symptomata* are adapted and condensed from Heiberg's edition of the Greek text and Taliaferro's subsequent translation. For brevity, I have omitted the full *ektheseis* and *kataskeuai*[4], though my diagrams retain all parts of those in Heiberg's edition. The diagrams are equivalent to those found in Taliaferro's translation, but for consistency, I retain the Greek letters.

[3]That is, if a triangle is situated inside the axial triangle subcontrariwise (ὑπερκείμεναι), similar but flipped, so that the third side is not parallel to the base.

[4]These are two of the six parts of most Greek proofs. See the appendix for definitions.

2.3.1 The Parabola

The square on the ordinate is equal to the rectangle contained by the abscissa and parameter (or latus rectum). In Figure 2.5, this is sq.(ΚΛ) = rect.(ΘΖ, ΖΚ), with ΖΚ the abscissa, and ΖΘ the parameter. Note that "parameter" is an abbreviation. Apollonius states in his *symperasma:* "let ΘΖ be called the straight line to which the the straight lines drawn ordinatewise to the diameter ΖΗ are applied in square (παρ' ἣν δύναται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν ΖΗ διάμετρον), and let it also be called the upright side (ὀρθία)."

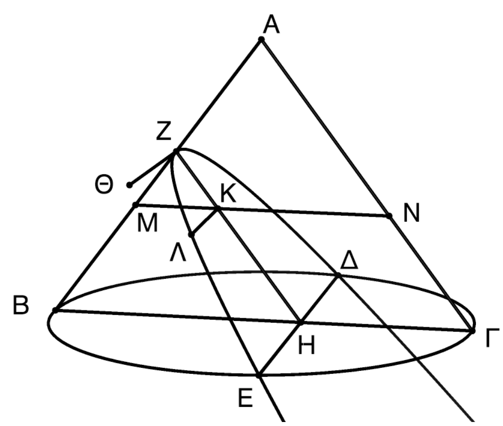

Figure 2.5: *Symptoma* of the Parabola

2.3.2 The Hyperbola

The square on the ordinate is equal to a rectangle applied to the upright side which has the abscissa as breadth, and this rectangle projects beyond the upright side by a rectangle similar to the rectangle contained by the upright and transverse sides. Again, the upright and transverse sides are abbreviations. The upright side is the abbreviation for the same Greek for which we use "parameter" as an abbreviation; the transverse side is essentially a "second" parameter. In terms of the diagram, Apollonius' *diorismos* is as follows: "I say that ΜΝ is equal in square to the parallelogram ΖΞ which is applied to

ZΛ, having ZN as breadth, and projecting beyond by a figure ΛΞ similar to the rectangle contained by ΘZ and ZΛ."

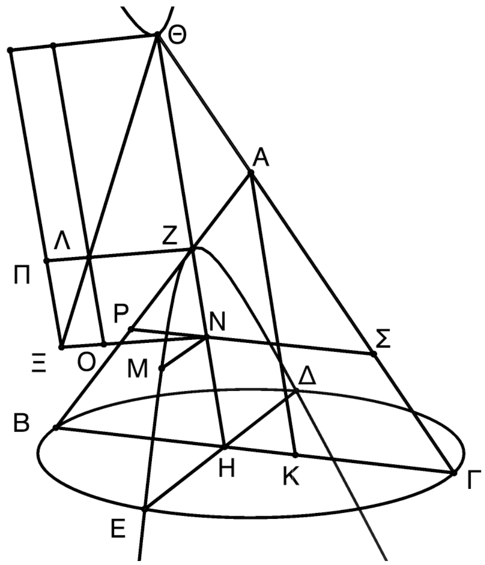

Figure 2.6: An Hyperbola

2.3.3 The Ellipse

The square on the ordinate is equal to a rectangle applied to the upright side which has the abscissa as breadth, and this rectangle is deficient by a rectangle similar to the rectangle contained by the upright and transverse sides. In terms of the figure, the ordinate is ΛM, the abscissa EM, the upright side EΘ, and the transverse side EΔ. The abbreviations "upright side" and "transverse side" are expressed the same as in the case of the hyperbola.

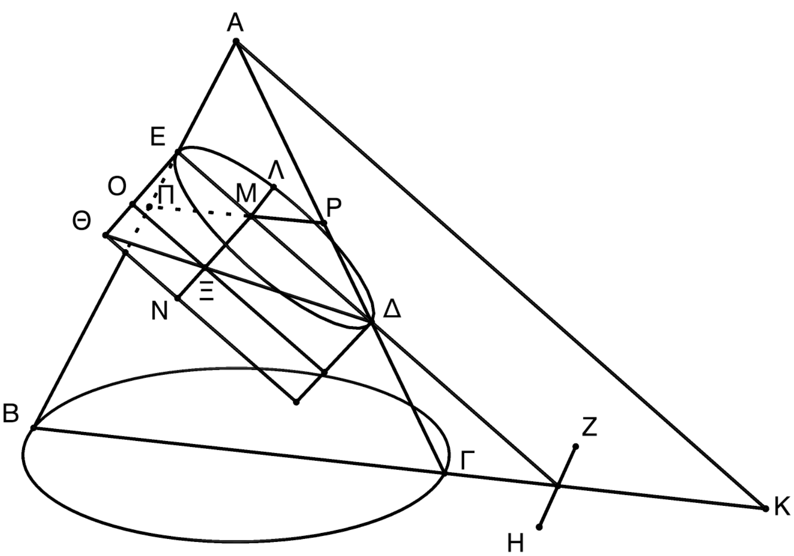

Figure 2.7: An Ellipse

CHAPTER 3
EUTOCIUS' COMMENTARY: TRANSLATION AND ANALYSIS

3.1 Eutocius' Introduction and To the First Definitions

(168.5) Apollonius the geometer, my dear friend Anthemius, was born in Perga in Pamphylia during the reign of Ptolemy Euergetes, as chronicled by Heraklius, who wrote *The Life of Archimedes*. He also says that Archimedes first thought of the conic theorems; but that Apollonius, having found them unpublished by Archimedes, made them his own. But he is not correct, in my opinion, at least. For both Archimedes seems to recall in many passages the *Elements of Conics* as more ancient, and Apollonius does not write his own thoughts: for he would not have said that he had worked these things out in full and more generally than the writings of others. But the very thing which Geminus says is true: that the ancients, defining a cone as the revolution of the right triangle, with one side about the right angle remaining fixed, naturally assumed that all cones are right, and that one section occurs in each: in the right-angled[1] cone, what is now called the parabola, in the obtuse-angled, the hyperbola, and in the acute-angled, the ellipse: and it is possible among them[2] to find the so-called sections. So just as the ancients theorized on the fact that in any triangle there are two right angles--first in the equilateral, in turn the isosceles, and last the scalene--their descendants proved a general theorem as follows: in every triangle, the three internal angles are equal to two right angles; likewise in the case of the sections of the cone. For the thing called a section of a right-angled cone they viewed only by means of a plane cutting orthogonal to one side of the cone, and they showed that the section of the obtuse cone occurs in an obtuse cone, and that of the acute cone in an acute one: likewise in all cones bringing planes orthogonal to one side of the cone: and he

---

[1]It is important that we distinguish - as does Apollonius - that a cone is *right* when it is formed from the revolution of a right triangle. The angle at the vertex of the cone, which subtends the rotating leg, determines if the cone is right-angled, obtuse-angled, or acute-angled.

[2]i.e. the different cones

[Geminus] shows also the ancient names of the curves themselves. But later, Apollonius of Perga theorized somewhat generally, that in every cone, both the right and the oblique, all the sections are according to a different application of the cutting plane to the cone; and his contemporaries, having marveled at the wonder of the conic theorems shown by him, called him a great geometer. So Geminus says these things in the sixth book of *The Theory of Mathematics*. But what he says, we will make clear in the diagrams below.

(170.28) The Parabola. Let there be the triangle ABΓ through the axis[3], and let ΔE be drawn at right angles to AB from the random point E, and let the plane produced through ΔE cut the cone orthogonal to AB: therefore each of the angles AEΔ, AEZ is right. Since the cone is right and the angle BAΓ is clearly right, as in the first diagram, the angles BAΓ, AEZ will be equal to two right angles: so that the line ΔEZ will be parallel to AΓ. And a section is produced on the surface of the cone which is called a *parabola*; thus named on account of the line ΔEZ , which is the section common to the cutting plane and the axial triangle, being parallel to the side AΓ of the triangle.

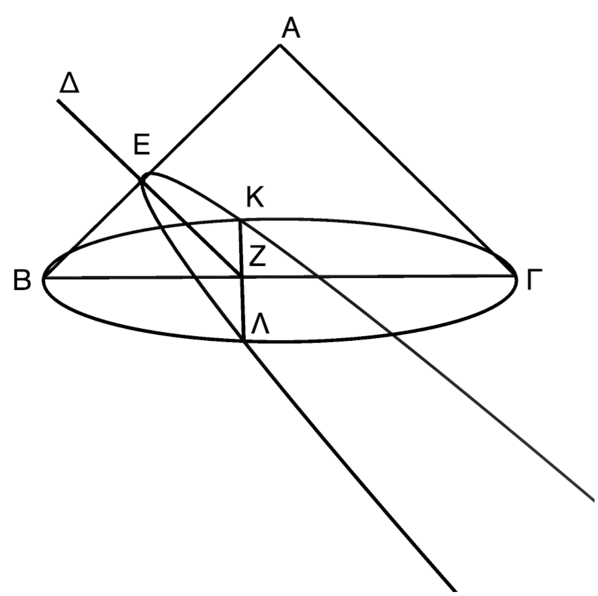

Figure 3.1: A Parabola

[3]i.e. the axial triangle

(172.18) The Hyperbola. But if the cone be obtuse-angled, as in the second diagram, with the angle ΒΑΓ manifestly being obtuse, but the angle ΑΕΖ right, the angles ΒΑΓ, ΑΕΖ will be greater than two right angles: so that ΔΕΖ will not intersect the side ΑΓ at the parts near Ζ, Γ; but rather, at the parts near Α, Ε, with ΓΑ manifestly being projected to Δ. So the cutting plane will make on the surface of the cone a section, the so-called *hyperbola*, being called thus from the fact that the aforementioned angles, i.e. ΑΕΖ, ΒΑΓ, exceed two right angles; or on account of ΔΕΖ projecting beyond the vertex of the cone and meeting ΓΑ outside.

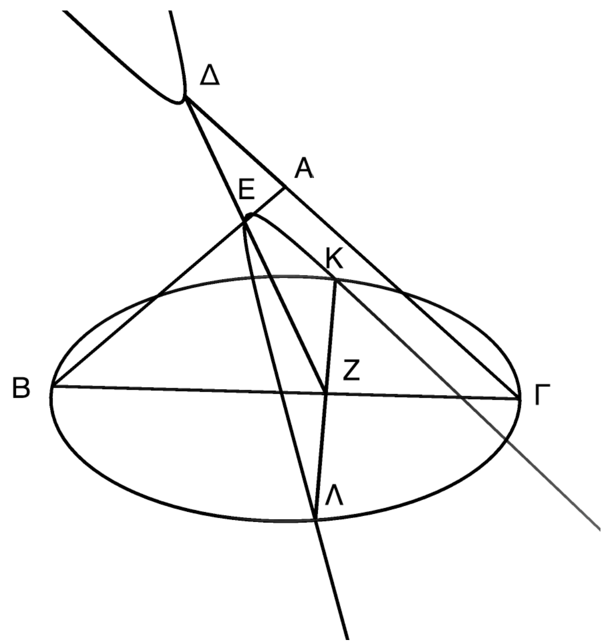

Figure 3.2: An Hyperbola, with its Opposite Section

(174.3) The Ellipse. But if the cone be acute, with the angle at ΒΑΓ manifestly being acute, the angles ΒΑΓ, ΑΕΖ will be less than two right angles: so that the segments ΕΖ, ΑΓ, being produced, will meet somewhere: for I am able to extend the cone. So there will be a section on the surface, which is called an *ellipse*, thus being called either on account of the aforementioned angles falling short of two right angles, or on account of the ellipse falling short of a circle.

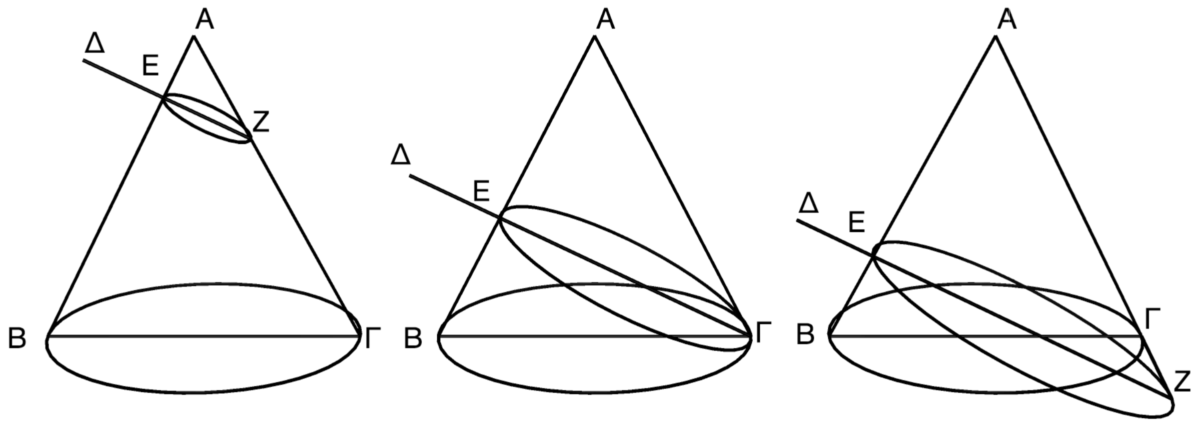

Figure 3.3: Three Possible Ellipses

(174.11) And so the ancients, having laid down the cutting plane, the one through ΔEZ, orthogonal to the side AB of the triangle through the axis of the cone[4], even so saw the cones as different, and in each its own section. But Apollonius, having laid down the right cone and the oblique one, made the different sections by the different inclination of the cutting plane.

(174.18) Let there again be, as in the same diagrams, the cutting plane through KEΛ, and let the section common to it and the base of the cone be KZΛ; and again, let the common section of the cutting plane KEΛ itself and the axial triangle ABΓ be EZ, which is called the *diameter* of the section. So of all the sections, he lays down KΛ orthogonal to the base BΓ of the triangle ABΓ; but it remains to produce, if EZ is parallel to AΓ, the parabolic section KEΛ on the surface of the cone; but if EZ meets AΓ beyond the vertex of the cone (as at Δ), to produce the hyperbolic section KEΛ; and if EZ meets AΓ inside, to produce the elliptical section, which they also call a shield. So in general, the diameter of a parabola is parallel to one side of the triangle; but the diameter of the hyperbola intersects the side of the triangle at the parts above the vertex of the cone; and

[4]i.e. the axial triangle.

the diameter of the ellipse meets the side of the triangle on the side of the base[5]. But it is necessary to see also, that the parabola and the hyperbola are things which increase towards infinity, but the ellipse not at all; for it converges back to itself completely, like a circle.

(176.16) But since there are so many editions, as he himself says in his letter, I considered it better to bring them together, from those available, placing the more manifest things[6] in the text side by side for the benefit of beginning students, and to note outside in the adjacent scholia the different courses of the proofs, as seemed reasonable.

(176.23) So then he says in his letter that the first four books encompass a course in the elements[7], of which the first encompasses the origins of the three sections of the cone, and of the sections called *opposite*, and the principle properties in them. And these are all the ones that follow from their first origin[8]. But the second book encompasses the properties deduced from the diameters and the axes of the sections, and also the *asymptotes*, and other things providing a fundamental and necessary function to their *diorismoi*.

But that the *diorismos* is twofold is clear to everyone; the one setting up after the *ekthesis* what is the thing being sought; the other, not agreeing that the *protasis* is general, but saying when and how and in how many ways it is possible to set up the proposition, such as the one in the twenty-second proposition of the first book of Euclid's *Elements* does: from three straight lines, which are equal to three given straight lines, to construct a triangle: indeed it is necessary that two sides chosen in any manner be greater than the remaining side; since it has been shown that in every triangle, the two sides are greater than the remaining side, being taken in any manner.

But the third book of the *Conics*, he says, encompasses many useful and amazing

[5]i.e. below the vertex of the cone.

[6]τὰ σαφέστερα; I take this to mean the clearer versions of proofs, nicer definitions, etc.

[7]i.e. the basics, which would have been in Euclid's *Elements of Conics*.

[8]i.e. the ἀρχικὰ συμτώματα follow directly from the origins of the sections.

theorems to the constructions of solid loci. It was customary for the ancient geometers to speak of planar loci whenever, in the problems, the problem arises not from one point alone, but rather from more; for example, when having been given some straight line, if someone enjoins you to find some point, from which the perpendicular being drawn to the given line makes a mean proportional of the pieces; they call things such as this a locus: for the thing making the problem is not only the one point, but rather a whole locus, which the perimeter of the circle, having the given straight line as its diameter, has. For if a semicircle is drawn around the given straight line, whatever point you take on the perimeter, and from it draw a perpendicular to the diameter, will solve the problem.

And likewise, given some straight line, if someone enjoins you to find a point outside it, from which the lines joining the endpoints of the [given] straight line will be equal to one another, in this case too, the thing making the problem is not only one point, but a locus, which the straight line being drawn at right angles at the midpoint of the given line provides. For if, having cut the given straight line in two, you also draw a straight line at right angles on the midpoint, and whatever point you take on it, will solve the assigned problem.

(180.11) And Apollonius himself writes similarly in the *Treasury of Analysis*[9] in the case of the following.

(180.13) When there are two given points[10] in a plane and an established ratio of unequal straight lines, it is possible to draw in the plane a circle so that the straight lines inflected from the given points to the perimeter of the circle have the same ratio as the given ratio.

(180.18) Let the given points be A and B, and the given ratio that of Γ to Δ with Γ being bigger: indeed it is necessary to solve the assigned problem. For let AB be joined, and let it be projected to the parts near B, and let it be contrived that, as Δ is to Γ, so too is

[9] See Heath's notes on Pappus.

[10] Heiberg has δύο δοθέντων [εὐθειῶν] ἐν ἐπιπέδῳ [καὶ] σημείων... There are not two given straight lines, just two given points. Note how Eutocius starts the construction in the next paragraph.

Γ to some other magnitude clearly being greater than Δ, and let, for example, it be $E\Delta$[11]:

$$\Delta : \Gamma = \Gamma : E\Delta;$$

and let again it be contrived, that

$$E : AB = \Delta : BZ = \Gamma : H.$$

It is manifest that both Γ is proportionally between $E\Delta$ and Δ[12] and H is proportionally between AZ, ZB[13]:

$$AZ : H = H : ZB;$$

With center Z and radius H, let the circle $K\Theta$ be drawn. It is manifest indeed, that the circumference $K\Theta$ cuts orthogonal to AB: for the straight line H is proportionally between AZ, ZB. Indeed, let a random point Θ be taken on the circumference, and let ΘA, ΘB, and ΘZ be joined. So then ΘZ is equal to H, and on account of this,

$$AZ : Z\Theta = Z\Theta : ZB.$$

[11]That is, E "+" Δ.

[12]This is true by hypothesis.

[13]I hardly found this point clear. As hypotheses, we have $\Delta : \Gamma = \Gamma : E\Delta$, $E : AB = \Delta : BZ$, and $E : AB = \Gamma : H$. Eutocius claims that H is a mean proportional between AZ and BZ. From the hypothesis that $E : AB = \Delta : BZ$, I say that $AZ : E\Delta = BZ : \Delta$.

$$E : \Delta = AB : BZ \text{ (\textit{Alternando})}$$

$$E\Delta : \Delta = AZ : BZ \text{ (\textit{Componendo})}$$

$$E\Delta : AZ = \Delta : BZ \text{ (\textit{Alternando})}$$

$$AZ : E\Delta = BZ : \Delta \text{ (\textit{Invertendo})}$$

as desired. But since it was hypothesized that $H : \Gamma = BZ : \Delta$, and the result just shown is that $AZ : E\Delta = BZ : \Delta$, we have that

$$H : \Gamma = AZ : E\Delta. \text{ Therefore}$$

$$AZ : H = E\Delta : \Gamma \text{ (\textit{Alternando}).}$$

But it was hypothesis that

$$E\Delta : \Gamma = \Gamma : \Delta, \text{ so,}$$

$$AZ : H = \Gamma : \Delta, \text{ and therefore}$$

$$AZ : \Gamma = H : \Delta \text{ (\textit{Alternando})}$$

We have as hypothesis that

$$\Gamma : H = \Delta : BZ, \text{ so \textit{ex aequali},}$$

$$AZ : H = H : BZ.$$

Therefore H is the mean proportional between AZ and BZ.

Also, [the sides] about the same angle ΘZB are proportional: then the triangle AZΘ is similar to the triangle ΘBZ, and the angle under ZΘB is equal to the one under ΘAB. Let BΛ, parallel to AΘ, be drawn through B. So since

$$\mathrm{AZ} : \mathrm{Z\Theta} = \mathrm{Z\Theta} : \mathrm{ZB},$$

as the first AZ is to the third ZB, so too is the square on AZ to the square on ZΘ:

$$\mathrm{AZ} : \mathrm{ZB} = \mathrm{sq.(AZ)} : \mathrm{sq.(Z\Theta)}.$$

But

$$\mathrm{AZ} : \mathrm{ZB} = \mathrm{A\Theta} : \mathrm{B\Lambda},$$

and so

$$\mathrm{sq.(AZ)} : \mathrm{sq.(Z\Theta)} = \mathrm{A\Theta} : \mathrm{B\Lambda}.$$

Again, when the the angle BΘZ is equal to the angle ΘAB, the angle AΘB is also equal to the angle ΘBΛ: for they are situated alternately[14]: and so the remaining angle is equal to the remaining angle, and the triangle AΘB is similar to the triangle B$\Theta\Lambda$, and the sides about equal angles are proportional:

$$\mathrm{A\Theta} : \mathrm{\Theta B} = \mathrm{\Theta B} : \mathrm{B\Lambda};$$

and

$$\mathrm{sq.(A\Theta)} : \mathrm{sq.(\Theta B)} = \mathrm{A\Theta} : \mathrm{B\Lambda}.$$

But it was previously shown also, that

$$\mathrm{A\Theta} : \mathrm{B\Lambda} = \mathrm{sq.(AZ)} : \mathrm{sq.(Z\Theta)} :$$

so

$$\mathrm{sq.(AZ)} : \mathrm{sq.(Z\Theta)} = \mathrm{sq.(A\Theta)} : \mathrm{sq.(\Theta B)};$$

and on account of this,

$$\mathrm{AZ} : \mathrm{Z\Theta} = \mathrm{A\Theta} : \mathrm{\Theta B}.$$

But

$$\mathrm{AZ} : \mathrm{Z\Theta} = \mathrm{E\Delta} : \Gamma = \Gamma : \Delta :$$

[14]That is, they are alternate interior angles in a transversal formed by ΘB cutting the parallels AΘ, BΛ.

and so

$$\Gamma : \Delta = \mathrm{A\Theta} : \mathrm{\Theta B}.$$

Similarly, all the straight lines from the points A, B, terminating at the perimeter of the circle, will be shown as having to one another the same ratio as Γ to Δ.

(184.3) Indeed, I say; that at another point not being on the perimeter, the ratio of the straight lines from the points A and B to it, does not become the same as that of Γ to Δ.

(184.7) For if possible, let it happen at the point M outside the perimeter; for indeed, if it be taken inside, the same contradiction will happen on account of the other hypotheses, and let MA, MB, MZ be joined; and let it be supposed that

$$\Gamma : \Delta = \mathrm{AM} : \mathrm{MB}.$$

So

$$\mathrm{E}\Delta : \Delta = \mathrm{sq.}(\mathrm{E}\Delta) : \mathrm{sq.}(\Gamma) = \mathrm{sq.}(\mathrm{AM}) : \mathrm{sq.}(\mathrm{MB}).$$

But

$$\mathrm{E}\Delta : \Delta = \mathrm{AZ} : \mathrm{ZB}.$$

and so

$$\mathrm{AZ} : \mathrm{ZB} = \mathrm{sq.}(\mathrm{AM}) : \mathrm{sq.}(\mathrm{MB}).$$

And because of the things having been shown before, if we draw from B a parallel to AM, it will be shown that

$$\mathrm{AZ} : \mathrm{ZB} = \mathrm{sq.}(\mathrm{AZ}) : \mathrm{sq.}(\mathrm{ZM}).$$

But it was also shown, that

$$\mathrm{AZ} : \mathrm{ZB} = \mathrm{sq.}(\mathrm{AZ}) : \mathrm{sq.}(\mathrm{Z\Theta}).$$

Therefore

$$\mathrm{Z\Theta} = \mathrm{ZM}:$$

the very thing which is impossible.

(184.21) So much, then, for the planar loci: but the so-called solid loci received that designation from the fact that the figures (through which the problems relating to them

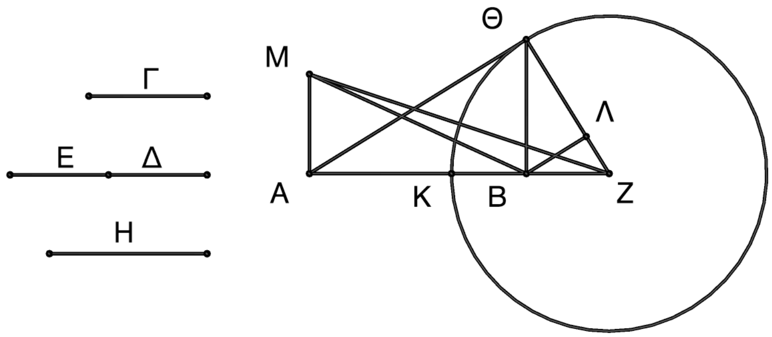

Figure 3.4: Eutocius' Locus Diagram

are drawn) have their origin from the section of solids, such as the sections of the cone and many others. But there are also other loci, being so-named according to a surface, which have their designation on account of the property concerning them.

(186.1) But he [Geminus] does not go on to blame Euclid, as Pappus and certain others believe, on account of his [Euclid] not having found two means proportional: for Euclid easily found one mean proportional, but not, as he himself says, by chance; and he absolutely did not try to investigate concerning the two means proportional in the *Elements*; and Apollonius himself seems to investigate nothing concerning two means proportional in his third book. But rather, as seems likely, he [Geminus] finds fault with another book written by Euclid concerning loci, which has not been carried down to us.

(186.11) But the subsequent things said about the fourth book are manifest. He says that the fifth book comprises things concerning the minima and maxima. For just as in the *Elements* we learned that in the case of the circle, there is a point outside, from which, when lines fall on the interior[15], the one through the center is largest, and when they fall to the perimeter[16], the one between the point and the diameter is least; so also he investigates the case of the sections of the cone in his fifth book. The theme of the sixth, seventh, and

[15]τὴν κοίλην περιφέρειαν, lit. the "concave perimeter".

[16]lit. the "convex perimeter".

eighth books has been stated clearly by him [Apollonius]. So much for his letter.

(186.22) Beginning with definitions, he sketches out the origin of the conic surface, but he did not give the definition of what it is: but it is permissible to take the definition from the origin itself for those wishing to do so. But what is being said by him, we will make manifest through a diagram.

"If, from some point to the circumference of a circle," etc.[17] For let the circle be AB, the center of which be Γ, and let there be some point Δ not in the plane of the circle, and let ΔB, having been joined, be projected to infinity in both directions as towards E, Z. Indeed, if, with Δ being fixed, ΔB be carried around, until B, having been carried around upon the perimeter of the circle AB, is restored to its original position from whence it began to be revolved, it will generate a certain surface, which is comprised of two surfaces being joined to one another at Δ, which he calls a *conic surface*. But he says, that also it extends to infinity, on account of the fact that the straight line drawing it--for instance, ΔB--is projected infinitely. And he calls Δ the *vertex* of the surface, and ΔΓ the *axis*.

(188.13) And he calls a *cone* the shape which is bounded by both the circle AB and the surface, which the straight line ΔB alone draws, and calls Δ the *vertex* of the cone, and ΔΓ the *axis*, and the circle AB the *base*.

(188.17) And if, on the one hand, ΔΓ be orthogonal to the circle AB[18], he calls the cone right; but if, on the other hand, it is not orthogonal, he calls the cone oblique: but an oblique cone will be generated, whenever upon taking the circle, we set up a straight line from the center itself not orthogonal to the plane of the circle, but from the raised point of the set-up line we join a straight line to the circle, and rotate the joined straight line around the circle, with the point at the point[19] on the set-up straight line remaining fixed:

[17]Eutocius is quoting Apollonius' definition here.

[18]i.e. the plane of the circle. The circle includes its interior.

[19]The Greek isn't terribly clear: here we have ἀπὸ δὲ τοῦ μετεώου σημείου τῆς ἀναταθείσης εὐθείας ἐπὶ τὸν κύκλον ἐπιζεύξωμεν εὐθεῖαν περὶ τὸν κύκλον τοῦ πρὸς τῷ μετεώρῳ σημείῳ τῆς ἀναταθείσης μένοντος. In the preceding materials, Eutocius specifically names the τοῦ μένοντος as the point Δ, even though here we have no specific names for any of the points or lines. For the genitive absolute

for the produced shape will be an oblique cone.

(190.1) But it is manifest that the rotating straight line becomes greater and smaller during this rotation, and in certain locations also will be equal at one point and another point of the circle. He proves this thus: if, straight lines be drawn from the vertex to the base of an oblique cone, of all the straight lines being drawn from the vertex to the base, one is least, and one is greatest; but two alone are equal, one on each side of the least and the greatest; but always the one nearer to the least is lesser than the one farther. Let there be an oblique cone, whose base is the circle ΑΒΓ, and whose vertex is the point Δ. And since the line drawn perpendicular from the vertex of the oblique cone to the underlying plane[20] will fall either at the perimeter of the circle ΑΒΓΖΗ, or outside it, or inside it. Let ΔΕ first fall at the perimeter, as in the first diagram, and let the center of the circle be taken, and let it be Κ, and let ΕΚ from Ε to Κ be joined, and projected to Β, and let ΒΔ be joined, and let there be taken two equal arcs ΕΖ, ΕΗ, one on each side of Ε, and on each side of Β as ΑΒ, ΒΓ, and let ΕΖ, ΕΗ, ΔΖ, ΔΗ, ΕΑ, ΕΓ, ΑΒ, ΒΓ, ΔΑ, and ΔΓ be joined. So since

$$\mathrm{EZ} = \mathrm{EH}$$

(for they are subtended by equal arcs), but also ΔΕ is common and perpendicular, therefore

$$\text{base } \Delta\mathrm{Z} = \text{base } \Delta\mathrm{H}.$$

Again, since

$$\mathrm{arc}(\mathrm{AB}) = \mathrm{arc}(\mathrm{B}\Gamma),$$

and the diameter of the circle is ΒΕ,

$$\text{remainder arc}(\mathrm{EZ}\Gamma) = \text{remainder arc}(\mathrm{EHA}):$$

τοῦ μένοντος, I read (a) that it refers to the same point which we selected not on the plane of the circle, and (b) it refers to what in the previous diagram is the point Δ. It seems reasonable to me that Eutocius is doubly-referring to this point (the point at the point) in order to remind us of the previously addressed case of the right cone.

[20]Again, the plane of the base circle.

so that also

$$AE = E\Gamma.$$

But also EΔ is common and orthogonal: therefore the base ΔA is equal to the base $\Delta\Gamma$. And likewise, all those equally distant from either ΔE or ΔB will be shown to be equal. Again, since of the triangle ΔEZ , the angle under ΔEZ is a right angle, ΔZ is greater than ΔE. And again, since the straight line EA is greater than the straight line EZ, and since the arc EZA is greater than the arc EZ, but also ΔE is common and orthogonal, ΔZ is therefore less than ΔA. Also on account of the same things, ΔA is lesser than ΔB. So since ΔE was shown to be less than ΔZ, ΔZ less than ΔA, and ΔA less than ΔB; ΔE is the smallest, ΔB is largest largest, but always the one nearer to ΔE is less than the one further.

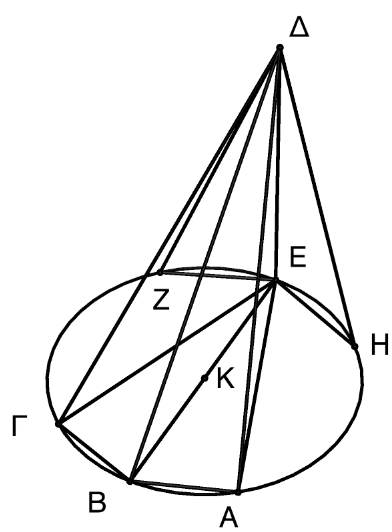

Figure 3.5: Maximum and Minimum Lines on a Cone with E on the Base Circle

(192.15) But indeed let the perpendicular ΔE fall outside of the circle ABΓHZ, as in the second diagram, and let again the center of the circle K be taken, and let EK be joined and projected to B, and let ΔB, $\Delta\Theta$ be joined, and let two equal arcs ΘZ, ΘH be taken, one on each side of Θ, and let two equal arcs AB, BΓ be taken, one on each side of B, and let EZ, EH, ZK, HK, ΔZ, ΔH, AB, BΓ, KA, KΓ, ΔK, ΔA, $\Delta\Gamma$ be joined.

So since the arc ΘZ is equal to the arc ΘH, the angle under ΘKZ is therefore also equal to the angle under ΘKH. So since the straight line ZK is equal to the straight line KH--for they are both radii--but KE is common, and the angle under ZKE is common to the angle under HKE, and the base ZE is equal to the base HE. So since the straight line ZE is equal to the straight line HE, and EΔ is common and orthogonal, the base ΔZ is therefore equal to the base ΔH. Again, since the arc BA is equal to the arc BΓ, the angle under AKB is also therefore equal to the angle under ΓKB, so that also the remaining angle under AKE is equal to the remaining angle under ΓKE. So since the straight line AK is equal to the straight like ΓK--for they are both radii--but KE is common, the two are equal to the two, and the angle under AKE is equal to the angle under ΓKE: and the base AE is therefore equal to the base ΓE. So since the straight line AE is equal to the straight line ΓE, and the straight line EΔ is both common and orthogonal, the base ΔA is therefore equal to the base ΔΓ. Likewise also all the straight lines that are equidistant from ΔB or ΔΘ will be shown to be equal. Also, since EΘ is less than EZ, but also EΔ is common and orthogonal, the base ΔΘ is therefore less than the base ΔZ. Again, since the segment from E touching the circle is bigger than all those falling towards the arc, and it was shown in the third book of the *Elements*[21], that the rectangle contained by AE, EΛ is equal to the square on EZ, whenever EZ is tangent, it is manifest that, as AE is to EZ, so too is EZ to EΛ. But EZ is greater than EΛ: for always the one nearer to the least is less than the one further: also AE is therefore greater than EZ. So since EZ is smaller than EA, and EΔ common and orthogonal, the base ΔZ is therefore smaller than the base ΔA. Again, since AK is equal to KB, but KE is common, the two straight lines AK, KE are therefore equal to EK, KB, i.e. to the whole EKB. But AK, KE are bigger than AE: and BE is therefore greater than AE. Again, since AE is smaller than EB, but EΔ is common and orthogonal, the base ΔA is therefore smaller than the base BΔ. So since ΔΘ is smaller than ΔZ, and ΔZ smaller than ΔA, and ΔA smaller than ΔB, ΔΘ is

[21] *Elements* III.36.

smallest and ΔB the biggest, and the nearer is always smaller than the further.

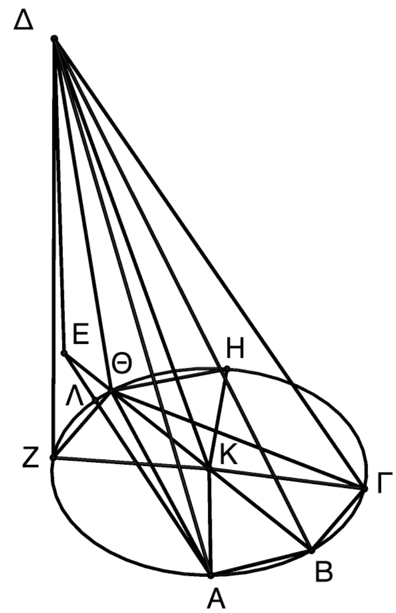

Figure 3.6: Maximum and Minimum Lines on a Cone with E outside the Base Circle

(196.7) But indeed let the perpendicular ΔE fall inside the circle ΑΒΓΗΖ as in the third diagram, and K be taken as the center of the circle, and let EK be joined and produced in each direction to B, Θ, and let ΔΘ, ΔB be joined, and let two equal arcs ΘZ, ΘH be taken on each side of Θ, and let EZ, EH, ZK, HK, ΔZ, ΔH, KA, KΓ, EA, EΓ, ΔA, ΔΓ, AB, BΓ be joined. So since the arc ΘZ is equal to the arc ΘH, the angle under ΘKZ is therefore also equal to the angle under ΘKH. And since KZ is equal to HK, and KE is common, also the angle under ZKE is equal to the angle under HKE, the base ZE is therefore equal to the base HE. So since ZE is equal to HE, and ΔE is common, and the angle under ZEΔ is equal to the angle under HEΔ, the base ΔZ is therefore equal to the base ΔH. Again, since the arc AB is equal to the arc BΓ, the angle under AKB is also therefore equal to the angle under ΓKB: so that also the remainder to two right angles AKE is equal to the remainder to two right angles ΓKE. So since AK is equal to KΓ, and EK is common, and the angle under AKE is equal to the angle under

ΓKE, the base AE is therefore equal to the base ΓE. So since AE is equal to ΓE, and EΔ is common, and the angle under AEΔ is therefore equal to the angle under ΓEΔ, the base ΔA is therefore equal to the base ΔΓ. Likewise also all the straight lines that are equidistant from either ΔB or ΔΘ will be shown to be equal. And since, in the circle ABΓ, the point E which is not the center of the circle has been taken on the diameter, EB is greatest, EΘ is least; but always the one nearer to EΘ is less than the one farther: so that EΘ is less than EZ. And since ΘE is less than ZE, but also EΔ is common and orthogonal, the base ΔΘ is therefore smaller than the base ΔZ. Again, since EZ is nearer than EΘ and AE is even further, EZ is less than AE. So since EZ is lesser than EA, and EΔ is common and orthogonal, the base ΔZ is therefore smaller than the base ΔA. Again, since AK is equal to KB, and KE is common, the two straight lines AK, KE are equal to the two straight lines BK, KE, i.e. to the whole BKE. But AK, KE are greater than AE: and EB is therefore bigger than EA. Again, since EA is less than EB, and EΔ is common and orthogonal to them, the base ΔA is therefore smaller than the base ΔB. So since ΔΘ is smaller than ΔZ, and ΔZ smaller than ΔA, and ΔA smaller than ΔB, ΔΘ is smallest, etc.

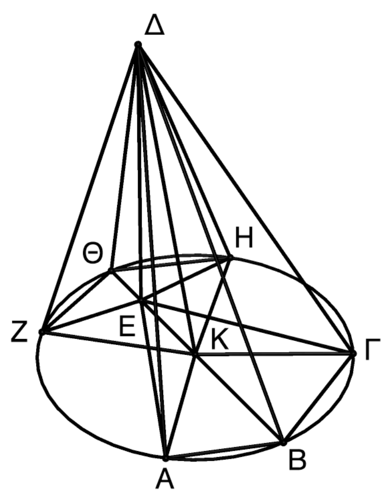

Figure 3.7: Maximum and Minimum Lines on a Cone with E inside the Base Circle

(198.26) "Of each curved figure, which is in one plane, I call the diameter," etc.[22] He said "in one plane" because of the helix of the cylinder and the sphere: for they are not in one plane. But what he means is as follows: let there be a curved figure ΑΒΓ and in it some parallel straight lines ΑΓ, ΔΕ, ΖΗ, ΘΚ, and let from Β the straight line ΒΛ be carried cutting them in two. So he says, that the *diameter* of the curve ΑΒΓ I call ΒΛ, the *vertex* Β, and each of the lines ΑΓ, ΔΕ, ΖΗ, ΘΚ have been drawn ordinatewise to ΒΛ[23]. But if ΒΛ bisects and cuts at right angles the parallels[24], it is called the *axis*.

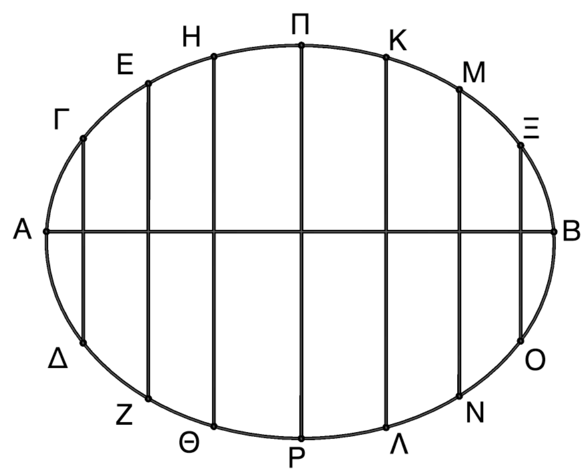

Figure 3.8: The Axis and Ordinates of an Ellipse

(200.10) "But similarly of two curved lines," etc. For if we consider the curves Α, Β and in them the parallels ΓΔ, ΕΖ, ΗΘ, ΚΛ, ΜΝ, ΞΟ and the straight line ΑΒ, having been produced in each direction and cutting the parallels in two, I call ΑΒ, he says, the *transverse diameter*; the points Α and Β the *vertices* of the curves; and the curves having been drawn ordinatewise to ΑΒ the lines ΓΔ, ΕΖ, ΗΘ, ΚΛ, ΜΝ, ΞΟ[25]. And if it (ΑΒ)

[22]Eutocius is again quoting Apollonius.

[23]τεταγμένως δὲ ἐπὶ τὴν ΒΛ κατῆχθαι. Taliaferro renders this as *having been drawn ordinatewise to the [diameter] ΒΛ.*

[24]i.e. these ordinates.

[25]In short, just *the ordinates*.

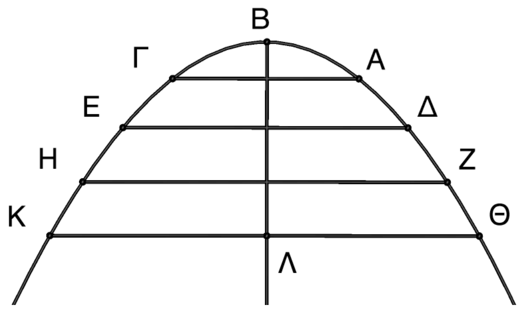

Figure 3.9: The Axis and Ordinates of a Parabola

also bisects them at right angles, it is called the *axis*. But if a certain straight line, such as ΠΡ, having been carried through ΓΞ, ΕΜ, ΗΚ bisects the parallels to ΑΒ, then ΠΡ is called the *upright diameter*, and each of the lines ΓΞ, ΕΜ, ΗΚ, [he says] have been drawn ordinatewise to the upright diameter ΠΡ. But if it bisects it at right angles, it is called *the upright axis*; and if ΑΒ, ΠΡ bisect the parallels of each other, they are called the *conjugate diameters*, but if they bisect at right angles, they are called the *conjugate axes*.

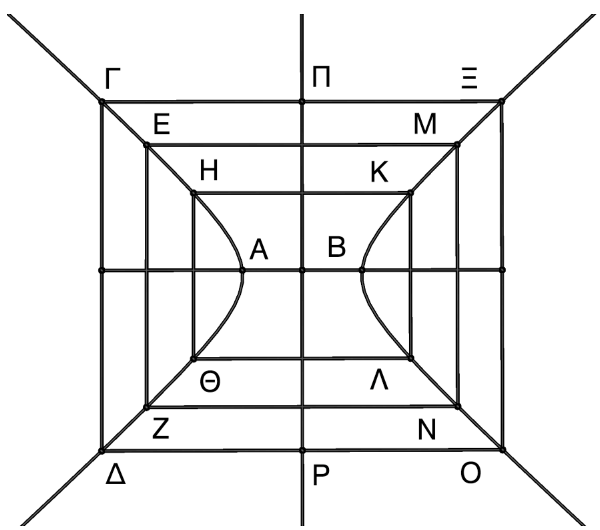

Figure 3.10: The Conjugate Axes of Two Opposite Sections

3.2 Theorem 1

(202.6) Concerning the different diagrams, one must see the many cases of the theorems, that a case is, whenever the things having been specified in the *protasis*, are given for a particular arrangement: for the taking of a different one of them, with the same *sumperasma*, makes the case. But similarly also a case occurs from the changing of the *kataskeue*. But when the theorem has many cases, the same proof fits all of them, and in the same elements save small differences; we will see this successively. For directly, the first theorem has three cases, on account of the fact that the point having been taken on the surface, i.e. B, is sometimes on the lower surface and this in two ways, either higher than the circle or lower; and sometimes on the part which, according to the vertex, lies opposite it.

He proposed this theorem to investigate that it is not true that any two points being taken on the surface, a straight line being joined between them is on the surface; rather only the inclination beyond the vertex, because of the fact that the conic surface is generated by a straight line having its boundary remaining fixed. But the second theorem shows that this is true.

3.3 Theorem 2

(202.26) The second theorem has three cases, because the chosen points Δ, E are either on the surface above the vertex, or below it in two ways: either inside or outside the circle. But it is necessary to understand that this theorem in some copies is found proved in full through the *reductio ad absurum*.

3.4 Theorem 3

(204.7) The third theorem does not have cases. But it is necessary to understand in it that the line AB is straight on account of being the common section of the cutting plane and the surface of the cone, that which was drawn by a straight line having its boundary remaining fixed at the vertex of the surface. For the whole surface, cutting the section by a means of a plane, does not make a straight line; nor does the cone itself, unless the cutting plane passes through the vertex.

3.5 Theorem 4

(204.16) There are three cases of this theorem, just as of both the first and the second.

3.6 Theorem 5

(204.19) The fifth theorem does not have cases. But beginning the *ekthesis* he says: "let the cone be cut by a plane through the axis at right angles to the base."[26]

But since in an oblique cone, the triangle through the axis is perpendicular to the base according to one arrangement only, we will show this thus: taking the center of the base, we will construct from it, at right angles to the plane of the base, and projecting a plane through it and the axis, we will have the thing being sought: for it is shown in the eleventh book of Euclid's *Elements*, that, if a straight line is orthogonal to some plane, then also all the planes through it [the line] will be orthogonal to the same plane. But he set forth the oblique cone, since in the isosceles cone[27], the plane parallel to the base is the same as the one drawn subcontrariwise.

(206.7) Still, he says: "But let it also be cut by another plane perpendicular to the triangle through the axis, and which cuts off a triangle near the vertex which is similar to the triangle ΑΒΓ, but which is lying subcontrariwise." But this occurs thus: for let there be the triangle through the axis ΑΒΓ, and let a random point Η be taken on ΑΒ, and let it the angle ΑΗΚ be supposed equal to the angle ΑΓΒ, i.e. the one near the straight line ΑΗ and the point Η on it; therefore the triangle ΑΗΚ is similar to ΑΒΓ, but lying subcontrariwise. Indeed, on ΗΚ, let a point Ζ be taken at random, and from Ζ let ΖΘ be set up at right angles to the plane of the triangle ΑΒΓ, and let the plane through ΗΚ, ΘΖ be projected. Indeed, this is orthogonal to the triangle ΑΒΓ on account of ΖΘ and

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[26] This is not an exact quote of the Greek as it appears in Heiberg, but more of a synopsis. The important parts match.

[27] It is interesting that he calls the non-oblique cone *isosceles* here, as opposed to *right*.

making the hypothesis.

(206.22) He says in the *sumperasma*, that on account of the similarity of the triangles ΔZH, EZK, the angle ΔZE is equal to the angle HKZ. But it is possible to show this without reference to the similarity of the triangles, saying that since each of the angles AKH, AΔE is equal to that at B, the points Δ, H, E, K are in the same section of the circumscribed circle. And since in the circle, the two straight lines ΔE, HK cut one another at Z, the rectangle ΔZE is equal to the rectangle HZK[28].

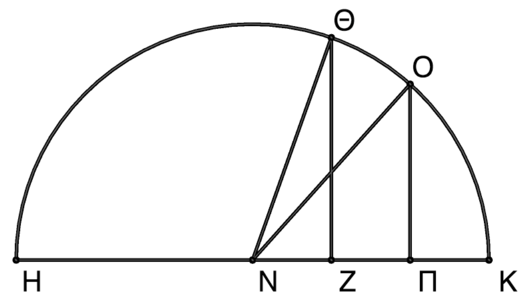

Figure 3.11: Eutocius' First Diagram for Theorem 5

(208.7) Similarly it will be shown, that also all the lines being drawn perpendicularly from the curve HΘ to HK are equal in square to the rectangle contained by the sections[29]. The section is therefore a circle, and the diameter of it is HK. And it is possible to conclude this through *reductio ad absurdum*. For if the circle, being drawn around KH, does not pass through the point Θ, the rectangle KZ, ZH will be equal either to the square on a segment bigger than ZΘ or one smaller: the very thing which is not supposed. But we will show this also in the case of a straight line.

(208.17) Let there be some curve HΘ, and let HK subtend it, and let the points Θ, O be taken at random on the curve, and from them [Θ, O] let ΘZ, OΠ be drawn to HK at

[28] *Elements* III.36.

[29] That is, the perpendicular meets the diameter at a point, and the rectangle is formed by the two segments of the diameter, each of which has one endpoint as an endpoint of the diameter, and the other the point where the perpendicular meets the diameter.

right angles, and let the square on ZΘ be equal to the rectangle HZK, and let the square on ΟΠ be equal to the rectangle ΗΠΚ. I say that the curve ΗΘΟΚ is a circle. For let HK be cut in two at N, and let NΘ, NO be joined. So since the straight line HK is cut into equal segments at N, and into unequal segments at Z, the rectangle HZK together with the square on NZ is equal to the square on NK[30]. But the rectangle HZK was supposed equal to the square on ΘZ; therefore the square on ΘZ together with the square on NZ is equal to the square on NK. But the squares on ΘZ, ZN are [together] equal to the square on NΘ: for the segment to Z is orthogonal: therefore the square on NΘ is equal to the square on NK. Similarly we will also show that the square on NO is equal to the square on NK . Therefore the curve HΘK is a circle, and HK is its diameter.

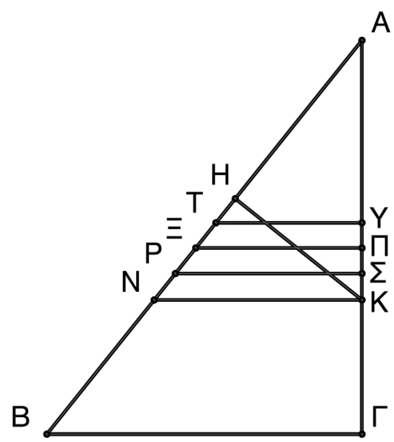

Figure 3.12: Eutocius' Second Diagram for Theorem 5

(210.8) But it is possible for the diameters ΔE, HK sometimes to be equal, other times unequal, but it is never possible for them to bisect one another. For let HK be drawn through K parallel to BΓ. So since BA is larger than AΓ, also NA is bigger than AK. Similarly also, KA [is bigger than] AH on account of the subcontrary section, so that the [line] being taken equal from AN to AK falls between the points H, N. Let it fall as AΞ: therefore the line being drawn parallel to BΓ through Ξ cuts HK. Let it cut as ΞΟΠ. And

[30]*Elements* II.5

since ΞA is equal to AK, as ΞA is to AΠ, so too is KA to AH on account of the similarity of the triangles HKA, ΞAΠ, and AH is equal to AΠ, also the remainder HΞ is equal to the remainder ΠK. And since the angles at Ξ, K are equal: for each of them is equal to that at B: but also the [angles] at O are equal: for they are vertical angles: therefore the triangle ΞHO is similar to the triangle ΠOK. And HΞ is equal to ΠK: so that also ΞO is equal to OK and HO to OΠ and the whole HK to ΞΠ. And it is manifest, that, if a point be taken between N, Ξ , such as P, and through P, the line PΣ be drawn parallel to NK, it will be larger than ΞΠ; and on account of this, [it will also be bigger than] HK; but if a point be taken between H, Ξ, such as T , and through it a parallel TΠ be drawn, it will be smaller than ΞΠ and KH. And since the angle ΞΠK is bigger than the angle AΞΠ, but the angle OΠK is equal to the angle OHΞ, therefore the angle OHΞ is bigger than the angle HΞO. Therefore ΞO is bigger than OH, and on account of this, KO is bigger than OΠ. But if one of them should ever be cut in two, the other will be cut in unequal [parts]

3.7 Theorem 6

(212.14) It is necessary to pay heed, that he set forth in the *protasis* not without purpose the necessity that the straight line drawn from the point on the surface be drawn parallel to some particular one of the straight lines in the base, being without a doubt orthogonal to the base of the triangle through the axis: for when this is not the case, it is not possible for it [the drawn straight line] to be bisected by the triangle through the axis: the very thing which is manifest from the diagram in the text. For if MN, regardless to whatever line ΔZH is parallel, is not orthogonal to BΓ, it is manifest that neither it [MN] nor KΛ is bisected. And this follows from the same ratios, that is,

$$\mathrm{K\Theta} : \Theta\Lambda = \Delta\mathrm{Z} : \mathrm{ZH},$$

and ΔH will therefore be been cut into unequal sections at Z.

(212.27) But it is possible to show the same things of the lower circle and in the case of the surface opposite the vertex[31].

[31] That is, the other surface of the conic surface. These two additional cases match Apollonius' diagrams very well.

3.8 Theorem 7

(214.2) The seventh theorem has four cases: for either ZH does not intersect AΓ, or it intersects in three ways: either outside the circle, or inside, or at the point Γ.

3.9 After Theorem 10

(214.6) It is necessary to realize, that these ten theorems depend on one another. But the first holds that the straight lines in the [conic] surface inclining towards the vertex stay in it. The second holds the converse. The third holds the section through the vertex of the cone; the fourth the [section] parallel to the base; the fifth, the subcontrary; the sixth, as if it were anticipating the seventh, shows that the common section of the circle and the cutting plane is in all cases necessarily at right angles to the diameter [of that circle], and that this being so, the parallels to it [the common section] are bisected by the triangle. But the seventh shows the other three sections, both that the diameter and the ordinates to it are parallel to the straight line in the base. In the eighth he shows, just as we said in the preceding materials, that the parabola and the hyperbola are of those things projecting to infinity; but in the ninth, that the ellipse, returning to itself like a circle does (on account of the cutting plane intersecting both sides of the triangle), is not a circle: for both the subcontrary section and the parallel [section] made circles. But it is necessary to understand, that the diameter of the section, in the case of the parabola, cuts one side of the [axial] triangle and the base [of it]; in the case of the hyperbola, both the side and the remaining side, being projected in a straight line from the vertex. But in the case of the ellipse, it cuts both the sides and the base. And the tenth, someone giving attention to it in a rather simple way might think that it is the same as the second, but this is not the case: for there, he says to take the two points on the whole surface, but here on the generated figure. In the following three theorems, he distinguishes more precisely each of the sections, along with specifying the principle peculiarities of them.

3.10 Theorem 11

(216.13) Let it have been made, that

$$\text{sq.}(\mathrm{B\Gamma}) : \text{rect.}(\mathrm{BA\Gamma}) = \Theta\mathrm{Z} : \mathrm{ZA}:$$

the thing being said is manifest, unless someone wants to comment on it. For let the rectangle ΟΠΡ be equal to the rectangle ΒΑΓ, and let it, being projected along ΠΡ, make ΠΣ as width and be equal[32] to the square on ΒΓ; and let it have happened that

$$\mathrm{O\Pi} : \Pi\Sigma = \mathrm{AZ} : \mathrm{Z}\Theta.$$

The thing being sought has therefore happened. For since it is the case, that

$$\mathrm{O\Pi} : \Pi\Sigma = \mathrm{AZ} : \mathrm{Z}\Theta;$$

invertendo,

$$\Sigma\Pi : \Pi\mathrm{O} = \Theta\mathrm{Z} : \mathrm{ZA}.$$

But

$$\Sigma\Pi : \Pi\mathrm{O} = \Sigma\mathrm{P} : \mathrm{PO} = \text{sq.}(\mathrm{B\Gamma}) : \text{rect.}(\mathrm{BA\Gamma}).$$

This is useful also in the following two theorems.

(218.1) But the square on ΒΓ has to the rectangle ΒΑΓ a ratio compounded from that which ΒΓ has to ΓΑ and ΒΓ to ΒΑ:

$$\text{sq.}(\mathrm{B\Gamma}) : \text{rect.}(\mathrm{BA\Gamma}) = (\mathrm{B\Gamma} : \Gamma\mathrm{A}) \text{ comp. } (\mathrm{B\Gamma} : \mathrm{BA}) :$$

it has been shown in the twenty-third theorem of the sixth book of the *Elements*, that equiangular parallelograms have to one another a ratio compounded out of that of the sides: but since it is discussed too inductively and not in the necessary manner by the commentators, we researched it; and it is written in our published work on the fourth theorem of the second book of Archimedes' *On the Sphere and the Cylinder*, and also in the scholia of the first book of Ptolemy's *Syntaxis*: but it is a good idea that this be written down here also, because readers do not always read it even in those works, and also because nearly the entire treatise of the *Conics* makes use of it.

(218.16) A ratio is said to be compounded from ratios, whenever the sizes of the

[32] i.e. the applied area is equal

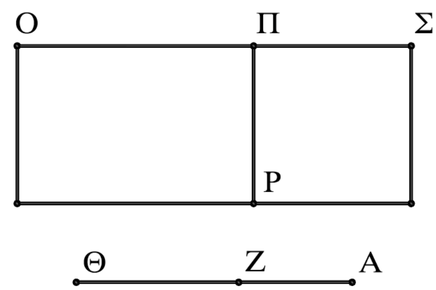

Figure 3.13: Eutocius' Diagram for Theorem 11

ratios, being multiplied into themselves, make something, with "size" of course meaning the number after which the ratio is named. So it is possible in the case of multiples that the size be a whole number, but in the case of the remaining relations it is necessary that the size must be a number plus part or parts, unless perhaps one wishes the relation to be irrational, such as are those according to the incommensurable magnitudes. But in the case of all the relations, it is manifest that the product of the size itself with the consequent of the ratio makes the antecedent.

(218.27) Accordingly, let there be a ratio of A to B, and let some mean of them be taken, as it chanced, as Γ, and let

$$\Delta = \text{size}(A : \Gamma),$$

and

$$E = \text{size}(\Gamma : B),$$

and let Δ, multiplying E, make Z. I say, that the size of the ratios A, B is Z,

$$Z = \text{size}(A : B),$$

that is that Z, multiplying B, makes A. Indeed, let Z, multiplying B, make H. So since Δ, multiplying E has made Z, and multiplying Γ has made A, therefore it is, that

$$E : Z = \Gamma : H$$

Alternando,

$$E : \Gamma = Z : H$$

But

$$\mathrm{E} : \Gamma = \mathrm{Z} : \mathrm{A}$$

Therefore

$$\mathrm{H} = \mathrm{A},$$

so that Z, multiplying B, has made A.

(220.17) But do not let this confuse those reading that this has been proved through arithmetic, for both the ancients made use of such proofs, being mathematical rather than arithmetical on account of the proportions, and that the thing being sought is arithmetical. For both ratios and sizes of ratios and multiplications, first begin by numbers, and through them by magnitudes, according to the speaker. As someone once said, "for these mathematical studies appear to be related."[33]

3.11 Theorem 13

(222.2) It is necessary to point out, that this theorem has three diagrams, as has been said often in the case of the ellipse: for $\Delta\mathrm{E}$ either falls on $\mathrm{A}\Gamma$ above Γ, or at Γ itself, or meets $\mathrm{A}\Gamma$, having been projected, outside.

3.12 Theorem 14

222.8) And it was necessary to likewise show, that,

$$\text{sq.}(\mathrm{A}\Sigma) : \text{rect.}(\mathrm{B}\Sigma\Gamma) = \text{sq.}(\mathrm{AT}) : \text{rect.}(\Xi\mathrm{TO}).$$

(222.11) For since $\mathrm{B}\Gamma$ is parallel to $\Xi\mathrm{O}$,

$$\Gamma\Sigma : \Sigma\mathrm{A} = \Xi\mathrm{T} : \mathrm{TA},$$

and on account of this,

$$\mathrm{A}\Sigma : \Sigma\mathrm{B} = \mathrm{AT} : \mathrm{TO}:$$

therefore, through equality,

$$\Gamma\Sigma : \Sigma\mathrm{B} = \Xi\mathrm{T} : \mathrm{TO},$$

[33] Much aught be said of this passage; some will regrettably be left for future work. Here Eutocius retains the Doric dialect of the quotation: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά. The quote is due to Archytas of Tarentum, preserved only in a fragment of his *Harmonics*. The quoted part reads: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι ἦμεν ἀδελφεά. For the full fragment, see fr. B.1 in Diels-Kranz ([6]) or Freeman's English translation ([2]). See also *Republic* VII 530d and Knorr ([7]), pp. 171 n. 22-23.

Therefore also

$$\text{sq.}(\Gamma\Sigma) : \text{rect.}(\Gamma\Sigma B) = \text{sq.}(\Xi T) : \text{rect.}(\Xi T O).$$

But on account of similar triangles,

$$\text{sq.}(A\Sigma) : \text{sq.}(\Sigma\Gamma) = \text{sq.}(AT) : \text{sq.}(\Xi T):$$

therefore, through equality,

$$\text{sq.}(A\Sigma) : \text{rect.}(B\Sigma\Gamma) = \text{sq.}(AT) : \text{rect.}(\Xi T O).$$

(222.20) And so,

$$\text{sq.}(A\Sigma) : \text{rect.}(B\Sigma\Gamma) = \Theta E : E\Pi,$$

but also,

$$\text{sq.}(AT) : \text{rect.}(\Xi T O) = \Theta E : \Theta\Pi:$$

and therefore,

$$\Theta E : E\Pi = E\Theta : \Theta\Pi.$$

Therefore

$$E\Pi = \Theta\Pi.$$

(222.24) But it does not have cases, and the investigation is manifestly continuous with the three before it: for similarly, by means of them, he investigates that the diameter of the opposite sections is the principle one, and also investigates the parameters[34].

3.13 Theorem 16

(224.2) The rectangle BKA is therefore equal to the rectangle AΛB--KA is therefore equal to BΛ--for since the rectangle BKA is equal to the rectangle AΛB, it will be proportionally, that

$$KB : A\Lambda = \Lambda B : AK.$$

Alternando,

$$KB : B\Lambda = \Lambda A : AK.$$

Componendo,

$$K\Lambda : \Lambda B = \Lambda K : KA.$$

[34] τὰς παρ' ἃς δύνανται: I take this as a referring to the upright and transverse sides collectively. See Eutocius' commentary to the next theorem.

Therefore

$$K A = B\Lambda.$$

(224.8) It is necessary to understand, that in the fifteenth and sixteenth theorems he had an investigation to find the so-called second and conjugate diameters of both the ellipse and the hyperbola or the opposite sections: for the parabola does not have this type of diameter. But one must note that the diameters of the ellipse fall inside, but of those of the parabola and the opposite sections outside. But it is necessary, when drawing the figures, to arrange the parameters or the upright sides at right angles, and also manifestly those [lines] parallel to them, but those being drawn ordinatewise and the second diameters not always: for most of all in an acute angle, it is necessary to drop them, so that they be clear to those encountering them as being different from the parallels to the upright side.

3.14 Second Definitions

(224.22) After the sixteenth theorem, he sets out the definitions concerning the so-called second diameter of the hyperbola and the ellipse, which we will make clear by means of a diagram.

(224.26) For let there be a hyperbola AB, and let ΓBA be a diameter of it, and let BE be the parameter[35]. So it is manifest that BΓ increases to infinity on account of the section, as is shown in the eight theorem, but BΔ, that is the one subtending the angle outside of the axial triangle, is finite. Indeed, bisecting it at Z and setting out from A the ordinate AH, but through Z, parallel to AH, setting out ΘZK and having made ΘZ equal to ZK, yet still also, the square on ΘK equal to to the rectangle ΔBE, we will have ΘK as the second diameter. For this is possible on account of the fact that ΘK, being outside of the section, is projected to infinity, and that it is possible to intercept one equal to a straight line extended from infinity. He calls Z the center, and names ZB and the lines similar to it which are drawn from the center Z to the section.

[35]Fully: "that according to which the ordinates being led to BΓ are equal in square." Henceforth I will consider this a standard abbreviation.

(226.16) So much for the cases of the hyperbola and the opposite sections: And it is clear, that each of the diameters is finite, the first obviously from the genesis of the section, the second, for the reason that it is a mean proportional between two finite straight lines, namely, the first diameter and the parameter.

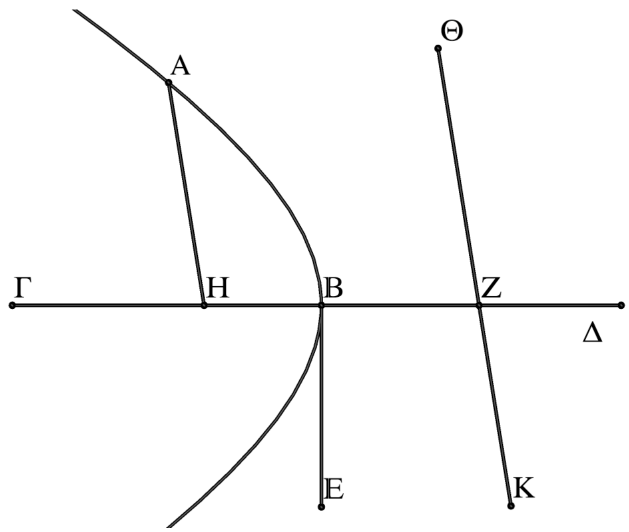

Figure 3.14: Two Diameters of an Hyperbola

(226.23) But in the case of the ellipse the thing said is not yet clear. For since it returns to itself, just as does the circle, and cuts off all the diameters inside and makes them bounded: so that not always in the case of the ellipse, the mean proportional of the sides of the figure, both being drawn through the center of the section and being bisected by the diameter, is bounded by the section.

(228.1) But it is possible to recapitulate it through the very things already said in the fifteenth theorem. For since, as it is shown there, the lines being drawn to ΔE parallel to AB are equal in square to the area applied along the third proportional to them[36], that is,

[36]*Conics* I.15: Here the diameter is AB, the "produced straight line" is ΔE, and the third proportional (τὴν τρίτην) to them is some straight line AN satisfying

$$\Delta E : AB : AB : AN.$$

The lines so drawn, being parallel to AB, are equal in square to an area having this third proportional AN as its width. It is critical to note that this third is a third proportional, not a third mean proportional.

AN. Therefore on account of this, the second diameter ΔE becomes a mean proportional between the sides of the figure BA and AN.

(228.13) But it is necessary to see also this on account of the utility of the diagrams: for since the diameters AB, ΔE are unequal (for in the circle only are they equal), it is clear that the line being dropped at right angles to the lesser of them, as at ΔZ, seeing as how it is the third term of the proportion between ΔE, AB, is the larger of the two; and that the line being drawn at right angles to the greater, as at AN, on account of being the third term of the proportion between AB, ΔE, is lesser of the two, so that the four terms are in continued proportion: for as AN is to ΔE, so too is ΔE to AB and AB to ΔZ.

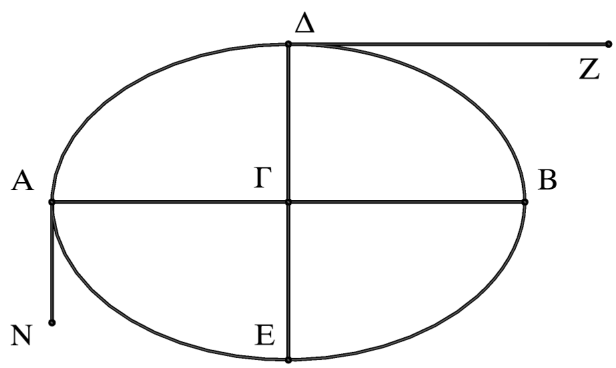

Figure 3.15: Two Diameters of an Ellipse

3.15 Theorem 17

(228.28) Euclid showed in the fifteenth theorem of the third book of the *Elements*, that the line being led at right angles from an endpoint of the diameter both falls outside and is tangent to the circle. But Apollonius in this work shows something more general, that it is possible to apply this to the the three sections of the cone and to the circle.

(230.5) The circle differs to this extent from the sections of the cone, that in this case the ordinates are drawn at right angles to the diameter: for no other straight lines parallel to themselves are bisected by the diameter of the circle, but in the case of the three sections, not always are they drawn at right angles, except to the axes alone.

3.16 Theorem 18

(230.13) In some copies, this theorem concerns a parabola and ellipse only, but it is better to have the protasis more general, unless it is because the case of the ellipse has been omitted by them as unambiguous: for ΓΔ, being inside the section (which is finite), cuts the section in both directions.

(230.19) But it is necessary to understand, that even if AZB cuts the section, the same proof is suitable.

3.17 Theorem 20

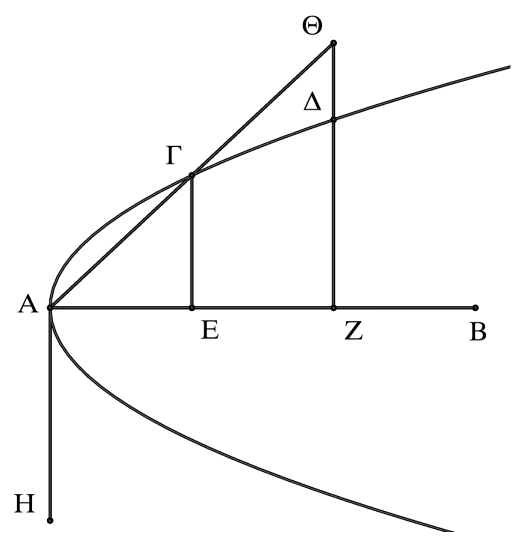

Figure 3.16: *Conics* I.20, with ΔΘ added

(230.22) Beginning from this theorem, successively in all of them he shows the *symptomata* of the parabola that belonging to it and not to any other [section], but for the most part, he shows the same *symptomata* belonging to the hyperbola and ellipse.

(230.27) But since it does not appear useless to those making mechanical drawings, on account of the difficulty concerning instruments[37], also often through successive points to draw the sections of the cone in the plane, through this theorem it is possible to provide successive points, through which the parabola will be drawn by means of an application

[37]As in mechanical instruments. "Gadgets" seems a fitting, though unacademic, word.

of a ruler. For if I set out a straight line, such as AB, and on it I take successive points, such as E, Z, and from them I make straight lines at right angles to AB, such as EΓ, ZΔ, taking on EΓ a random point Γ (if I should wish to make the parabola wider, taking Γ further from E; if narrower, closer), and I make the proportion

$$\mathrm{AE} : \mathrm{AZ} = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{sq.}(\mathrm{Z}\Delta),$$

then the points Γ and Δ will be on the section. But similarly we will also choose other points, through which the parabola will be drawn[38].

3.18 Theorem 21

(232.16) The theorem is set forth manifestly, and does not have cases: nevertheless, it is necessary to understand that the parameter, that is the upright side, is equal to the

[38]What application of what ruler? Eutocius unfortunately does not elaborate on this point. Based on his use of the word "mechanical," I can think of two possibilities for just how this "application" would be performed.

1. He means to use a marked ruler, much like a modern one. If this is the case, a degree of flexibility with arithmetic would be required, in order to solve the proportion

$$\mathrm{AE} : \mathrm{AZ} = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{sq.}(\mathrm{Z}\Delta).$$

This is certainly plausible, as similar arithmetical problems appear in the *Almagest* of Ptolemy. Certainly engineers or architects would be capable of this sort of arithmetic, and able to instruct their subordinates as to the desired measurements. It is worth recalling that Anthemius, the person to whom Eutocius addresses this work, was selected by the Byzantine emperor Justinian I to be one of the architects of the Hagia Sophia (the other being Isidore of Miletus).

2. That Eutocius has a more mechanical and mathematical procedure in mind, similar in spirit to some of the cube duplication procedures that he gives in his commentary on Archimedes.

The construction is straightforward up until Eutocius requires the proportion

$$\mathrm{AE} : \mathrm{AZ} = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{sq.}(\mathrm{Z}\Delta).$$

All of the terms in this proportion are known, with the exception of the position of Δ on $\mathrm{Z}\Delta$. Let $\mathrm{Z}\Delta$ be produced to Θ, and let $\mathrm{A}\Gamma$ be joined and produced to Θ. Let $\mathrm{Z}\Delta$ be the mean proportional between $\Gamma\mathrm{E}$ and $\mathrm{Z}\Theta$, so that

$$\Gamma\mathrm{E} : \mathrm{Z}\Delta = \mathrm{Z}\Delta : \mathrm{Z}\Theta.$$

From the standpoint of constructing $\mathrm{Z}\Delta$, this is easily accomplished by applying *Elements* VI.13. Therefore

$$\mathrm{sq.}(\mathrm{Z}\Delta) = \mathrm{rect.}(\Gamma\mathrm{E}, \mathrm{Z}\Theta), \text{ and so}$$

$$\mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{sq.}(\mathrm{Z}\Delta) = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{rect.}(\Gamma\mathrm{E}, \mathrm{Z}\Theta). \text{ But}$$

$$\mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{rect.}(\Gamma\mathrm{E}, \mathrm{Z}\Theta) = \mathrm{E}\Gamma : \mathrm{Z}\Theta,$$

and by virtue of the similarity of the triangles $\mathrm{AZ}\Theta$, $\mathrm{AE}\Gamma$, we have that

$$\mathrm{E}\Gamma : \mathrm{Z}\Theta = \mathrm{AE} : \mathrm{AZ}, \text{ so}$$

$$\mathrm{AE} : \mathrm{AZ} = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{rect.}(\Gamma\mathrm{E}, \mathrm{Z}\Theta).$$

Since $\mathrm{sq.}(\mathrm{Z}\Delta) = \mathrm{rect.}(\Gamma\mathrm{E}, \mathrm{Z}\Theta)$, we get the required proportion

$$\mathrm{AE} : \mathrm{AZ} = \mathrm{sq.}(\mathrm{E}\Gamma) : \mathrm{sq.}(\mathrm{Z}\Delta).$$

It should be noted, though, that this does not give a *construction* of a parabola, but merely of the point Δ on a parabola given the position of Γ on it and the intrinsic fact that AZ is the diameter. To fully construct a parabola in this way would require an infinite number of steps, one for each point Δ, and as such is not a true Euclidean construction. For practical uses, though, constructing several points might be sufficient for a given desired degree of accuracy.

diameter in the case of a circle. For if it is, that

$$\text{sq.}(\Delta\text{E}) : \text{rect.}(\text{AEB}) = \Gamma\text{A} : \text{AB};$$

but

$$\text{sq.}(\Delta\text{E}) = \text{rect.}(\text{AEB})$$

in the case of the circle alone, and so

$$\Gamma\text{A} = \text{AB}.$$

(232.23) But it is necessary also to see this, that the ordinates in the perimeter of the circle are always orthogonal to the diameter, and meet at right angles the parallels to AΓ.

(232.27) But through this theorem, continuing in the same manner as those mentioned in the case of the parabola, we draw a hyperbola and an ellipse by an application of a ruler. For let a straight line AB be set out and projected towards infinity through H, and from A let AΓ be led at right angles to this line, and let BΓ be joined and projected, and let some points be taken on AH, such as E, H, and from E and H, let EΘ, HK be drawn parallel to AΓ, and let it be contrived that

$$\text{ZH} : \text{AHK} = \Delta\text{E} : \text{AE}\Theta\text{:}$$

for the hyperbola will have come to be through A, Δ, Z. Similarly we will construct the things in the case of the ellipse.

3.19 Theorem 23

(234.12) But it is necessary to understand, that in the protasis, by "two diameters", he means not simply random ones, but rather the so-called *conjugate* diameters, of which each is drawn ordinatewise and is a mean proportional between the sides of the figure and the conjugate diameter[39], and on account of this they bisect the parallels to each other, as is shown in the fifteenth theorem. For if it is not taken in this way, it will happen that the intermediate straight line of the two diameters will be parallel to the other of them: which is not supposed.

[39]That is, the other diameter in the pair called conjugate.

(234.21) And when H is nearer than the midpoint of AB, which is Θ, and when

$$\text{rect.(BHA)} + \text{sq.}(\Theta\text{M}) = \text{sq.(AM)},$$

and

$$\text{rect.}(\text{A}\Theta\text{B}) + \text{sq.}(\Theta\text{M}) = \text{sq.(AM)}.$$

But

$$\text{sq.}(\Theta\text{M}) > \text{sq.(HM)},$$

so

$$\text{rect.(BHA)} > \text{rect.}(\text{B}\Theta\text{A}).$$

3.20 Theorem 25

(236.4) In some copies there is also this proof: let some point Θ be taken on the section, and let ZΘ be joined: therefore ZΘ, being projected, intersects ΔΓ: so that also ZE, being projected, intersects ΔΓ. Again, let it be taken, and let KZ be joined and projected: therefore it will intersect BA, being projected: so that also ZH, being projected, will intersect BA.

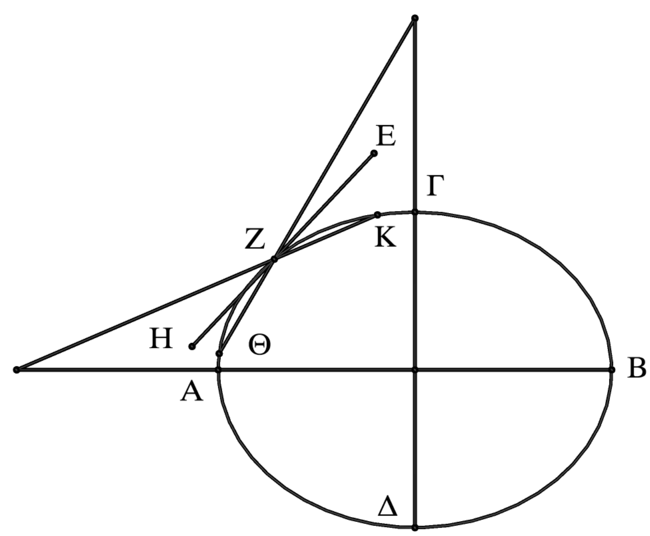

Figure 3.17: Eutocius' Figure for Theorem 25

3.21 Theorem 26

(236.11) This theorem has many cases, the first, that EZ is taken on the convex part of the section, as it is here, or on the concave part; next, that the straight line drawn ordinatewise from E, indiscriminately intersects the diameter (which is infinite) inside at one point, but when it is outside, especially in the case of the hyperbola, having this particular arrangement, it intersects either beyond B or at B or between A and B.

3.22 Theorem 27

(236.20) In some copies of the twenty-seventh theorem, the following proof is transmitted:

(236.22) Let there be a parabola, whose diameter is AB, and let it be cut by some straight line HΔ inside the section. I say, that HΔ, being projected in both directions, will intersect the section.

(238.3) For let some line AE be drawn ordinatewise through A; therefore AE will fall outside the section.

Then either HΔ is parallel to AE or not.

So if it is parallel, it has been drawn ordinatewise: so that being projected both ways, since it is bisected by the diameter, it will intersect the section. Therefore let it not be parallel to AE, but being projected, let it intersect AE at E as HΔE.

(238.11) So it is manifest that it intersects the section in the direction of E: for if projected to AE, it cuts the section long before.

(238.14) I say also that being projected in the other direction, it cuts the section.

(238.16) For let the parameter be MA, and let AZ be projected at right angles to it: therefore MA is orthogonal to AB. Let it be contrived, that

$$\text{sq.(AE)} : \text{tri.(AE}\Delta) = \text{MA} : \text{AZ},$$

and through M and Z, let ZK and MN be drawn parallel to AB. So since ΛAΔH is a quadrilateral, and ΛA is in this particular arrangement, let ΓKB be drawn parallel to ΛA, cutting the triangle ΓKH equal to the quadrilateral ΛAΔH, and through B let ΞBN be drawn parallel to ZAM.

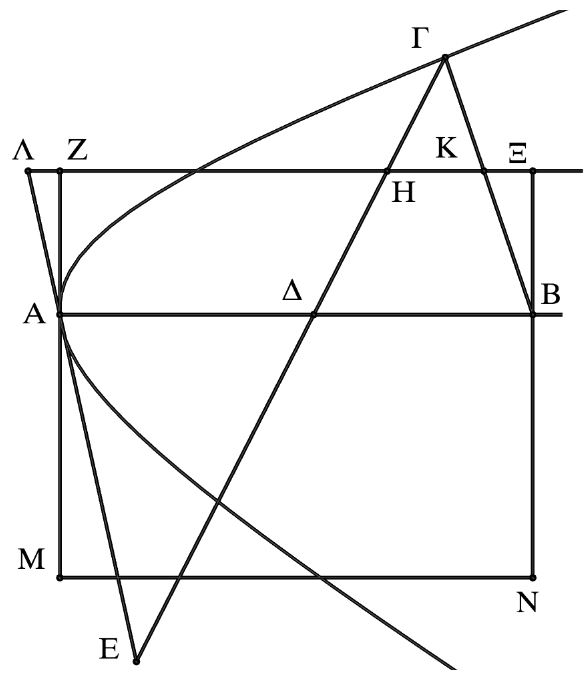

Figure 3.18: Eutocius' Figure for Theorem 27

And since

$$\text{sq.(AE)} : \text{tri.(AE}\Delta) = \text{MA} : \text{AZ},$$

but

$$\text{sq.(AE)} : \text{tri.(AE}\Delta) = \text{sq.(}\Gamma\text{B)} : \text{tri.(}\Delta\Gamma\text{B)}:$$

for AE is parallel to ΓB, and ΓE, AB join them[40]. But

$$\text{MA} : \text{AZ} = \text{pllg.(AMNB)} : \text{pllg.(A}\Xi\text{)};$$

therefore

$$\text{sq.(}\Gamma\text{B)} : \text{tri.(}\Gamma\Delta\text{B)} = \text{pllg.(AMNB)} : \text{pllg.(AZ}\Xi\text{B)}.$$

So *alternando*,

$$\text{sq.(}\Gamma\text{B)} : \text{pllg.(AMNB)} = \text{tri.(}\Gamma\Delta\text{B)} : \text{pllg.(AZ}\Xi\text{B)}.$$

But

$$\text{pllg.(ZABE)} = \text{tri.(}\Gamma\text{BA)} :$$

[40] i.e. AE, ΓB, forming a quadrilateral.

for since

$$\text{tri.}(\Gamma\text{HK}) = \text{quad.}(\text{A}\Lambda\text{H}\Delta),$$

and the quadrilateral HΔBK is common,

$$\text{pllg.}(\Lambda\text{ABK}) = \text{tri.}(\Gamma\Delta\text{B}).$$

But

$$\text{pllg.}(\Lambda\text{ABK}) = \text{pllg.}(\text{ZAB}\Xi) :$$

for they are on the same base AB and in the same parallels AB, ZK. Therefore

$$\text{tri.}(\Gamma\Delta\text{B}) = \text{pllg.}(\text{ZAB}\Xi) :$$

so that

$$\text{sq.}(\Gamma\text{B}) = \text{pllg.}(\text{AMNB}).$$

But

$$\text{pllg.}(\text{MABN}) = \text{rect.}(\text{MAB}),$$

for MA is orthogonal to AB: therefore

$$\text{rect.}(\text{MAB}) = \text{sq.}(\Gamma\text{B}),$$

and MA is the upright side of the figure, and AB the diameter, and ΓB an ordinate (for it is parallel to AE): therefore Γ is on the section. Therefore ΔHΓ intersects the section at Γ: the very thing which it was necessary to show.

(240.23) Scholia to the preceding theorem.

(240.24) [Let it be contrived, that as the square on AE is to the triangle AEΔ, so too is MA to AZ.] This is shown in the scholia to the eleventh theorem. For having drawn the square on AE, and being applied to its side equal to the triangle AEΔ, I will have the desired property.

(242.1) To the same.

(242.2) [Since the quadrilateral is ΛAΔH, let ΓKB be drawn parallel to ΛA, cutting off the triangle ΓHK equal to the quadrilateral ΛAΔH.] We will show this as follows: for if, as we learned in the *Elements*, we construct the same ΣTY to be equal to the given rectilineal figure, the quadrilateral ΛAΔH, and similar to another given figure, the triangle

AEΔ, so that ΣY is in the same proportion to AΔ:

$$\Sigma TY : \text{tri.}(AE\Delta) = \Sigma Y : A\Delta,$$

and if we take

$$HK = \Sigma Y$$

and

$$H\Gamma = TY$$

and if we join ΓK, it will be the thing being sought. For since the angle at Y is equal to that at Δ--that is, to the angle at H--for this reason ΓHK is both equal to and similar to ΣTY. And the angle Γ is equal to E, and they are vertical angles: therefore ΓK is parallel to AE.

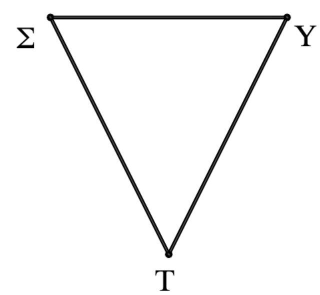

Figure 3.19: Eutocius' Figure for the Scholia of Theorem 27

(242.16) But it is manifest that, whenever AB is an axis, MA lies tangent on the section, but whenever it is not an axis, it cuts; that is, if it is led at right angles to the diameter.

3.23 Theorem 28

(242.20) That, even if ΓΔ cuts the hyperbola, the same things will follow, just as in the eighteenth theorem.

3.24 Theorem 30

(242.23) [And therefore, as *componendo* in the case of the ellipse, *invertendo* and *convertendo* in the case of the opposite sections.]

So in the case of the ellipse, we will say the following: since

$$\text{rect.}(\mathrm{AZB}) : \text{sq.}(\Delta\mathrm{Z}) = \text{rect.}(\mathrm{AHB}) : \text{sq.}(\mathrm{HE});$$

but

$$\text{sq.}(\Delta\mathrm{Z}) : \text{sq.}(\mathrm{Z}\Gamma) = \text{sq.}(\mathrm{EH}) : \text{sq.}(\mathrm{H}\Gamma),$$

and so *ex aequali*,

$$\text{rect.}(\mathrm{AZB}) : \text{sq.}(\mathrm{Z}\Gamma) = \text{rect.}(\mathrm{AHB}) : \text{sq.}(\mathrm{H}\Gamma);$$

and so *componendo*,

$$\text{rect.}(\mathrm{AZB}) + \text{sq.}(\mathrm{Z}\Gamma) : \text{sq.}(\mathrm{Z}\Gamma) = \text{sq.}(\mathrm{A}\Gamma) : \text{sq.}(\Gamma\mathrm{Z}) = \text{sq.}(\Gamma\mathrm{B}) : \text{sq.}(\Gamma\mathrm{H}),$$

for AB has been cut into equal sections at Γ and unequal sections at Z. And *alternando*,

$$\text{sq.}(\mathrm{A}\Gamma) : \text{sq.}(\Gamma\mathrm{B}) = \text{sq.}(\mathrm{Z}\Gamma) : \text{sq.}(\Gamma\mathrm{H});$$

But in the case of the opposite sections: since

$$\text{rect.}(\mathrm{BZA}) : \text{sq.}(\mathrm{Z}\Gamma) = \text{rect.}(\mathrm{AHB}) : \text{sq.}(\Gamma\mathrm{H}),$$

through equality, and *invertendo*

$$\text{sq.}(\mathrm{Z}\Gamma) : \text{rect.}(\mathrm{BZA}) = \text{sq.}(\Gamma\mathrm{H}) : \text{rect.}(\mathrm{AHB});$$

and *convertendo*,

$$\text{sq.}(\mathrm{Z}\Gamma) : \text{sq.}(\Gamma\mathrm{A}) = \text{sq.}(\mathrm{H}\Gamma) : \text{sq.}(\Gamma\mathrm{B}) :$$

for since some straight line AB has been bisected at Γ, and ZA is attached, and

$$\text{rect.}(\mathrm{BZA}) + \text{sq.}(\mathrm{A}\Gamma) = \text{sq.}(\Gamma\mathrm{Z}),$$

so that the square on ΓZ exceeds the rectangle BZA by the square AΓ; and rightly it has been called *componendo*.

3.25 Theorem 31

[*Separando*, the square on ΓB has a ratio to the rectangle AHB bigger than that of the square on ΓB to the rectangle AΘB.] For since the straight line AB has been bisected at Γ, and BH is attached to it,

$$\text{rect.}(\mathrm{AHB}) + \text{sq.}(\Gamma\mathrm{B}) = \text{sq.}(\Gamma\mathrm{H}) :$$

so that the square on ΓH exceeds the rectangle AHB by the square on ΓB. But for the same reason also the square on ΓΘ exceeds the rectangle AΘB by the square on ΓB: so that rightly it has been called *separando*.

3.26 Theorem 32

(246.4) In the seventeenth theorem he showed more clearly that the line having been drawn ordinatewise touches (the section), but the proof there in the *Elements* in the case of the circle alone; he shows this more generally for each section of a cone.

(246.9) But it is necessary to understand the very thing which was also shown there [*Conics* I.17], that it is perhaps in no way strange that a curved line falls between the straight line and the section, but a straight line cannot: for it itself cuts the section and is not tangent, for it is not possible for two lines to be tangent through the same point.

(246.15) But since this theorem has been proved with a lot of twists and turns in the different editions, we made the proof simpler and more manifest.

3.27 Theorem 34

(246.18) It is necessary to understand, that the ordinate to the diameter ΓΔ, in the case of the hyperbola, defining ΔB, ΔA, leaves BA needing to be cut into the ratio of BΔ to ΔA; but in the case of the ellipse and the circle the inverse: cutting BA into the defined ratio of BΔ to ΔA, makes us find the ratio of BE to EA: for it is in no way difficult, having been given one ratio, to provide another equal to it.

(248.1) But it is necessary also to know, that according to each section, there are two diagrams, the point Γ either being taken inside or outside with respect to the point Z: so that there are six total cases.

(248.5) But he also needs two lemmas, which we will write in turn.

(248.6) [Therefore the rectangle ANΞ is bigger than the rectangle AOΞ: therefore NO has to ΞO a ratio greater than that of OA to AN.] For since

$$\text{rect.}(\mathrm{AN}, \mathrm{N\Xi}) > \text{rect.}(\mathrm{AO}, \mathrm{O\Xi}),$$

let it happen that the rectangle formed by AO and some other line ΞΠ is equal to the rectangle AN, NΞ:

$$\text{rect.}(\mathrm{AO}, \Xi\Pi) = \text{rect.}(\mathrm{AN}, \mathrm{N\Xi}) :$$

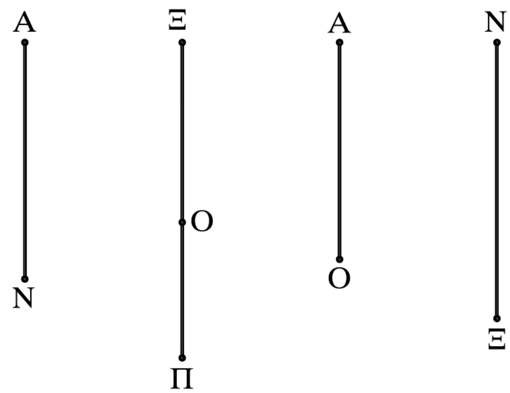

Figure 3.20: Eutocius' Figure for Theorem 34

this line ΞΠ will be greater than ΞO: Therefore

$$\mathrm{OA} : \mathrm{AN} = \mathrm{N\Xi} : \Xi\Pi.$$

But

$$\mathrm{N\Xi} : \Xi\mathrm{O} > \mathrm{N\Xi} : \Xi\Pi,$$

and therefore

$$\mathrm{OA} : \mathrm{AN} < \mathrm{N\Xi} : \Xi\mathrm{O}.$$

(248.15) The *alternando* is also manifest, that also if

$$\mathrm{N\Xi} : \Xi\mathrm{O} > \mathrm{OA} : \mathrm{AN},$$

then

$$\mathrm{rect}(\Xi\mathrm{N}, \mathrm{NA}) > \mathrm{rect.}(\mathrm{AO}, \mathrm{O\Xi}).$$

(248.18) For let it happen, that as OA is to AN, so too is NΞ to some line which is manifestly greater than ΞO, say, ΞΠ:

$$\mathrm{OA} : \mathrm{AN} = \mathrm{N\Xi} : \Xi\Pi :$$

therefore

$$\mathrm{rect.}(\Xi\mathrm{N}, \mathrm{NA}) = \mathrm{rect.}(\mathrm{AO}, \Xi\Pi),$$

so that

$$\mathrm{rect.}(\Xi\mathrm{N}, \mathrm{NA}) = \mathrm{rect.}(\mathrm{AO}, \mathrm{O\Xi}).$$

(248.22) To the same.

(248.23) [But as the rectangle BK, AN is to the square on ΓE, so too is the rectangle BΔA to the square on EΔ.] So since, on account of the lines AN, EΓ, and KB being parallel,

$$\mathrm{AN} : \mathrm{E\Gamma} = \mathrm{A\Delta} : \mathrm{\Delta E};$$

but

$$\mathrm{E\Gamma} : \mathrm{KB} = \mathrm{E\Delta} : \mathrm{\Delta B}.$$

Therefore *ex aequali*,

$$\mathrm{AN} : \mathrm{KB} = \mathrm{A\Delta} : \mathrm{\Delta B} :$$

and therefore

$$\text{sq.}(\mathrm{AN}) : \text{rect.}(\mathrm{AN}, \mathrm{KB}) = \text{sq.}(\mathrm{A\Delta}) : \text{rect.}(\mathrm{A\Delta B}).$$

But

$$\text{sq.}(\mathrm{E\Gamma}) : \text{sq.}(\mathrm{AN}) = \text{sq.}(\mathrm{E\Delta}) : \text{sq.}(\mathrm{\Delta A}) :$$

therefore *ex aequali*,

$$\text{sq.}(\mathrm{E\Gamma}) : \text{rect.}(\mathrm{AN}, \mathrm{KB}) = \text{sq.}(\mathrm{E\Delta}) : \text{rect.}(\mathrm{A\Delta B});$$

and *alternando*,

$$\text{rect.}(\mathrm{KB}, \mathrm{AN}) : \text{sq.}(\mathrm{E\Gamma}) = \text{rect.}(\mathrm{B\Delta A}) : \text{sq.}(\mathrm{E\Delta}).$$

3.28 Theorem 37

It is clear by these theorems in what way it is possible to draw a tangent through a given point on the diameter and through the vertex of the section.

3.29 Theorem 38

(250.16) In some copies this theorem is found proved only in the case of the hyperbola, but here it is shown more generally: for the same things occur in the cases of the other sections. And it seems that to Apollonius, not only the hyperbola has a second diameter, but also the ellipse, as we often heard from him in the proceeding materials.

(250.23) And with respect to the ellipse, there are not cases, but with respect to the hyperbola there are three: for the point Z, at which the tangent intersects the second diameter, is either before Δ, at Δ, or after Δ; and for this reason the point Θ similarly will have three possible locations, and it is necessary that one attend to the fact that either Z will fall before Δ, and Θ will be before Γ, or that Z will be on Δ, and Θ on Γ, or that Z will be after Δ, and Θ will be beyond Γ.

3.30 Theorem 41

(252.8) This theorem does not have cases for the hyperbola, but with the ellipse, if the ordinate passes through the center, and the remaining things become the same, the figure on the ordinate will be equal to the figure on the line from the center.

(252.13) For let there be an ellipse, whose diameter is AB, center Δ, and let ΓΔ be drawn ordinatewise, and let equiangular figures AZ, ΔH be set up on the bases ΓΔ and AΔ, and let

$$\Delta\Gamma : \Gamma\mathrm{H} = (\mathrm{A}\Delta : \Delta\mathrm{Z}) \text{ comp. (the upright : the transverse).}$$

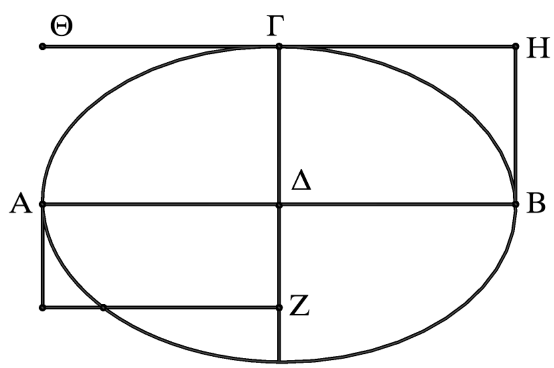

Figure 3.21: Eutocius' Figure for Theorem 41

(252.19) I say that the figure AZ is equal to the figure ΔH.

(252.20) For since it is shown in this text, that

$$\text{sq.}(\mathrm{A}\Delta) : \text{fig.}(\mathrm{AZ}) = \text{rect.}(\mathrm{A}\Delta\mathrm{B}) : \text{fig.}(\Delta\mathrm{H}),$$

I say that also *alternando*,

$$\text{sq.}(\mathrm{A}\Delta) : \text{rect.}(\mathrm{A}\Delta\mathrm{B}) = \text{fig.}(\mathrm{AZ}) : \text{fig.}(\Delta\mathrm{H}).$$

But

$$\text{sq.}(\mathrm{A}\Delta) = \text{rect.}(\mathrm{A}\Delta\mathrm{B}),$$

therefore

$$\text{fig}(\mathrm{AZ}) = \text{fig.}(\Delta\mathrm{H}).$$

3.31 Theorem 42

(252.26) This theorem has 11 cases, one, if Δ be taken inside of Γ: for it is clear, that also the parallels inside will fall inside of $\mathrm{A}\Gamma\Theta$. But another five are these: if Δ be taken outside of Γ, manifestly the parallel $\Delta\mathrm{Z}$ will fall outside of $\Theta\Gamma$; but $\Delta\mathrm{E}$ will fall either between A and B, or at B, or between $\mathrm{B}\Theta$, or at Θ, or outside Θ: for it is not possible for it to fall outside A; since Δ is outside of Γ, and manifestly also the line being drawn parallel through it to $\mathrm{A}\Gamma$. But if Δ be taken on the other [parts] of the section, either the two parallels will be terminated between Θ, B, or $\Delta\mathrm{Z}$ inside Θ, but E at Θ, or when $\Delta\mathrm{Z}$ likewise remaining outside of Θ, will pass through E: but again, when E falls outside, Z will either fall at Θ, so as to be the single straight line $\Gamma\Theta\Delta$, unless, strictly speaking, the peculiarity of the parallel is preserved then; or Z will fall outside of Θ. But it is necessary in the case of the proof of the final five cases to project $\Delta\mathrm{Z}$ right up to the section and the parallel $\mathrm{H}\Gamma$ in this way to make the proof.

(254.20) But it is possible also think of one other diagram from these things, whenever a different point is taken, the original straight lines make the thing being said; but this is a theorem rather than a case.

3.32 Theorem 43

(254.25) In some copies, the following proof of this theorem appears:

(256.1) For since

$$\text{rect.}(\mathrm{Z}\Gamma\Delta) = \text{sq.}(\Gamma\mathrm{B}),$$

therefore,

$$\mathrm{Z}\Gamma : \Gamma\mathrm{B} = \Gamma\mathrm{B} : \Gamma\Delta:$$

and therefore

$$\text{fig.}(\Gamma\mathrm{Z}) : \text{fig.}(\Gamma\mathrm{B}) = \mathrm{Z}\Gamma : \Gamma\Delta.$$

But

$$\text{sq.}(\Gamma Z) : \text{sq.}(\Gamma B) = \text{tri.}(EZ\Gamma) : \text{tri.}(\Lambda\Gamma B):$$

therefore

$$\text{tri.}(E\Gamma Z) : \text{tri.}(B\Lambda\Gamma) = \text{tri.}(E\Gamma Z) : \text{tri.}(E\Gamma\Delta).$$

Therefore

$$\text{tri.}(E\Gamma\Delta) = \text{tri.}(B\Gamma A).$$

And therefore, as in the case of the hyperbola, *convertendo*, but in the case of the the ellipse *alternando* and *separando*,

$$\text{tri.}(EZ\Gamma) : \text{quad.}(E\Lambda BZ) = \text{tri.}(E\Gamma Z) : \text{tri.}(E\Delta Z);$$

therefore

$$\text{tri.}(E\Delta Z) = \text{quad.}(E\Lambda BZ).$$

And since

$$\text{sq.}(\Gamma Z) : \text{sq.}(\Gamma B) = \text{tri.}(EZ\Gamma) : \text{tri.}(A\Gamma B),$$

in the case of the hyperbola, *convertendo*, but in the case of the ellipse *alternando*, *separando*, and *alternando* again,

$$\text{rect.}(AZB) : \text{sq.}(B\Gamma) = \text{quad.}(E\Lambda BZ) : \text{tri.}(B\Lambda\Gamma).$$

Similarly also,

$$\text{sq.}(\Gamma B) : \text{rect.}(AKB) = \text{tri.}(A\Gamma B) : \text{quad.}(M\Lambda BK) :$$

therefore *ex aequali*,

$$\text{rect.}(AZB) : \text{rect.}(AKB) = \text{quad.}(E\Lambda BZ) : \text{quad.}(E\Lambda BZ).$$

But

$$\text{rect.}(AZB) : \text{rect.}(AKB) = \text{sq.}(EZ) : \text{sq.}(HK);$$

and

$$\text{sq.}(EZ) : \text{sq.}(HK) = \text{tri.}(E\Delta Z) : \text{tri.}(H\Theta K) :$$

and therefore,

$$\text{tri.}(E\Delta Z) : \text{sq.}(H\Theta K) = \text{quad.}(E\Lambda BZ) : \text{quad.}(M\Lambda BK).$$

Alternando,

$$\text{tri.}(\mathrm{E\Delta Z}) : \text{quad.}(\mathrm{E\Lambda BZ}) = \text{tri.}(\mathrm{H\Theta K}) : \text{quad.}(\mathrm{M\Lambda BK}).$$

But

$$\text{tri.}(\mathrm{E\Delta Z}) = \text{quad.}(\mathrm{E\Lambda BZ})^{41} :$$

therefore also

$$\text{tri.}(\mathrm{H\Theta K}) = \text{quad.}(\mathrm{M\Lambda BK}).$$

Therefore the triangle MΓK differs from the triangle HΘK by the triangle ΛBΓ.

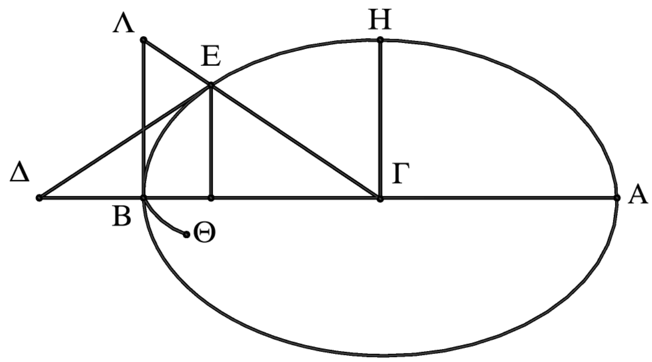

Figure 3.22: Eutocius' Figure for Theorem 43

(258.4) But it is necessary to understand by this proof: for it has a little confusion in the proportions of the ellipse: in order that we separate the things said together for the conciseness of the text, for instance--for he says: since

$$\text{sq.}(\mathrm{Z\Gamma}) : \text{sq.}(\mathrm{\Gamma B}) = \text{tri.}(\mathrm{E\Gamma Z}) : \text{tri.}(\mathrm{\Lambda B\Gamma}),$$

and *invertendo*, *convertendo* and *invertendo* again,

$$\text{sq.}(\mathrm{B\Gamma}) : \text{sq.}(\mathrm{\Gamma Z}) = \text{tri.}(\mathrm{AB\Gamma}) : \text{tri.}(\mathrm{EZ\Gamma}).$$

Convertendo, as the square on BΓ is to the rectangle AZB, that is, the excess of the square on ΓB to the square on ΓZ on account of Γ being the midpoint of AB, so too is the triangle ABΓ to the quadrilateral ΛBZE:

$$\text{sq.}(\mathrm{B\Gamma}) : \text{sq.}(\mathrm{AZB}) = \text{tri.}(\mathrm{AB\Gamma}) : \text{quad.}(\mathrm{\Lambda BZE}).$$

[41]This was shown previously.

So *invertendo*,

$$\text{rect.}(\text{AZB}) : \text{sq.}(\text{B}\Gamma) = \text{quad.}(\text{E}\Lambda\text{BZ}) : \text{tri.}(\text{AB}\Gamma).$$

(258.17) In the case of the hyperbola, it has eleven cases, as many as the one before it had for the parabola, and another one, whenever the chosen point H be the same as E: for then, it follows that the triangle EΔZ together with the triangle ΛBΓ is equal to the triangle ΓEZ: for the triangle EΔZ has been shown equal to the quadrilateral ΛBZE, but the quadrilateral ΛBZE exceeds the triangle ΓZE by the triangle ΛBΓ. But in the case of the ellipse, either H is the same as E or is taken outside of E: for it is clear, that both the parallels will fall between Δ, Z, as it is in the text. But if H be taken outside of E, and the line from it parallel to EZ falls between Z, Γ, the point Θ makes five cases: for either it falls between Δ, B or at B or between BZ or at Z or between Z, Γ. But if the parallel to the ordinate through H intersects at the center Γ, the point Θ will again make another five cases in the same manner: and it is necessary, pertaining to this, to point out that the triangle being formed from the parallels to EΔ, EZ is equal to the triangle ABΓ: for since it is, that

$$\text{sq.}(\text{EZ}) : \text{sq.}(\text{H}\Gamma) = \text{tri.}(\text{E}\Delta\text{Z}) : \text{tri.}(\text{H}\Theta\Gamma),$$

for they are similar, but

$$\text{sq.}(\text{EZ}) : \text{sq.}(\text{H}\Gamma) = \text{tri.}(\text{BZA}) : \text{rect.}(\text{B}\Gamma\text{A}),$$

that is, the square on BΓ; therefore

$$\text{tri.}(\text{E}\Delta\text{Z}) : \text{tri.}(\text{H}\Theta\Gamma) = \text{rect.}(\text{BZA}) : \text{sq.}(\text{B}\Gamma):$$

But as the rectangle BZA is to the square on BΓ, so too it has been shown that the quadrilateral ΛBZE is to the triangle ΛBΓ: and therefore

$$\text{tri.}(\text{E}\Delta\text{Z}) : \text{tri.}(\text{H}\Theta\Gamma) = \text{quad.}(\Lambda\text{BZE}) : \text{tri.}(\text{AB}\Gamma):$$

And *invertendo*. But it is also possible for those saying these things to show them in another way, that in the case of the double, parallelograms themselves, these things have been shown in the scholia to the forty-first theorem.

(260.23) But if the line being drawn through H parallel to EZ falls between Γ, A, it will be projected until ΓE intersects it; but the point Θ will make 7 cases: for it is either between B, Δ, or falls at B, or between B, Z, or at Z, or between Z, Γ, or at Γ, or beween Γ, A: and in these cases it follows that the difference between the triangles ΛBΓ, HΘK forms a smaller triangle on the line segment AB by the line ΛΓ being projected[42].

(262.7) But if H be taken on the other parts of the section, and the line from H parallel to EZ falls between B, Z, it will be projected on account of the proof, until it cuts ΛΓ; but the point Θ will make seven cases: either it is between B, Z, or falling at Z, or between Z, Γ, or at Γ, or between Γ, A, or at A, or beyond A. But if the line through H parallel to EZ falls at Z, so that EZH is one straight line, the point Θ will make five cases: for either it will fall between Z, Γ, or at Γ, or between Γ, A, or at A, or beyond A. But if HK falls beyond Z, Γ, the point Θ will make five cases: for it will fall either between Z, Γ, or at Γ, or between Γ, A, or at A, or beyond A. But if HK intersects the center Γ, the point Θ will make three cases: it falling either between Γ, A or at A or beyond A: and in these cases it will again follow that the triangle HΘK becomes equal to the triangle ΛBΓ. But if HK falls between Γ, A, the point Θ will fall either between Γ, A, or at A, or beyond A.

(262.28) So it follows in the case of an ellipse that there are forty-two cases in all, and in the case of the perimeter of a circle the same number; so that there are, all told, 96 cases of this theorem.

3.33 Theorem 44

(264.4) [So since the opposite sections are ZA, BE, whose diameter is AB, but the line through the center [is] ZΓE and the lines ZH, ΔE are tangent to the sections, ZH is parallel to EΔ.] For since AZ is a hyperbola, and ZH is tangent, and ZO is an ordinate, the rectangle OΓH is equal to the square on ΓA by the thirty-seventh theorem: and indeed similarly, the rectangle ΞΓΔ is equal to the rectangle ΓB. Therefore it is, as the rectangle OΓH is to the square on AΓ, so too is the rectangle ΞΓΔ to the square on

[42] i.e. we've made a triangle on the segment AB, whose third vertex is on the line segment ΛΓ.

BΓ, and *invertendo*, as the rectangle OΓH is to the rectangle ΞΓΔ, so too is the square AΓ to the square ΓB. But the square on AΓ is equal to the square on ΓB: therefore also the rectangle OΓH is equal to the rectangle ΞΓΔ. And OΓ is equal to ΓΞ: therefore also HΓ is equal to ΓΔ: but also, ZΓ is equal to ΓE by the thirtieth theorem: therefore the [sections] ZΓH are equal to the [sections] EΓΔ. And equal angles are at Γ: for it is at a vertex, so that also ZH is equal to EΔ and the angle ΓZH is equal to the angle ΓEΔ. And they are alternate angles: therefore ZH is parallel to EΔ.

(264.22) There are twelve cases of it, just as there are in the case of the hyberbola in the thirty-third theorem, and the proof is the same.

3.34 Theorem 45

(264.25) It is necessary to understand that this theorem has very many cases. For in the case of the hyperbola there are twenty: for the point chosen instead of B is either the same as A or the same as Γ: for then it follows that the triangle on AΘ, similar to the triangle ΓΔΛ, is the same as the triangle being cut by the parallels to ΔΛΓ. But if B be chosen between A, Γ, and Δ, Λ be beyond the endpoints of the second diameter, three cases occur: for Z, E are carried beyond these endpoints or at them or before them. But if the points Δ, Λ be at the endpoints of the second diameter, the points Z, E will be carried inside. But similarly also if B be chosen beyond Γ[43], and ΘΓ is projected to Γ, it follows that another three cases thus arise: for when the point Δ is carried either beyond the endpoint of the second diameter or at it or before it, the point Z similarly being carried will make these three cases. But if the point B be taken on the other parts of the section, ΓΘ will be projected to Θ according to the proof, but BZ, BE make three cases, since the point Λ is either carried to the endpoint of the second diameter, or beyond it, or before it.

(266.21) But in the case of the ellipse and the perimeter of the circle, we will say nothing complicated, but only so much as was said in the preceding theorem: so that there are 104 cases of this theorem.

[43]Heiberg notes that there is a serious textual issue here, and so this translation should be treated as tentative pending review of the manuscripts.

(268.1) But the proofs of the protasis are possible also in the case of the opposite sections.

3.35 Theorem 46

(268.4) This theorem has many cases, which we will show, paying attention to those of the forty-second [theorem].

(268.6) For the sake of an illustration, if Z falls at B, we will obviously say: since BΔΛ is equal to ΘBΔM, let the common part NMΔB be be subtracted: therefore the remainder ΛNM is equal to the remainder NΘB.

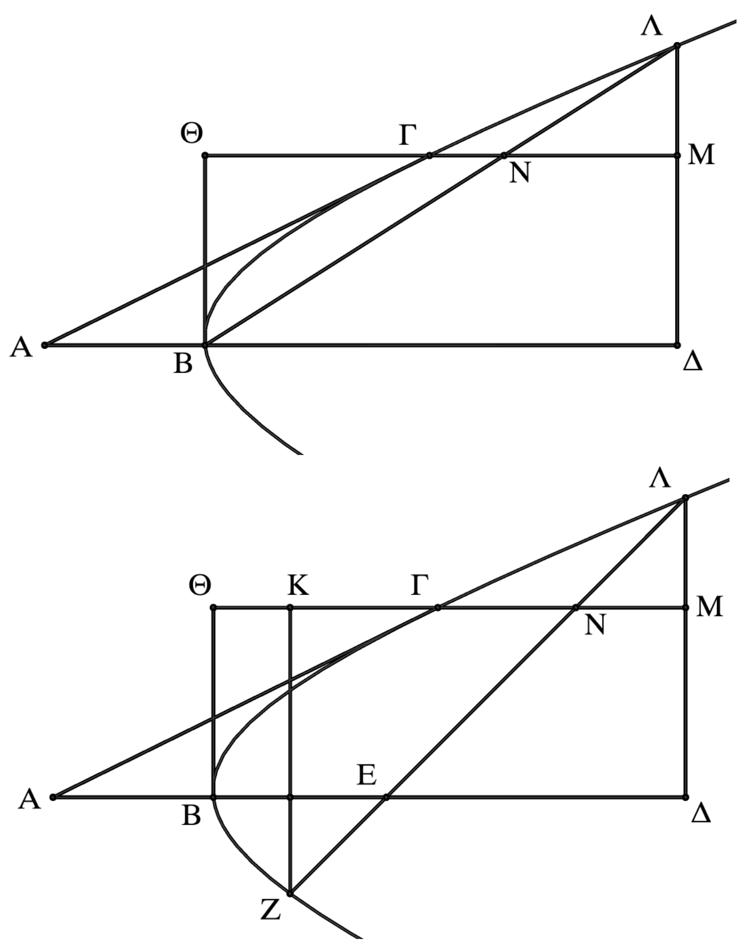

Figure 3.23: Eutocius' Figures for Theorem 46

(268.10) But in the case of the remaining, we will say: since ΛEΔ is equal to

ΘΒΔΜ, that is to ΚΗΔΜ and ΗΖΕ, that is to ΖΚΝ and ΝΕΔΜ, let the common part ΝΕΔΜ be subtracted: therefore also the remainder ΛΝΜ is equal to the remainder ΚΖΝ.

3.36 Theorem 47

(268.15) This theorem has cases with respect to the hyperbola, as many as the preceding theorem had for the case of the parabola, but we will show the proofs of them, paying attention to the cases of the forty-third, but also in the case of the ellipse, we will show the proofs from the cases of the forty-third theorem, for instance in the case of the diagram below, with the point Η having been taken outside. Since the triangle ΛΑΓ is equal to the triangles ΘΗΩ, ΩΛΜ, that is to the triangles ΟΘΓ, ΟΗΜ, but both the triangle ΞΠΓ and the quadrilateral ΛΔΠΞ are equal to the triangle ΛΑΓ, that is the triangle ΝΘΠ by what was shown in the forty-third theorem; and therefore the triangles ΞΠΓ and ΝΘΠ are equal to the triangles ΟΘΓ and ΟΜΗ. Let the common part ΘΟΓ be subtracted: therefore the remainder ΞΟΝ is equal to the remainder ΗΟΜ. And ΝΞ is parallel to ΜΗ: therefore ΝΟ is equal to ΟΗ.

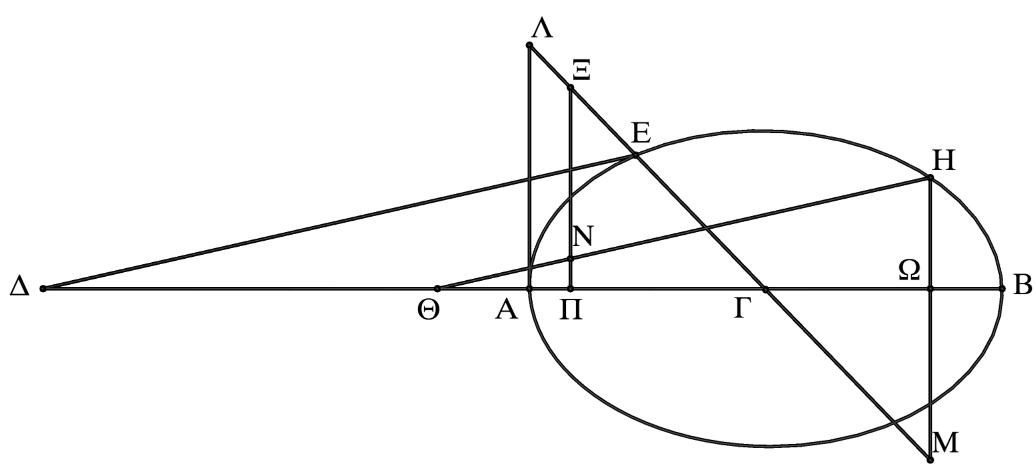

Figure 3.24: Eutocius' Figure for Theorem 47

3.37 Theorem 48

(270.15) And the cases of this theorem are similar to those of the preceding in the case of the 47th, according to the diagram of the hyperbola.

3.38 Theorem 49

(270.19) [Therefore the remainder, triangle ΚΛΝ, is equal to the parallelogram ΔΛΠΓ. And the angle ΔΛΠ is equal to the angle ΚΛΝ: therefore the rectangle ΚΛΝ is double the rectangle ΛΔΓ.] For let the triangle ΚΛΝ and the parallelogram ΔΛΠΓ be set out independently. And since the triangle ΚΛΝ is equal to the parallelogram ΔΠ, let ΝΡ be drawn through Ν parallel to ΛΚ, and let ΚΡ be drawn through Κ parallel to ΛΝ: therefore ΛΡ is a parallelogram double the triangle ΚΛΝ: so that it is also double the parallelogram ΔΠ. Indeed, let the lines ΔΓ, ΛΠ be projected to Σ, Τ, and let ΓΣ be equal to ΔΓ, and ΠΤ to ΛΠ, and let ΣΤ be joined: therefore ΔΤ is a parallelogram double ΔΠ: so that ΛΡ is equal to ΛΣ. But it is also equiangular to it, by virtue of the angles at Λ being vertical: but of the equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional: therefore it is, that as ΚΛ is to ΛΤ, that is ΔΣ, so too is ΔΛ to ΛΝ, and the rectangle ΚΛΝ is equal to the rectangle ΛΔΣ. And since ΔΣ is double ΔΓ, the rectangle ΚΛΝ is double the rectangle ΛΔΓ.

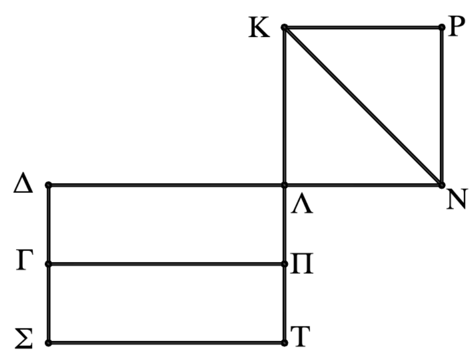

Figure 3.25: Eutocius' Figure for Theorem 49

(272.17) But if ΔΓ is parallel to ΛΠ, and ΓΠ is parallel to ΛΔ, it is manifest that

ΔΓΠΛ is a trapezoid; but I also say, that the rectangle ΚΛΝ is equal to the rectangle ΔΛ and the sum of ΓΔ, ΛΠ. For if ΛΡ be restored, as has been said before, and the lines ΔΓ, ΛΠ also be projected, and ΓΣ be cut equal to ΛΠ, and ΠΤ be cut equal to ΔΓ, and ΣΤ be joined, the parallelogram ΔΤ will be double the parallelogram ΔΠ, and the same proof will suffice. And this will be useful to the following [theorem].

3.39 Theorem 50

(272.28) The cases of this theorem number the same as those of the forty-third, but also similarly in the case of the fifty-first.

(274.1) To the conclusion.

(274.2) By "the original diameter," he means the section that came into being in the cone, common to the cutting plane and the triangle through the axis: but he also calls this the principal diameter. But he also says that all the demonstrated *symptomata* of the sections, where we have supposed the principle diameters as in the aforementioned theorems, are able to occur when all the other diameters are supposed, as well.

3.40 Theorem 54

(274.11) [And let a plane be erected on ΑΒ at right angles to the given plane, and in it let the circle ΑΕΒΖ be drawn around ΑΒ, so that the section of the diameter of the circle within the sector ΑΕΒ has to the section of the diameter within the sector ΑΖΒ a ratio not greater than that which ΑΒ has to ΒΓ.] For let there be two straight lines, ΑΒ and ΒΓ, and let it be required to draw a circle around ΑΒ, so that the diameter of it is cut by ΑΒ in such a way that the part of it towards Γ has a ratio to the remaining part not bigger than that of ΑΒ to ΒΓ.

(274.22) Now let it be supposed[44], and let ΑΒ be bisected at Δ, and through it, at right angles to ΑΒ, let ΕΔΖ be drawn, and let it be contrived, that as ΑΒ is to ΒΓ, so too is ΕΔ to ΔΖ, and let ΕΖ be bisected: it is manifest that, if ΑΒ is equal to ΒΓ and ΕΔ to ΔΖ, then Δ will be the midpoint of ΕΖ; but if ΑΒ is bigger than ΒΓ and ΕΔ is bigger

[44] i.e. that they have the same ratio.

than ΔZ, the midpoint is beyond Δ, but if AB is lesser than BΓ, the midpoint is closer.

(276.7) For now let it be beyond, at say H, and with center H and diameter HZ let a circle be drawn: it is necessary that it will pass through the points A, B, or inside, or outside. And if it goes through the points A, B, the enjoined would have occured: but let it fall beyond A, B, and let AB, being projected both ways, intersect the perimeter at the points Θ, K, and let ZΘ, OE, EK, KZ be joined, and through let MB be drawn through B parallel to ZK, and let BΛ be drawn through B parallel to KE, and let MA, AΛ be joined: and they themselves will be parallel to ZΘ, ΘE on account of AΔ being equal to ΔB, and ΔΘ being equal to ΔK, and ZΔE being at right angles to ΘK. And since the angle at K is right, and MB, BΛ are parallel to ZK, KE, therefore the angle at B is right: and on account of the same things, the one at A is also [right], so that the circle being drawn about MΛ will pass through A, B. Let it be drawn as MAΛB. And since MB is parallel to ZK, it is, that as ZΔ is to ΔM, so too is KΔ to ΔB. Similarly also, as KΔ is to ΔB, so too is EΔ to ΔΛ. And *alternando,* as EΔ is to ΔZ, that is AB to BΓ, so too is ΛΔ to ΔM.

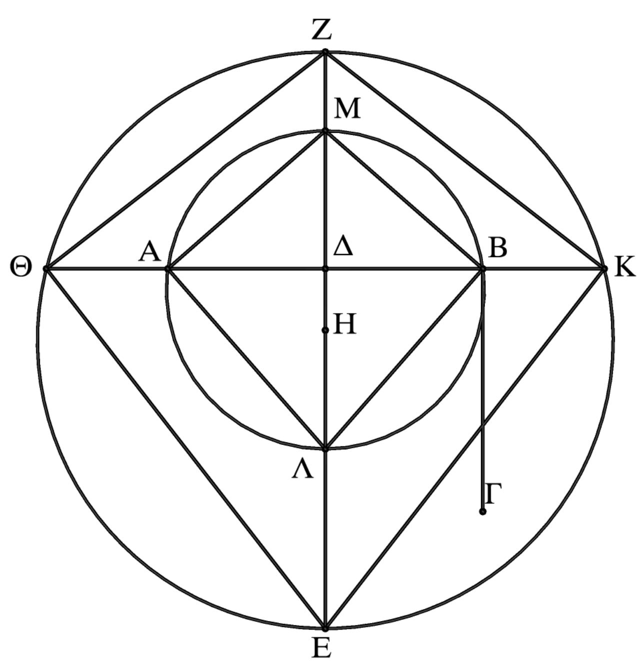

Figure 3.26: Eutocius' Figure for Theorem 54

(278.3) But similarly, even if the circle being drawn about ZE cuts AB, the same thing will be shown.

3.41 Theorem 55

(278.6) [And let the semicircle AΔZ be drawn on AΔ, and let some line ZH be drawn to the semicircle parallel to AΘ, making the ratio of the square on ZH to the rectangle ΔHA the same as that of ΓA to the double of AΔ[45]. Let there be a semicircle ABΓ on the diameter AΓ, and let the given ratio be that of EZ to ZH, and let it be required to do the enjoined.

(278.13) Let ZΘ be supposed equal to EZ, and let ΘH be bisected at K, and let some chance straight line ΓB be drawn in the circle in the angle AΓB, and let ΛΣ be drawn through from the center Λ and at right angles to it (ΓB), and being projected, let it (ΛΣ) intersect the circumference at N, and through N let NM be drawn parallel to ΓB: therefore it will be tangent to the circle. And let it be contrived, that as ZΘ is to ΘK, so too is MΞ to ΞN, and let it happen that NO is equal to ΞN, and let ΛΞ, ΛO be joined, cutting the semicircle at Π, P; and let ΠPA be joined.

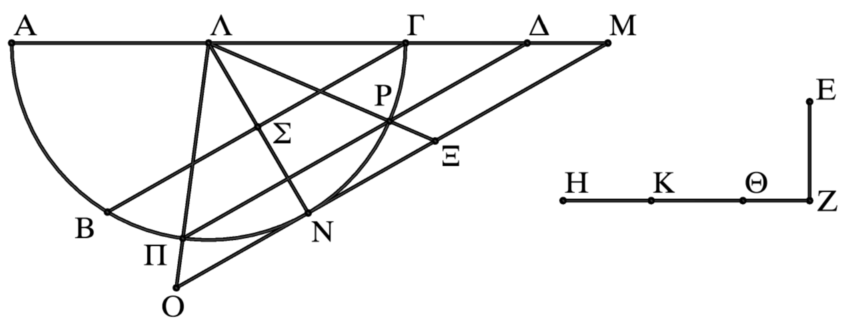

Figure 3.27: Eutocius' Figure for Theorem 55

(278.23) Then since ΞN is equal to NO, and NΛ is common and orthogonal, therefore also ΛO is equal to ΛΞ. But also ΛΠ is equal to ΛP: and therefore the remainder ΠO is equal to the remainder ΠΞ. Therefore ΠPA is parallel to MO. And it is, that as

[45]i.e. to AB, since in Apollonius' proposition, Δ is the midpoint of AB.

ZΘ is to ΘK, so too is MΞ to NΞ: but as ΘK is to ΘH, so too is NΞ to ΞO: therefore by equality, as ΘZ is to ΘH, so too is MΞ to ΞΘ: *invertendo*, as HΘ is to ΘZ, so too is OΞ to ΞM; *componendo*, as HZ is to ZΘ, so too is OΞ to ΞM, that is, ΠΔ to ΔP. But as ΠΔ is to ΔP, so too is the rectangle ΠΔP to the square on ΔP; but the rectangle ΠΔP is equal to the rectangle AΔΓ: therefore as HZ is to ZE, so too is the rectangle AΔΓ to the square on ΔP. Therefore *invertendo,* as EZ is to ZH, so too is the square on ΔP to the rectangle AΔΓ.

3.42 Theorem 58

(280.13) And on AE let the semicircle AEZ have been drawn, and in it, let ZH be drawn parallel to AΔ, making

$$\text{sq.(ZH)} : \text{rect.(AHE)} = \Gamma\text{A} : \text{double(AE)}.$$

For let there be a semicircle ABΓ, and in it, let there be some straight line AB, and let two unequal straight lines ΔE and EZ be given, and let EZ be produced to H, and let it happen that

$$\text{ZH} = \Delta\text{E},$$

and let the whole EH be bisected at Θ, and let the center of the circle be taken as K, and from it, at right angles to AB, let a straight line be drawn, and let it intersect the circumference at Λ, and through Λ parallel to AB let ΛM be drawn, and let KA being produced, intersect ΛM at M, and let it have been contrived, that

$$\Theta\text{Z} : \text{ZH} = \Lambda\text{M} : \text{MN}.$$

And let

$$\Lambda\Xi = \text{AN};$$

and let NK, KΞ be joined and produced; and let the circle, being completed, cut them at Π and O; and let OPΠ be joined.

(282.7) Then since

$$\text{Z}\Theta : \text{ZH} = \Lambda\text{M} : \text{MN},$$

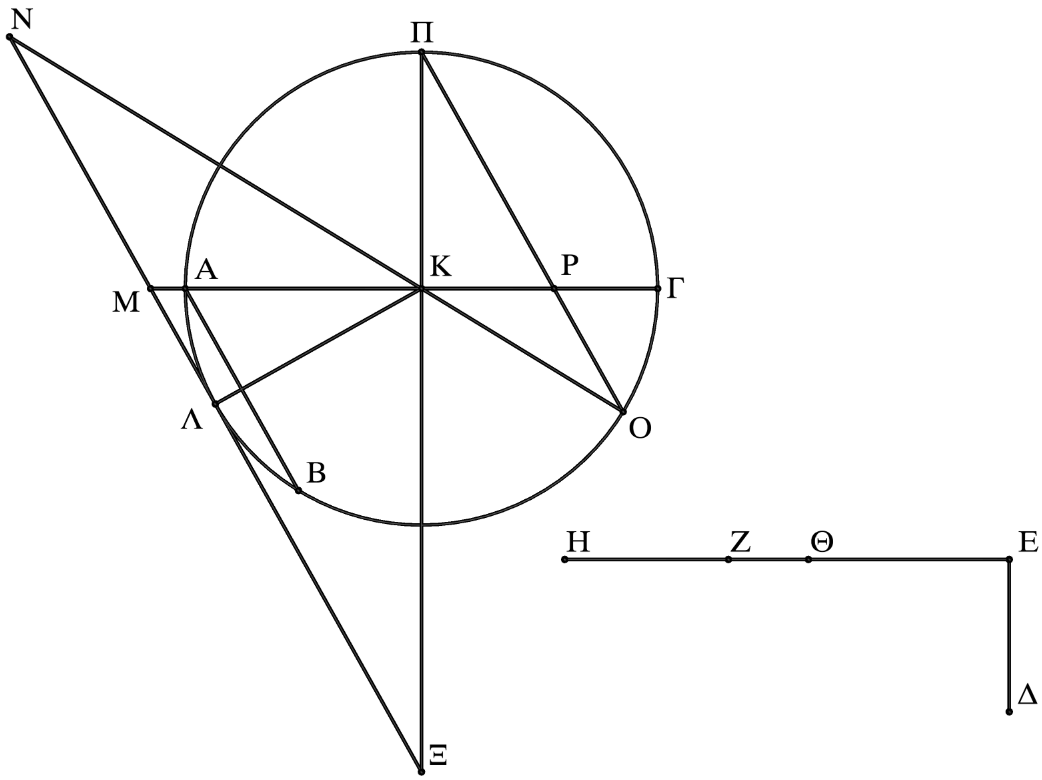

Figure 3.28: Eutocius' Figure for Theorem 58

componendo,

$$H\Theta : HZ = \Lambda N : MN;$$

invertendo,

$$ZH : H\Theta = NM : N\Lambda.$$

But

$$ZH : HE = MN : N\Xi,$$

so *separando*,

$$ZH : ZE = NM : M\Xi.$$

And since

$$N\Lambda = N\Xi,$$

and ΛK is orthogonal and common, therefore

$$KN = K\Xi.$$

But

$$\mathrm{K O} = \mathrm{K\Pi} \text{ also,}$$

therefore NΞ and OΠ are parallel. Therefore the triangle KMN is similar to the triangle OKP, and the triangle KMΞ is similar to the triangle ΠPK. Therefore,

$$\mathrm{KM} : \mathrm{KP} = \mathrm{MN} : \mathrm{PO}.$$

But also,

$$\mathrm{KM} : \mathrm{KP} = \mathrm{M\Xi} : \mathrm{\Pi P},$$

therefore

$$\mathrm{NM} : \mathrm{M\Xi} = \mathrm{HZ} : \mathrm{ZE} = \Delta\mathrm{E} : \mathrm{EZ};$$

but

$$\mathrm{OP} : \mathrm{P\Pi} = \text{sq.}(\mathrm{OP}) : \text{rect.}(\mathrm{OP\Pi});$$

and therefore

$$\Delta\mathrm{E} : \mathrm{EZ} = \text{sq.}(\mathrm{OP}) : \text{rect.}(\mathrm{OP\Pi}).$$

But

$$\text{rect.}(\mathrm{OP\Pi}) = \text{rect.}(\mathrm{AP\Gamma}).$$

Therefore

$$\Delta\mathrm{E} : \mathrm{EZ} = \text{sq.}(\mathrm{OP}) : \text{rect.}(\mathrm{AP\Gamma}).$$

3.43 Eutocius' Epilogue to the First Book

(284.1) In the scholia after the tenth theorem, the aim of the first thirteen theorems is stated, and in the [scholia] to the sixteenth the [aim of the] following three; but it is necessary to understand, that in the sixteenth, he says that the line drawn through the vertex to an ordinate intersects [the ordinate] outside [the section]. But in the eighteenth, he says that the parallel to any tangent line, being drawn inside the section, will intersect the section.

In the nineteenth, he says that the line being drawn ordinatewise from some point on the diameter intersects the section.

In the twentieth and twenty-first, he seeks the ordinates of the sections, how they are to one another and how the sections of the diameter being defined by them are in relation to one another.

In the twenty-second and twenty-third he speaks concerning the straight line intersecting the section at two points.

In the twenty-fourth and twenty-fifth, he speaks concerning the straight line intersecting the section at one point, that is a tangent.

In the twenty-sixth he speaks concerning the line being drawn parallel to the diameter of the parabola and the hyperbola.

In the twenty-seventh he speaks concerning the line cutting the diameter of the parabola, that it intersects section in both directions.

In the twenty-eighth he speaks concerning the line being drawn parallel to a tangent of one of the opposite sections.

In the twenty-ninth, he speaks concerning the line being projected through the center of the opposite sections.

In the thirtieth, he says that the line being projected through the center of the ellipse and the opposite sections is bisected[46].

In the thirty-first he says that in the case of the hyperbola, the tangent line cuts the diameter between the vertex and the center.

In the thirty-second, thirty-third, thirty-fourth, thirty-fifth, and thirty-sixth, he speaks concerning the tangents, he gives an account of the ratio.

In the thirty-seventh, he speaks concerning the tangents and the ordinates from the point of contact of the ellipse and the hyperbola.

In the thirty-eighth, he speaks he speaks concerning the tangents of the hyperbola and the ellipse, how they are in relation to the second diameter.

In the thirty-ninth and fortieth, concerning the same, he gives an account of the

[46]i.e. the the line connecting two points on the sections, through the center, is bisected at the center.

ratio, seeking the ratios compounded from these[47].

In the forty-first, he speaks concerning the parallelograms being drawn from the ordinate and the line from the center of the hyperbola and ellipse.

In the forty-second, concerning the parabola, he says that the triangle being formed from the tangent and the ordinate is equal to the parallelogram having the same height as it, but having half the base.

In the forty-third, in the cases of the hyperbola and the ellipse, he seeks how the triangles formed by tangents and ordinates are to one another; in the forty-fourth, the same for the opposite sections; in the forty-fifth, the same in the case of the second diameter of the hyperbola and the ellipse.

In the forty-sixth, he speaks concerning the other [diameters] after the principle diameter of the parabola; in the forty-seventh concerning the other diameters of the hyperbola and the ellipse; in the forty-eighth concerning the other diameters of the opposite sections;

In the forty-ninth he speaks concerning the straight lines to which the straight lines being drawn ordinatewise to the different diameters of the parabola are equal in square; in the fiftieth concerning the same thing of the hyperbola and the ellipse; in the fifty-first concerning the same thing of the opposite sections.

Having said these things, and having appended an epilogue to the things being said, he shows a problem in the fifty-second and fifty-third, how it is possible to draw the parabola in a plane.

In the fifty-fourth and fifty-fifth he says how it is necessary to draw the hyperbola; in the fifty-sixth, fifty-seventh, and fifty-eighth, how it is necessary to draw the ellipse; in the fifty-ninth he says how it is possible to draw opposite sections.

In the sixtieth he speaks concerning the conjugate sections.

---

[47] i.e. the diameters of the ellipse and the hyperbola.

CHAPTER 4
THE DUPLICATION PROBLEM

As in the last section, let us recall that the circle may be thought of as the locus which finds one mean proportional between two given finite straight lines. This again was well known even before Euclid, appearing in *Elements* VI.13. However, a much more difficult question is the matter of finding *two* means proportional. The desire to find this second proportional was not merely an attempt at mathematical abstraction; it came from the desire to solve this problem came from another more practical one: the duplication of the cube.

4.1 An Historical Overview of the Problem

The problem comes to us via two quotes, both of Eratosthenes. One of the quotes is given by Theon of Smyrna, the other by Eutocius in his commentary on Proposition II.1 of Archimedes' *On the Sphere and the Cylinder* ([15, 16]). Theon quotes:

> Eratosthenes in his work entitled *Platonicus* relates that, when the god proclaimed to the Delians by the oracle that, if they would get rid of a plague, they should construct an altar double of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made double of a (similar) solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar double the size, but that he wished, in setting them to the task, to shame the Greeks for their neglect of mathematics and their contempt for Geometry.

Eutocius, however, quotes him thus in a letter to king Ptolemy III Euergetes. Heath believed this letter to be a forgery; Netz, on the other hand, believes it is genuine.

> Eratosthenes to king Ptolemy, greetings.
>
> They say that one of the old tragic authors introduced Minos, building a tomb to Glaucos, and, hearing that it is to be a hundred cubits long in each directon, saying:
>
>> "You have mentioned a small precinct of a tomb royal;
>> Let it be double, and, not losing its beauty,
>> Quickly double each side of the tomb."

> He seems, however, to have been mistaken; for, the sides doubled, the plane becomes four times[1], while the solid becomes eight times. And this was investigated by the geometers, too: in which way one could double the given solid, the solid keeping the same shape; and they called this problem "duplication of a cube:" for, assuming a cube, they investigated how to double it. And, after they were all puzzled by this for a long time, Hippocrates of Chios was the first to realize that, if it is found how to take two mean proportionals, in continuous proportion, between two straight lines (of whom the greater is double the smaller), then the cube shall be doubled, so that he converted the puzzle into another, no smaller puzzle. After a while, they say, some Delians, undertaking to fulfill an oracle demanding that they double one of their altars, encountered the same difficulty, and that they sent messengers to the geometers who were with Plato in the Academy, asking of them to find that which was asked. Of those who dedicated themselves to this diligently, and investigated how to take two mean proportionals between two given lines, it is said that Archytas of Tarentum solved this with the aid of semicylinders, while Eudoxus[2] did so with so-called curved lines; as it happens, all of them wrote demonstratively, and it was impossible practically to do this by hand (except Menaechmus, by the *shortness*[3], and this with difficulty). But we have conceived of a certain easy mechanical way of taking two proportionals through which, given two lines, means - not only two, but as many as one may set forth - shall be found.

Interestingly enough, there is a passage in Plato (*Republic* VII) that matches these stories somewhat. While Plato does not specifically mention the cube duplication problem, he does refer to solid geometry as an area of mathematics not fully explored:

> (528.A7) "After plane surfaces," said I, "we went on to solids in revolution before studying them in themselves. The right way is next in order after the second dimension to take the third. This, I suppose, is the dimension of cubes and everything that has depth." "Why, yes, it is," he said; "but this subject, Socrates, does not appear to have been investigated yet." "There are two causes of that," said I: "first, inasmuch as no city holds them in honour, these inquiries are languidly pursued owing to their difficulty. And secondly, the investigators need a director, who is indispensible for success and who, to begin with, is not easy to find, and then, if he could be found, as things are now, seekers in this field would be too arrogant to submit to his guidance. But if the state as a whole should join in superintending these studies and honour them, these specialists would accept advice, and continuous and strenuous investigation would bring out the truth. Since even now, lightly esteemed as

[1] *Cf.* Plato, *Meno* [17].

[2] Lamentably, Eudoxus' solution did not pass the scrutiny of Eutocius, who derides it, and thus it does not appear in his collection of solutions. I suspect, however, that the solution Eutocius had was a corruption or forgery; though I admit that currently extant texts give no evidence either way.

[3] A question for future study: what is the "shortness"?

> they are by the multitude and hampered by the ignorance of their students as to the true reasons for pursuing them, they nevertheless in the face of all these obstacles force their way by their inherent charm and it would not surprise us if the truth about them were made apparent." "It is true," he said, "that they do possess an extraordinary attractiveness and charm. But explain more clearly what you were just speaking of. The investigation of plane surfaces, I presume, you took to be geometry?" "Yes," said I. "And then," he said, "at first you took astronomy next and then you drew back." "Yes," I said, "for in my haste to be done I was making less speed. For while the next thing in order is the study of the third dimension or solids, I passed it over because our absurd neglect to investigate it, and mentioned next after geometry astronomy, which deals with the movements of solids."

We do know that Archytas' solution was the first, and that it involves a complicated construction involving the intersection of a torus (really just the top half of one), cylinder, and cone; though oddly, the letter here refers to multiple semicylinders, whereas the construction itself uses only one. Later, Eudoxus' and Archytas' student, Menaechmus, solved the problem in two ways, by considering the intersections of conics; other authors solve the problem by means of mechanical arguments that often depend on rulers or the like. We will consider three solutions: that of Archytas, Eratosthenes, and the first of Menaechmus. Before this, however, we should examine the relationship of duplicating a cube and finding two means proportional.

4.2 Reduction of the Problem due to Hippocrates of Chios

As Eratosthenes states in the quote above, Hippocrates of Chios reduced the problem of finding a duplicate cube into one finding two means proportional. We can see shadows of it in the propositions of *Elements* XI, echoing, in a way, similar propositions in *Elements* VI. For example, Euclid demonstrates in VI.1 that "triangles and parallelograms which are under the same height are to one another as their bases," and in VI.19-20, he shows that "similar triangles are to one another in the duplicate ratio of the corresponding sides." Of course our problem is about cubes, but this is dealt with firmly by XI.32 and 33, which are, in a sense, parallels of VI.1 and VI.19-20, respectively. XI.32 states that "parallelepipedal solids which are of the same height are to one another as their bases," while XI.33 states that "Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides."

In our case, the two parallelepipedal solids are cubes, which are clearly similar to one another. It would be worth noting Euclid's porism to XI.33, which states that "from this it is manifest that, if four straight lines be [continuously] proportional, as the first is to the fourth, so will a parallelipipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second, inasmuch as the first has to the fourth the triplicate ratio of that which it has to the second." The last part is really just V. def. 10. In symbols, if we have that $A : B = B : \Gamma = \Gamma : \Delta$, then the cube on A is to the cube on B as A is to Δ. In the case of our duplication, we should take Δ to be double A, so that the cube on B is double the cube on A. Thus duplicating the cube requires finding the magnitudes B and Γ. (It should be noted that finding one immediately gives the other, since the problem then is to find a single mean proportional). For the similar case within Euclid's books on number, *cf. Elements* VIII.19.

4.3 Finding Two Means Proportional

Elements XI.33 shows us that the matter of duplicating a cube is equivalent to that of finding two means proportional. (Pseudo?) Eratosthenes puts it quite right: Hippocrates exchanged one puzzle for another no less difficult. Eutocius, fortunately, collected together many ancient solutions to the problem, beginning with Archytas and ending with Pappus, in his commentary to II.1 of Archimedes' *On the Sphere and the Cylinder*. The variety of different solutions itself is striking: we have everything from a "standard" construction involving conics to excerpts from a work by Hero on the construction of missile-throwing machines to solutions so beautiful to their discoverer (Eratosthenes) that they write in epigrams and attach a model of the mechanical apparatus to a pillar dedicated to King Ptolemy. I do find it quite interesting that two solutions (Hero's and Eratosthenes') make explicit reference to the military sciences. And if we are to believe Eratosthenes, it should not be surprising that an air of mysticism and divinity should pervade the subject. For Archytas and the lot, having by their various means solved the problem, have fulfilled the demands of the Delians by their oracle. Indeed, even, they have done the work of God himself! For if we read the *Timaeus* ([18]),

> ...God placed water and air in the mean between fire and earth, and made them to have the same proportion so far as was possible (as fire is to air so is air to water, and as air is to water so is water to earth); and thus he bound and put together a visible and tangible heaven. And for these reasons, and out of such elements which are in number four, the body of the world was created, and it was harmonised by proportion, and therefore has the spirit of friendship; and having been reconciled to itself, it was indissoluble by the hand of any other than the framer.

4.4 Menaechmus' Solutions

The solutions are in an analysis-synthesis form. First we suppose that the solution has been found, and then investigate necessary conditions. Then we reverse the process to construct the solution in a rigorous way. Modern mathematicians do the same thing, if not explicitly: consider, for example, how we "find" the necessary δ in a proof involving limits.

Menaechmus' First Solution

Analysis. Let the two given magnitudes be A, E. Suppose that the two means proportional have been found, say, B, Γ. Thus

$$\mathrm{A} : \mathrm{B} = \mathrm{B} : \Gamma = \Gamma : \mathrm{E}$$

In particular, A : B = B : Γ, so that

$$\text{rect.}(\mathrm{A}, \Gamma) = \text{sq.}(\mathrm{B}).$$

Let the line ΔH be set out, and at Δ , let ΔZ be set equal to Γ, and let ZΘ be constructed at right angles to ΔZ and equal to B. Hence it follows that

$$\text{rect.}(\mathrm{A}, \Gamma) = \text{sq.}(\mathrm{Z}\Theta).$$

Therefore the point Θ is on a parabola through Δ (i.e. with vertex Δ) with a parameter (or *latus rectum*) A [*Conics* I.11].

Let now ΔK, ΘK be drawn as parallels (to ΘZ, ΔZ, respectively). Again, by hypothesis,

$$\mathrm{A} : \mathrm{B} = \mathrm{B} : \Gamma = \Gamma : \mathrm{E},$$

so that in particular,

$$\mathrm{A} : \mathrm{B} = \Gamma : \mathrm{E},$$

whence

$$\text{rect.}(\mathrm{A}, \mathrm{E}) = \text{rect.}(\mathrm{B}, \Gamma).$$

But by hypothesis, $\mathrm{B} = \mathrm{Z}\Theta$, $\Gamma = \Delta\mathrm{Z}$, and $\mathrm{Z}\Theta$ is perpendicular to $\Delta\mathrm{Z}$, it follows that

$$\text{rect.}(\mathrm{A}, \mathrm{E}) = \text{rect.}(\mathrm{B}, \Gamma) = \text{rect.}(\Delta\mathrm{Z}, \mathrm{Z}\Theta).$$

Therefore Θ is on a hyperbola with asymptotes $\mathrm{K}\Delta$, $\Delta\mathrm{Z}$ [*Conics* II.12]. Since Θ is now given, being the intersection of a parabola and an hyperbola, Z is now also given. Hence both B, Γ are now given.

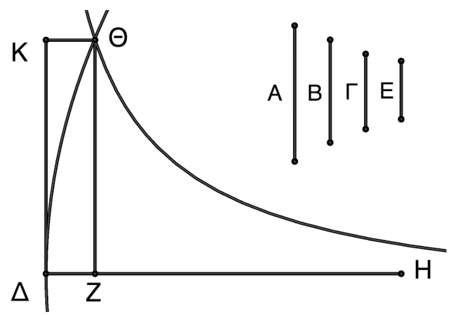

Figure 4.1: Menaechmus' First Solution

Synthesis. Let again the two given lines be A, E. Let $\Delta\mathrm{H}$ be drawn, limited at Δ (so that H can be as far as needed from Δ). With Δ as vertex, and $\Delta\mathrm{H}$ as axis, let a parabola be constructed whose parameter (latus rectum) is A (the construction of a parabola given the parameter is provided in *Conics* I.52). Let this parabola be $\Delta\Theta$, and let $\Delta\mathrm{K}$ be drawn at right angles to $\Delta\mathrm{Z}$. Now let an hyperbola be drawn with $\mathrm{K}\Delta$, $\Delta\mathrm{Z}$ as asymptotes, and on this hyperbola, the lines drawn parallel to $\mathrm{K}\Delta, \Delta\mathrm{Z}$ make the rectangular area contained by them equal to the rectangle contained by the given lines A, E (*Conics* II.12). Therefore also the hyperbola will cut the parabola; let this happen at Θ. Now let $\mathrm{K}\Theta, \Theta\mathrm{Z}$ be drawn as perpendiculars. Since Θ is on the parabola, we know that

$$\text{sq.}(\mathrm{Z\Theta}) = \text{rect.}(\mathrm{A}, \mathrm{\Delta Z}),$$

and thus

$$\mathrm{A} : \mathrm{Z\Theta} = \mathrm{\Theta Z} : \mathrm{Z\Delta}.$$

Also, since

$$\text{rect.}(\mathrm{A}, \mathrm{E}) = \text{rect.}(\mathrm{\Theta Z}, \mathrm{Z\Delta}),$$

we have

$$\mathrm{A} : \mathrm{Z\Theta} = \mathrm{Z\Delta} : \mathrm{E}.$$

But then

$$\mathrm{A} : \mathrm{Z\Theta} = \mathrm{\Theta Z} : \mathrm{Z\Delta} = \mathrm{Z\Delta} : \mathrm{E}.$$

Let now $\mathrm{B} = \mathrm{\Theta Z}$, and $\Gamma = \mathrm{\Delta Z}$, so that

$$\mathrm{A} : \mathrm{B} = \mathrm{B} : \Gamma = \Gamma : \mathrm{E},$$

which it was required to find.

Menaechmus' Second Solution

Analysis. Let the two given lines be AB, $\mathrm{B\Gamma}$, at right angles to one another, and let the means be $\mathrm{\Delta B}$, BE, so that

$$\mathrm{\Gamma B} : \mathrm{B\Delta} = \mathrm{B\Delta} : \mathrm{BE} = \mathrm{BE} : \mathrm{BA}.$$

Let also $\mathrm{\Delta Z}$, EZ be drawn at right angles [to each other, and also to the given lines AB, $\mathrm{B\Gamma}$). Since $\mathrm{\Gamma B} : \mathrm{B\Delta} = \mathrm{B\Delta} : \mathrm{BE}$,

$$\begin{aligned} \text{rect.}(\mathrm{\Gamma B}, \mathrm{BE}) &= \text{sq.}(\mathrm{B\Delta}) \\ &= \text{sq.}(\mathrm{EZ}). \end{aligned}$$

Therefore Z lies on a parabola with axis BE, parameter $\mathrm{B\Gamma}$, and vertex B.

Similarly, since $\mathrm{B\Delta} : \mathrm{BE} = \mathrm{BE} : \mathrm{BA}$,

$$\begin{aligned} \text{rect.}(\mathrm{AB}, \mathrm{B\Delta}) &= \text{sq.}(\mathrm{BE}) \\ &= \text{sq.}(\mathrm{\Delta Z}). \end{aligned}$$

Therefore Z also lies on a parabola with axis $\mathrm{B\Delta}$, parameter AB, and vertex B. But since it also lies on the other parabola, with the same vertex, axis BE, and parameter BΓ, the point Z is given. Therefore the means, EB and BΔ, being perpendiculars, are also given.

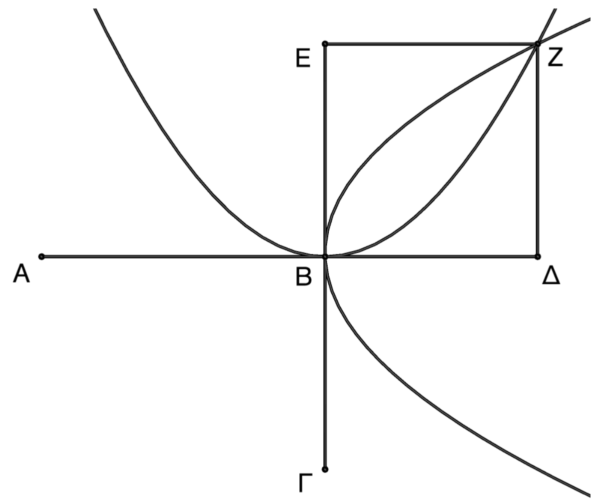

Figure 4.2: Menaechmus' Second Solution

Synthesis. Let again the given lines be AB, BΓ, set out at right angles to each other. With axis BΓ produced, vertex B, and parameter AB, set out a parabola. Similarly, with axis AB produced, vertex B, and parameter BΓ, set out a second parabola. By construction, they will meet, for their axes are perpendicular and they share a common vertex. Let them meet at Z. Let EZ be drawn parallel to AB produced, and also ZΔ parallel to ΓB produced. Now since Z is on each parabola, we have

$$\text{rect.}(\Gamma B, BE) = \text{sq.}(EZ)$$

and

$$\text{rect.}(AB, B\Delta) = \text{sq.}(\Delta Z).$$

Therefore

$$AB : \Delta Z = \Delta Z : B\Delta$$

and

$$\Gamma B : EZ = EZ : BE.$$

But EZ is equal to BΔ, so *invertendo,*

$$\mathrm{BE} : \mathrm{B\Delta} = \mathrm{B\Delta} : \Gamma\mathrm{B},$$

whence

$$\mathrm{AB} : \mathrm{EB} = \mathrm{EB} : \mathrm{B\Delta} = \mathrm{B\Delta} : \Gamma\mathrm{B},$$

which it was required to find. [Note the uses of *Conics* I.11 and I.52.]

4.5 Eratosthenes' Solution

Eratosthenes' solution involves the motion of three rectangles in the plane. But it is best to imagine the solution having a third dimensional component, albeit small. Netz suggests we imagine each rectangle as a door[4]. The object is to find the two means between AE and ΘΔ, and the solution will result by moving the doors into a certain position.

We begin with three rectangular doors, AZ, ΛH, and IΘ, with the diagonals also connected and parallel. The middle door will remain fixed, but the other two may move over one another, with door AZ on top, door ΛH in the middle, and door IΘ at the bottom. They still should be regarded as on the same plane, however, as if the doors themselves are made of exceedingly thin paper. Define the point B to be where the right hand edge of door one meets the diagonal of door two; so that B starts as Λ. Similarly, the point Γ is the intersection of the right hand edge of door two with the diagonal of door three; again, Γ starts as I. By moving the right hand door left, and the left hand door right, the segments AB, BΓ, and ΓΔ will eventually all be colinear, forming the straight line ABΓΔK, with point K the intersection of that line with the line EΘ.

Once the doors and lines are in this configuration, we are left with a series of similar triangles. The desired means then are ZB and HΓ. So by similar triangles,

$$\mathrm{AK} : \mathrm{KB} = \mathrm{EK} : \mathrm{KZ} = \mathrm{ZK} : \mathrm{KH}.$$

Similarly,

$$\mathrm{BK} : \mathrm{K}\Gamma = \mathrm{ZK} : \mathrm{KH} = \mathrm{HK} : \mathrm{K}\Theta.$$

[4]I have produced an interactive diagram for the case of three doors. The Processing source code is included with this thesis.

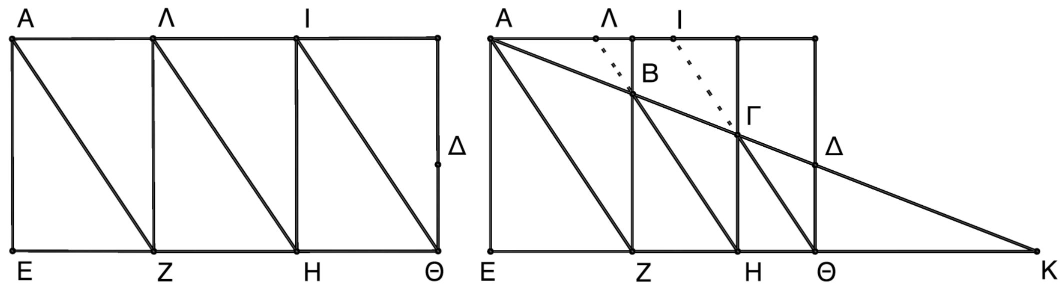

Figure 4.3: Eratosthenes' "Moving Doors" Solution

But

$$\mathrm{ZK} : \mathrm{KH} = \mathrm{EK} : \mathrm{KZ},$$

therefore also

$$\mathrm{EK} : \mathrm{KZ} = \mathrm{ZK} : \mathrm{KH} = \mathrm{HK} : \mathrm{K\Theta}.$$

But

$$\mathrm{EK} : \mathrm{KZ} = \mathrm{AE} : \mathrm{BZ},$$

and

$$\mathrm{ZK} : \mathrm{KH} = \mathrm{BZ} : \Gamma\mathrm{H}.$$

And

$$\mathrm{HK} : \mathrm{K\Theta} = \Gamma\mathrm{H} : \Delta\Theta,$$

therefore

$$\mathrm{AE} : \mathrm{ZB} = \mathrm{ZB} : \Gamma\mathrm{H} = \Gamma\mathrm{H} : \Delta\Theta,$$

which was enjoined.

The usefulness of Eratosthenes solution goes further, though. For if it is enjoined to find any number of means proportional, say n, then it suffices to carry out a similar procedure with $n+1$ doors instead. The doors all are of course the same, with parallel diagonals, so that the argument reduces to a multitude of similar triangles, all sharing the vertex K.

The letter goes on to give a briefer description of the proof, which appeared on the pillar he dedicated to King Ptolemy III. He also mentions that this task will find multiple means, again, by using one more "door" than means required. And lastly, he gives an epigram, which was also on the pillar. Netz' analysis of it indicates a fusion of literary style; mathematical and epic, in a sense. I reproduce the epigram below for the benefit of the reader:

Εἰ κύβον ἐξ ὀλίγου διπλήσιον, ὦγαθέ, τεύχειν
φράζεαι ἢ στερεὴν πᾶσαν ἐς ἄλλο φύσιν
εὖ μεταμορφῶσαι, τόδε τοι πάρα, κἂν σύ γε μάνδρην
ἢ σιρὸν ἢ κοίλου φρείατος εὐρὺ κύτος
τῇδ' ἀναμετρήσαιο, μέσας ὅτε τέρμασιν ἄκροις
συνδρομάδας δισσῶν ἐντὸς ἕλῃς κανόνων.
μηδὲ σύ γ' Ἀρχύτεω δυσμήχανα ἔργα κυλίνδρων
μηδὲ Μεναιχμείους κωνοτομεῖν τριάδας
διζήσῃ, μηδ' εἴ τι θεουδέος Εὐδόξοιο
καμπύλον ἐγ γραμμαῖς εἶδος ἀναγράφεται.
τοῖσδε γὰρ ἐν πινάκεσσι μεσόγραφα μυρία τεύχοις
ῥεῖά κεν ἐκ παύρου πυθμένος ἀρχόμενος.
εὐαίων, Πτολεμαῖε, πατὴρ ὅτι παιδὶ συνηβῶν
πάνθ', ὅσα καὶ Μούσαις καὶ βασιλεῦσι φίλα,
αὐτὸς ἐδωρήσω· τὸ δ' ἐς ὕστερον, οὐράνιε Ζεῦ,
καὶ σκήπτρων ἐκ σῆς ἀντιάσειε χερός.
καὶ τὰ μὲν ὣς τελέοιτο, λέγοι δέ τις ἄνθεμα λεύσσων
τοῦ Κυρηναίου τοῦτ' Ἐρατοσθένεος.

If you plan, of a small cube, its double to fashion,
Or - dear friend - any solid to change to another
In Nature: it's yours. You can measure, as well:
Be it byre, or corn-pit, or the space of a deep,
Hollow well. As they run to converge, in between
The two rulers - seize the means by their boundary-ends.
Do not seek the impractical works of Archytas'
Cylinders; nor the three conic-cutting Menaechmics;
And not even that shape which is curved in the lines
That Divine Eudoxus constructed.
By these tablets, indeed, you may easily fashion -
With a small base to start with - even thousands of means.
O Ptolemy, happy! Father, as useful as son:
You have given him all that is dear to the muses
And to kings. In the future - O Zeus! - may *you* give him,
From your hand, this, as well: a sceptre.
May it all come to pass. And may him, who looks, say:
"Eratosthenes, of Cyrene, set up this dedication."

The epigram does raise in my mind several questions. Why does Eratosthenes again mention the *cylinders* of Archytas, when in fact only one cylinder is explicitly mentioned in his solution? We might imagine a second (semi)cylinder, which has the same radius and axis as the cone of his construction, but Archytas makes no mention of it whatsoever (as we shall see). He also refers to three conic-cutting Menaechmus; perhaps he is not explicitly referring to Menaechmus' two constructions, neither of which make use of the ellipse. And what does he mean by the lines that Divine Eudoxus constructed? Indeed, there is some dissonance here with what Eutocius himself says, for Eutocius explicitly excludes Eudoxus' solution on the grounds that it does not actually use curved lines, and falsely that a discrete proportion is a continuous one[5]. Netz, however, refers to the Eudoxus to whom Eutocius refers as pseudo-Eudoxus, apparently, then, believing that the proof Eutocius examined was a forgery. Unfortunately, of course, we do not have this proof of (pseudo?)-Eudoxus, and Netz says nothing further on the matter.

4.6 Archytas' Solution

Archytas' solution is quite a bit more bold, perhaps not surprisingly, given that it is the first known solution. It involves the following parts: a triangle, which rotates and thus sweeps out a cone; a semicircle, which rotates around one of its vertices, sweeping out a torus with no middle hole; and a cylinder. The rotating figures intersect the cylinder, forming two curves, which themselves intersect at a special point of consideration.[6]

Let the given straight lines be $\mathrm{A\Delta}$ and Γ, and let a circle be described with diameter $\mathrm{A\Delta}$ in the plane of reference. Let a point B be taken on the circle, and let AB be joined, so that AB is equal to Γ. Let AB be produced to Π, and the line $\Pi\Delta\mathrm{O}$ be made tangent to the circle at Δ. Through B, let the straight line BEZ be drawn perpendicular to the diameter AB, with Z on the circumference of the circle. With diameter $\mathrm{A\Delta}$, let a semicircle be constructed in the plane through $\mathrm{A\Delta}$ which is perpendicular to BEZ. Lastly, let a right cylinder be constructed on the circle $\mathrm{A\Delta}$.

[5]Netz suggests that Eutocius means that Eudoxus gave a proportion of the form a : b = c : d instead of one like a : b = b : c = c : d.

[6]I have produced an animated 3D diagram of his construction. The Processing source code is attached to this thesis.

Now the motion begins. Let the triangle ΠΔA be rotated about the side AΔ. Its intersection with the cylinder as it moves traces out a curved line ("Eugene's Line," under Netz's explanation), which starts at B and moves upwards, eventually to a point above Δ, and then falls back in the same manner towards Z. The semicircle rotates about the point A, sweeping out a torus; its intersection with the cylinder traces another curved line ("Tatiana's Line") which starts at Δ and moves upwards, before falling again to rest at A. These two curves will intersect at a point, say K. Let this occur when the triangle is in position AΔΛ, and when the semicircle in the position AΔK. Let the semicircle intersect the base circle at I . Now the motion of the point B sweeps out a semicircle on the cone. Let this semicircle intersect the other at M. Lastly, let KI and MI be joined, and let MΘ be produced through M and fall perpendicularly to Θ, the intersection of AI and BZ.

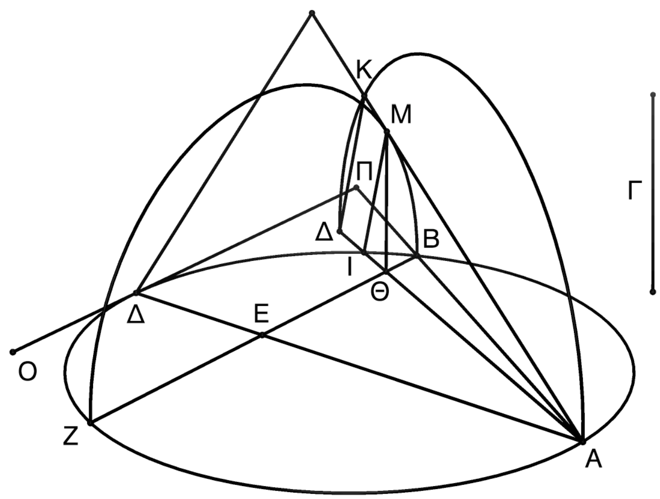

Figure 4.4: Archytas' Solution

The line KI, being on the cylinder, is perpendicular to the base plane. It is therefore parallel to the other perpendicular, MΘ. Now

$$\begin{aligned} \text{rect.}(B\Theta, \Theta Z) &= \text{rect.}(A\Theta, \Theta I) \quad (\textit{Elements}\ \text{III.35}) \\ &= \text{sq.}(M\Theta) \qquad\quad (\textit{Elements}\ \text{III.31, VI.8}). \end{aligned}$$

Imagine a semicircle with diameter AI. Since MΘ was set up perpendicular to the plane, and rect.$(A\Theta, \Theta I) = $ sq.$(M\Theta)$, it is the case that the point M is actually on such a semicircle. Therefore the angle AMI is right. For the same reason, the angle ΔKA is right, so that the lines KΔ and MI are parallel. Therefore each of the triangles AMI, MIΘ, and MAΘ are similar. On account of this,

$$\begin{aligned} \Delta A : AK &= KA : AI \\ &= IA : AM \end{aligned}$$

Therefore

$$\Delta A : AK = AK : AI = AI : AM.$$

But AM is equal to AB, since the cone is right, whence

$$\Delta A : AK = AK : AI = AI : AB.$$

But AB was constructed equal to Γ, so

$$\Delta A : AK = KA : AI = AI : \Gamma;$$

and that which was enjoined has been done.

CHAPTER 5
ON COMPOUND RATIOS

5.1 Introduction

In this chapter, I wish to revisit some of the issues surrounding compound ratios, specifically those discussed by Ken Saito in *Compound ratio in Euclid and Apollonius* ([26], itself building upon [25]). His paper, as the title indicates, examines the role of compound ratios both in the *Elements* and *Data* of Euclid, along with the *Conics* of Apollonius and selections from the *Collection* of Pappus. Throughout it, Saito shows how we should not think that compounding ratios, at least in his source texts, should be thought of in some sort of algebraic way. In my own studies of compound ratios, I began to look towards other authors, specifically Eutocius, who provides commentaries to both Archimedes' *On the Sphere and the Cylinder* and the aforementioned *Conics*. While it is certainly true that Eutocius is quite a late author within the timeline of Greek mathematics, his commentaries of the *Conics* certainly played a role in the transmission of the Greek text, and so it should be studied together the "older" text of the *Conics* itself and other "old" mathematical works. Saito's paper even invites this revisiting; he concludes with a citation of Archimedes' use of compound ratios in the analysis of *SC* II.4, saying:

> I believe more overlooked peculiarities remain in ancient theories concerning geometric magnitudes. I have establised only one example of the inadequacy of the algebraic interpretation. Further reexamination of the ancient technique concerning geometric magnitudes is needed to understand Greek mathematics in the proper sense of the word.

By examining compound ratios as seen in Eutocius, I am to add to the discussion begun by Saito.

In my study of Eutocius' commentary to *Conics* I, I came upon his discussion of compound ratios. This in itself is nothing noteworthy, but the processes of translating this commentary led me to consult the Greek (and Netz' translation) of Eutocius' commentary on *The Sphere and the Cylinder*. At first glance, the very passage on compound ratios in Eutocius' commentary of *Conics* I.11 is, *mutatis mutandis*, that which appears in his commentary of *On the Sphere and the Cylinder* II.4. This I endeavor to analyze and compare the two passages to discover exactly what these differences are.

It should be noted however that a comparative analysis of the two passages already exists: given by Knorr in chapter 7 of *Textual Studies in Ancient and Medieval Geometry* ([7]). However, the purpose of his analysis is not strictly speaking to compare Eutocius' two works together or to concentrate on compound ratios in them. Instead, he seeks to refute the thesis of J. Mogenet, expressed in his *l'Introduction à l'Almageste*, that the unattributed text *Introduction to the Syntaxis* is a work of Eutocius. Part of this unattributed text deals with compound ratios; Knorr observes that the style and mathematical understanding displayed by the author just does not fit with Eutocius' two passages. In doing so, he places Eutocius' passages side-by-side and compares the flow of the passage in the *Introduction* with them. Part of my analysis will concentrate on just the parts of Knorr's analysis concerning Eutocius himself; in doing so, exploring a new direction with Knorr's analysis as a foundation.

I this begin with a look at how compound ratios are used in several Hellenistic authors, specifically Euclid, Archimedes, and Apollonius.

5.2 Compound Ratios as Used in Hellenistic Authors

5.2.1 Euclid

The notion of compounded ratios only appears twice in the *Elements*. In VI.23, Euclid proves that equiangular parallelograms are to one another in the ratio compounded out of the ratios of their corresponding sides; VIII.5 is an analogous proposition for the case of numbers. Disturbingly, nowhere is the idea of compounded ratios explicitly defined[1], though it could be regarded as a generalization of the duplicate and triplicate ratios. Euclid defines these concepts as follows:

V def. 9: When three magnitudes are proportional, the first is said to have to

[1] At least in the non-Theonine editions of the *Elements*. We shall address this soon.

the third the *duplicate ratio* of that which it has to the second.

V def. 10: When four magnitudes are [continuously] proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

In symbols, if we have that $\mathrm{A} : \mathrm{B} = \mathrm{B} : \Gamma = \Gamma : \Delta$, then the ratio $\mathrm{A} : \Gamma$ is said to be the duplicate ratio of $\mathrm{A} : \mathrm{B}$; similarly, the ratio $\mathrm{A} : \Delta$ is the triplicate ratio of $\mathrm{A} : \mathrm{B}$. The first definition is used in *Elements* VI.19, for example, to prove that similar triangles are to one another as the duplicate ratio of their corresponding sides. Euclid extends this in VI.20 to any pair of similar rectilineal figures, by virtue of dividing the rectilineal figures into two set of similar triangles, equal in multitude, and applying VI.19.

Moving on to VI.23, we see the debut use of the idea of compounded ratios. In this proposition, as we have said, Euclid proves that equiangular parallelograms have to one another the ratio compounded of the ratios of their sides. That is, if we have two equiangular parallelograms, $\mathrm{AB\Gamma\Delta}$ and $\mathrm{\Gamma EZH}$, then

$$\text{pllg.}(\mathrm{AB\Gamma\Delta}) : \text{pllg.}(\mathrm{\Gamma EZH}) = (\mathrm{AB} : \mathrm{\Gamma E})\text{comp. } (\mathrm{B\Gamma} : \mathrm{\Gamma H}).$$

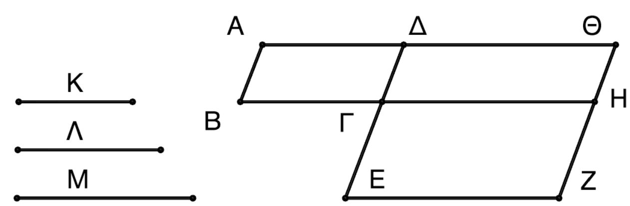

Figure 5.1: *Elements* VI.23

Now if it so happened to be the case that the parallelograms $\mathrm{AB\Gamma\Delta}$ and $\mathrm{\Gamma EZH}$ were also similar, so that $\mathrm{AB} : \mathrm{B\Gamma} = \mathrm{\Gamma E} : \mathrm{EZ}$, then VI.20 applies, so that

$$\text{pllg.}(\mathrm{AB\Gamma\Delta}) : \text{pllg.}(\ \mathrm{\Gamma EZH}) = \text{dup.}(\mathrm{AB} : \mathrm{\Gamma E}).$$

But similar parallelograms are in particular equiangular, so the conclusion of VI.23 still applies. Hence VI.11 yields that

$$\text{dup.}(\mathrm{AB} : \Gamma\mathrm{E}) = (\mathrm{AB} : \Gamma\mathrm{E})\text{comp. }(\mathrm{B}\Gamma : \Gamma\mathrm{H}).$$

But because the parallelograms are supposed similar, *alternando*, AB : ΓE = BΓ : EZ. And EZ = ΓH, so that AB : ΓE = BΓ : ΓH. Hence

$$\text{dup.}(\mathrm{AB} : \Gamma\mathrm{E}) = (\mathrm{AB} : \Gamma\mathrm{E})\text{comp. }(\mathrm{AB} : \Gamma\mathrm{E}).$$

For this reason we should see the ratio compounded of two ratios as a generalization of a duplicate ratio; a similar argument using XI.33 shows that the ratio compounded of three ratios is a generalization of a triplicate ratio. This may be a reason why Euclid never formally defines compound ratio, taking it to be such a generalization of duplicate and triplicate ratios.

Saito, however, gives a reconstructed definition of compounded ratios, saying[2]

> (1) Let A, B, C be magnitudes of the same kind. The ratio A : C is said to be compounded out of the ratios A : B and B : C.
>
> (2) Further, if A : B = D : E and B : C = F : G then A : C is said to be compounded of D : E and F : G.

The first definition is close in spirit to the case of duplicate ratios, though notice that it is not assumed that A : B = C : D, only that they are two arbitrary ratios. If they are proportional, however, then the compounded ratio that Saito gives corresponds precisely to the duplicate ratio. In the second definition, the special case when B : C = E : F yields that A : C = D : F *ex aequali*. Therefore we note that D : F, and hence A : C, are each the ratio compounded out of the ratios D : E and E : F.

Saito observes that Euclid really never uses compounded ratio in the *Elements*, even though VI.23 and XI.33 are about compounded ratio. And in the *Data*, he points out that in situations that could involve compounded ratio, Euclid prefers instead to reduce the matter to linear ratios by means of the application of areas. This method is first apparent

[2] I have preserved Saito's Latin letters in this quote.

in VI.19, where Euclid introduces a mean proportional between in order to move the given problem (about ratios of areas) to a question about linear ratios. This, of course, is necessary under his definition of duplicate ratios, since he needs a proportion of three terms in order to invoke the definition. While VI.19 by itself would not by itself to point to an aversion on the part of Euclid for using compounded ratios, Saito argues that it is the case throughout the *Data* that Euclid uses a method of reduction to linear ratios instead of compounded ratio.

5.2.2 Apollonius and Archimedes

The next question, of course, is what about Apollonius? To what extent did he use compounded ratio or this Euclidean method of reduction to linear areas? For it certainly is the case that compounded ratio appears quite often in *Conics* I, as early as, for example, I.12 (the hyperbola). But Saito argues that compounded ratios here are not truly part of Apollonius' analysis; rather, he contends, Apollonius uses compounded ratio as a way to simplify arguments and expressions, where other more cumbersome expressions (such as the use of ratios *ex aequali*) would be needed. But in terms of the analysis, at least for I.41-43, Saito argues that Apollonius followed Euclid in using a reduction to linear ratios.

He contrasts this Euclidean and Apollonian aversion to compounded ratio with Pappus, who writing much later, makes far more extensive use of them. In fact, Saito says that Pappus uses compounding as an operation of sorts on ratios, and thus his arguments do have a quasi-algebraic feel that is not present in those of Euclid or Apollonius. Interestingly enough, Saito mentions in a footnote that compounded ratios are indispensable in Archimedes' analysis in *On the Sphere and the Cylinder* II.4, but nothing more on the matter is said. Fortunately we now have Netz. He also relegates the point to a footnote, though a quite interesting one:

> The operation of "composition of ratios" was never fully clarified by the Greeks: see Eutocius for an honest attempt.

And later, in his translation of Eutocius' commentary, he adds (in another footnote):

> What Eutocius says is that as far as the mathematical consensus is concerned, Archimedes' argument [SC II.4] is clear and even obvious. However, since the mathematical consensus itself seems to be at fault here, a commentary is required. First we had a spirit of philological enterprise, in the catalogue of two means proportionals [Eutocius' commentary to SC II.1], and now a mathematical independence. Eutocius has grown considerably since the commentary to the first book. The composition of ratios is indeed a sore point in Greek mathematics: let's see how much sense he will make out of it (Eutocius himself clearly was happy with his own discussion, and he has recycled it in his later commentary to Apollonius' *Conics*, II. pp. 218 ff.).

And so the discussion comes full circle. If anyone would have a clear idea as to the matter of compounded ratios, it should be Eutocius, who writes by far the latest. The fact that he understands Archimedes well enough to write a valuable commentary must mean that he in particular understood Archimedes' use of compounded ratios. But Netz says that he wrote his commentary to *On the Sphere and the Cylinder* before his commentary on *Conics*. And as Netz also says, the character of Eutocius' commentary grows from the commentary on SC I to that on SC II. But as Eutocius' commentary on the *Conics* is, beyond a few fragments untranslated by others, and my own translation does not yet extend beyond Eutocius' introduction, we are left at the moment with a profound and open question. *Just what does Eutocius say about compounded ratios? Just what is the definition of compound ratio?*

5.3 The Definition in Theon's Edition of the *Elements*

As I mentioned above, compound ratio is not defined in the non-Theonine editions of the *Elements*. By this, I mean editions which do not descend from the edition of Theon. The manuscript tradition, though, is quite a mess. See Heath and Heiberg for details of the manuscript tradition of the *Elements*, and Cameron ([1]) for details on the practice of editing mathematical texts.

5.3.1 Theon's Definition

In Theon's edition, we do have the following (as VI def. 5: bracketed in Heiberg, absent from the Green Lion edition):

> A ratio is said to be compounded of ratios, whenever the sizes of the ratios, having been multiplied into themselves, produce something.
> Λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ᾽ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα.

In many respects, this definition is disappointing, since it seems to raise more question than it answers. What precisely are the sizes (αἱ πηλικότητες) of the ratios, how are they multiplied into themselves (πολλαπλασιασθεῖσαι), and just what something (τίνα) do they produce? The definition is inherently non-geometric, and feels completely out of place given the context of Book VI. It would make much more sense if it appeared in Book V, given the nature of that book; however, since it is first *used* in Book VI, perhaps Theon felt it to the be appropriate place for the definition.

5.3.2 Theon's Impact on Eutocius

It is to be noted that Theon wrote earlier than Eutocius, and by a sufficient amount that it is most likely that Eutocius thought Theon's edition was the *Elements*. Eutocius deals with compound ratios twice in his works; the first being in his commentary on *On the Sphere and the Cylinder*, the second being that on the *Conics*. In the next section, I will show what Eutocius says on the matter in each case, and the combined weight of his words in both commentaries will show the influence that Theon had on Eutocius.

5.4 Further Definitions: Exposition and Analysis

Each subsection here will present original source material (in translation). I postpone discussion until the next major section.

5.4.1 Eutocius' Exposition in his *Commentary on* On the Sphere and the Cylinder

"Now since the ratio of ΡΛ to ΛΧ is combined of both: the ratio <ratio> which ΡΛ has to ΛΔ, and <of> ΛΔ to ΛΧ." It is obvious that, once ΛΔ is taken as a mean, the synthesis of ratios is taken (as this is taken in the *Elements*, too). Since, however, the discussion of the subject has been somewhat confused, and not such as to make the concept satisfactory, (as can be found reading Pappus and Theon and Arcadius, who, in many treatises, present the operation not by arguments, but by examples), there will be no incongruity if we linger briefly on this subject so as to present the operation more clearly.

So: I say that if some middle term is taken between two numbers (or magnitudes), the ratio of the initially taken numbers is composed of the ratio, which the first has to the mean and of the ratio which the mean has to the third.

So first it aught to be recalled how a ratio is said to be compounded of ratios. For as in the *Elements*: "when the quantities of the ratios, multiplied, produce a certain <quantity>," where "quantity" clearly stands for "the number" whose cognate is the given ratio (as say several authors as well as Nichomachus in the first book of *On Music* and Heronas in the commentary to the *Arithmetical Introduction*) which is the same as saying: "the number which, multiplied on the consequent term of the ratio, produces the antecedent as well." And the quantity would be taken in a more legitimate way in the case of multiples, while in the case of the superparticulars, superpartients, it is no longer possible for the quantity to be taken without the unit remaining undivided - which, even if this does not belong to what is proper in arithmetic, yet it does belong to what is proper in calculation. And the unit is divided by the part of by the parts by which the ratio is called, so that (to say this in a clearer way), the quantity of the half-as-large again is, added to the unit, half the unit; and <the quantity of the> four-thirds is, added to the unit, one third the unit, so that, as has been said above as well, the quantity of the ratio, multiplied on the consequent term, produces the antecedent. For the quantity of nine to six, being the unit and the half, multiplied on 6, produces 9, and it is possible to observe the same in other cases as well.

Having clarified these first, let us return to the enunciated proposition. For let the two given numbers be Α, Β, and let a certain mean be taken between them, Γ. So it is to be proved that the ratio of Α to Β is combined of the ratio which Α has to Γ, and Γ to Β.

For let the quantity of the ratio Α, Γ be taken, namely Δ, and let the quantity of the ratio Γ, B be taken, namely E; therefore Γ, multiplying Δ, produces A, while B, multiplying E, produces Γ. So let Δ, multiplying E, produce Z. I say that Z is a quantity of the ratio of A to B, that is, that Z, multiplying B, produces A. For let B, multiplying Z, produce H. Now since B, multiplying Z, has produced H, and multiplying E, has produced Γ, it is therefore: as Z to E, H to Γ. Again, since Δ, multiplying E, has produced Z, while, multiplying Γ, it has produced A, it is therefore: as E to Γ, Z to A. Alternately: as E to Z, Γ to A, inversely also: as Z to E, so A to Γ. But as Z to E, H was proved to be Γ, therefore also: as H to Γ, A to Γ; therefore A is equal to H. But B, multiplying Z, has produced H; therefore B, multiplying Z, produces A as well; therefore Z is the quantity of the ratio of A to B. And Z is: Δ, multiplied on E, that is: the quantity of the ratio A, Γ, multiplied on the quantity of the ratio Γ, B; therefore the ratio of A to B is composed of both: the ratio which A has to Γ, and Γ to B; which it was required to prove.

[Eutocius concludes this passage with numerical examples, which I omit.]

5.4.2 Eutocius' Exposition in his *Commentary on the* Conics

But the square on ΒΓ has to the rectangle ΒΑΓ a ratio compounded from that which ΒΓ has to ΓΑ and ΒΓ to ΒΑ:

$$\text{sq.}(\mathrm{B\Gamma}) : \text{rect.}(\mathrm{BA\Gamma}) = (\mathrm{B\Gamma} : \mathrm{\Gamma A}) \text{ comp. } (\mathrm{B\Gamma} : \mathrm{BA}) :$$

it has been shown in the twenty-third theorem of the sixth book of the *Elements*, that equiangular parallelograms have to one another a ratio compounded out of that of the sides: but since it is discussed too inductively and not in the necessary manner by the commentators, we researched it; and it is written in our published work on the fourth theorem of the second book of Archimedes' *On the Sphere and the Cylinder*, and also in the scholia of the first book of Ptolemy's *Syntaxis*: but it is a good idea that this be written down here also, because readers do not always read it even in those works, and also because nearly the entire treatise of the *Conics* makes use of it.

(218.16) A ratio is said to be compounded from ratios, whenever the sizes of the

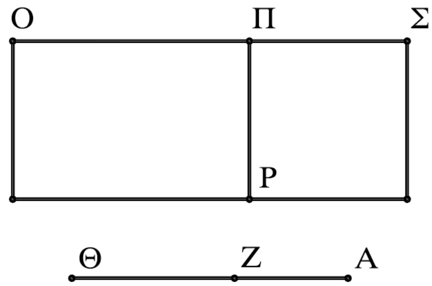

Figure 5.2: Eutocius' Diagram for Theorem 11

ratios, being multiplied into themselves, make something, with "size" of course meaning the number after which the ratio is named. So it is possible in the case of multiples that the size be a whole number, but in the case of the remaining relations it is necessary that the size must be a number plus part or parts, unless perhaps one wishes the relation to be irrational, such as are those according to the incommensurable magnitudes. But in the case of all the relations, it is manifest that the product of the size itself with the consequent of the ratio makes the antecedent.

(218.27) Accordingly, let there be a ratio of A to B, and let some mean of them be taken, as it chanced, as Γ, and let

$$\Delta = \text{size}(A : \Gamma),$$

and

$$E = \text{size}(\Gamma : B),$$

and let Δ, multiplying E, make Z. I say, that the size of the ratios A, B is Z,

$$Z = \text{size}(A : B),$$

that is that Z, multiplying B, makes A. Indeed, let Z, multiplying B, make H. So since Δ, multiplying E has made Z, and multiplying Γ has made A, therefore it is, that

$$E : Z = \Gamma : H$$

Alternando,

$$E : \Gamma = Z : H$$

But

$$E : \Gamma = Z : A$$

Therefore

$$H = A,$$

so that Z, multiplying B, has made A.

(220.17) But do not let this confuse those reading that this has been proved through arithmetic, for both the ancients made use of such proofs, being mathematical rather than arithmetical on account of the proportions, and that the thing being sought is arithmetical. For both ratios and sizes of ratios and multiplications, first begin by numbers, and through them by magnitudes, according to the speaker. As someone once said, "for these mathematical studies appear to be related."[3]

5.4.3 Theon's Definition and Exposition in his *Commentary on the* Syntaxis

Lemma[4]. One ratio is said to be compounded out of two or more ratios, whenever the sizes of the ratios, being multiplied into themselves, produce some size of a ratio.

For let the ratio of AB to ΓΔ be given, and that of ΓΔ to EZ: I say that the ratio of AB to EZ is compounded out of the ratios AB to ΓΔ and ΓΔ to EZ, that is, if the size of the ratio AB to ΓΔ is multiplied into that of ΓΔ to EZ, it makes the size of the ratio AB to EZ.

[Theon continues with numerical examples, which I omit.]

5.5 Conclusion

To conclude, we must observe a few things.

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[3]Much aught be said of this passage; some will regrettably be left for future work. Here Eutocius retains the Doric dialect of the quotation: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά. The quote is due to Archytas of Tarentum, preserved only in a fragment of his *Harmonics*. The quoted part reads: ταῦτα γὰρ τὰ μαθήματα δοκοῦντι ἦμεν ἀδελφεά. For the full fragment, see fr. B.1 in Diels-Kranz ([6]) or Freeman's English translation ([2]). See also *Republic* VII 530d and Knorr ([7]), pp. 171 n. 22-23.

[4]This is my translation of A. Rome's edition of the Greek, in [13].

The first: to Eutocius, Theon's definition of compound ratios *was* part of the *Elements*.

The second, and much more important: note how Eutocius develops mathematically between his two passages. In his first passage, Eutocius has seemingly no conception that the "size" of a ratio might be what we call an irrational number; in essence, his understanding at that point in time is only of "rational" ratios. Or, perhaps, he simply felt that mentioning irrational magnitudes was not necessary within the context of his exposition. That irrational magnitudes existed is not new; Eutocius, as an educated mathematician, would be well versed in approaching them. His two proofs are essentially the same: nowhere does he actually need these sizes to be rational.

It is clear, too, that Eutocius takes the indefinite τίνα to mean one of the πηλικότητα, based purely on how his proofs are structured. Given that Theon's definition from the *Syntaxis* commentary adds "size of a ratio" after τίνα, and that it is entirely likely that Eutocius was aware of this commentary, I am not surprised that Eutocius reads this for τίνα. It also does not surprise me that Eutocius quoted the "simpler" definition, that is, the one from the *Elements*, though it would have certainly have been helpful for his readers (myself included!) to know how to take τίνα. For the purpose of reading Apollonius, at least in Book I, single-compounding is adequate: the multiple-compounding needed for reading the *Syntaxis* is just not relevant here.

In future work (post-graduate), I intend to further explore this area by expanding the discussion to as many Hellenistic authors as possible, with the aid of the various extant manuscripts.

APPENDIX A
THE GREEK TEXT OF EUTOCIUS' COMMENTARY ON *CONICS* I

168 **Εἰς τὸ πρῶτον.**

Ἀπολλώνιος ὁ γεωμέτρης, ὦ φίλε ἑταῖρε Ἀνθέμιε, γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίᾳ ἐν χρόνοις τοῦ Εὐεργέτου Πτολεμαίου, ὡς ἱστορεῖ Ἡράκλειος ὁ τὸν βίον Ἀρχιμήδους γράφων, ὃς καί φησι τὰ κωνικὰ θεωρήματα ἐπινοῆσαι μὲν πρῶτον τὸν Ἀρχιμήδη, τὸν δὲ Ἀπολλώνιον αὐτὰ εὑρόντα ὑπὸ Ἀρχιμήδους μὴ ἐκδοθέντα ἰδιοποιήσασθαι, οὐκ ἀληθεύων κατά γε τὴν ἐμήν. ὅ τε γὰρ Ἀρχιμήδης ἐν πολλοῖς φαίνεται ὡς παλαιοτέρας τῆς στοιχειώσεως τῶν κωνικῶν μεμνημένος, καὶ ὁ Ἀπολλώνιος οὐχ ὡς ἰδίας ἐπινοίας γράφει· οὐ γὰρ ἂν ἔφη ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργάσθαι ταῦτα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα. ἀλλ᾽ ὅπερ φησὶν ὁ Γεμῖνος ἀληθές ἐστιν, ὅτι οἱ παλαιοὶ κῶνον ὁριζόμενοι τὴν τοῦ ὀρθογωνίου τριγώνου περιφορὰν μενούσης μιᾶς τῶν περὶ τὴν ὀρθὴν εἰκότως καὶ τοὺς κώνους πάντας ὀρθοὺς ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἑκάστῳ, ἐν
170 μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλουμένην παραβολήν, ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολήν, ἐν δὲ τῷ ὀξυγωνίῳ τὴν ἔλλειψιν· καὶ ἔστι παρ᾽ αὐτοῖς εὑρεῖν οὕτως ὀνομαζομένας τὰς τομάς. ὥσπερ οὖν τῶν ἀρχαίων ἐπὶ ἑνὸς ἑκάστου εἴδους τριγώνου θεωρησάντων τὰς δύο ὀρθὰς πρότερον ἐν τῷ ἰσοπλεύρῳ καὶ πάλιν ἐν

τῷ ἰσοσκελεῖ καὶ ὕστερον ἐν τῷ σκαληνῷ οἱ μεταγενέστεροι καθολικὸν θεώρημα ἀπέδειξαν τοιοῦτο· παντὸς τριγώνου αἱ ἐντὸς τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· οὕτως καὶ ἐπὶ τῶν τοῦ κώνου τομῶν· τὴν μὲν γὰρ λεγομένην ὀρθογωνίου κώνου τομὴν ἐν ὀρθογωνίῳ μόνον κώνῳ ἐθεώρουν τεμνομένῳ ἐπιπέδῳ ὀρθῷ πρὸς μίαν πλευρὰν τοῦ κώνου, τὴν δὲ τοῦ ἀμβλυγωνίου κώνου τομὴν ἐν ἀμβλυγωνίῳ γινομένην κώνῳ ἀπεδείκνυσαν, τὴν δὲ τοῦ ὀξυγωνίου ἐν ὀξυγωνίῳ, ὁμοίως ἐπὶ πάντων τῶν κώνων ἄγοντες τὰ ἐπίπεδα ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κώνου· δηλοῖ δὲ καὶ αὐτὰ τὰ ἀρχαῖα ὀνόματα τῶν γραμμῶν. ὕστερον δὲ Ἀπολλώνιος ὁ Περγαῖος καθόλου τι ἐθεώρησεν, ὅτι ἐν παντὶ κώνῳ καὶ ὀρθῷ καὶ σκαληνῷ πᾶσαι αἱ τομαί εἰσι κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κῶνον προσβολήν· ὃν καὶ θαυμάσαντες οἱ κατ’ αὐτὸν γενόμενοι διὰ τὸ θαυμάσιον τῶν ὑπ’ αὐτοῦ δεδειγμένων κωνικῶν θεωρημάτων μέγαν γεωμέτρην ἐκάλουν. ταῦτα μὲν οὖν ὁ Γεμῖνος ἐν τῷ ἕκτῳ φησὶ τῆς τῶν μαθημάτων θεωρίας. ὃ δὲ λέγει, σαφὲς ποιήσομεν ἐπὶ τῶν ὑποκειμένων καταγραφῶν.

ἔστω τὸ διὰ τοῦ ἄξονος τοῦ κώνου τρίγωνον τὸ
172 ΑΒΓ, καὶ ἤχθω τῇ ΑΒ ἀπὸ τυχόντος σημείου τοῦ Ε πρὸς ὀρθὰς ἡ ΔΕ, καὶ τὸ διὰ τῆς ΔΕ ἐπίπεδον ἐκβληθὲν ὀρθὸν πρὸς τὴν ΑΒ τεμνέτω τὸν κῶνον· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΑΕΔ, ΑΕΖ γω-

νιῶν. ὀρθογωνίου μὲν ὄντος τοῦ κώνου καὶ ὀρθῆς δηλονότι τῆς ὑπὸ ΒΑΓ γωνίας ὡς ἐπὶ τῆς πρώτης καταγραφῆς δύο ὀρθαῖς ἴσαι ἔσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ γωνίαι· ὥστε παράλληλος ἔσται ἡ ΔΕΖ τῇ ΑΓ. καὶ γίνεται ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομὴ ἡ καλουμένη παραβολὴ οὕτω κληθεῖσα ἀπὸ τοῦ παράλληλον εἶναι τὴν ΔΕΖ, ἥτις ἐστὶ κοινὴ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου, τῇ ΑΓ πλευρᾷ τοῦ τριγώνου.

ἐὰν δὲ ἀμβλυγώνιος ᾖ ὁ κῶνος ὡς ἐπὶ τῆς δευτέρας καταγραφῆς ἀμβλείας δηλονότι οὔσης τῆς ὑπὸ ΒΑΓ, ὀρθῆς δὲ τῆς ὑπὸ ΑΕΖ, δύο ὀρθῶν μείζους ἔσονται αἱ ὑπὸ ΒΑΓ, ΑΕΖ γωνίαι· ὥστε οὐ συμπεσεῖται ἡ ΔΕΖ τῇ ΑΓ πλευρᾷ ἐπὶ τὰ πρὸς τοῖς Ζ, Γ μέρη, ἀλλὰ ἐπὶ τὰ πρὸς τοῖς Α, Ε προσεκβαλλομένης δηλονότι τῆς ΓΑ ἐπὶ τὸ Δ. ποιήσει οὖν τὸ τέμνον ἐπίπεδον ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομὴν τὴν καλουμένην ὑπερβολὴν οὕτω κληθεῖσαν ἀπὸ τοῦ ὑπερβάλλειν τὰς εἰρημένας γωνίας, τουτέστι τὰς ὑπὸ ΑΕΖ,
174 ΒΑΓ, δύο ὀρθὰς ἢ διὰ τὸ ὑπερβάλλειν τὴν ΔΕΖ τὴν κορυφὴν τοῦ κώνου καὶ συμπίπτειν τῇ ΓΑ ἐκτός.

ἐὰν δὲ ὀξυγώνιος ᾖ ὁ κῶνος ὀξείας δηλονότι οὔσης τῆς ὑπὸ ΒΑΓ, αἱ ΒΑΓ, ΑΕΖ ἔσονται δύο ὀρθῶν ἐλάσσονες· ὥστε αἱ ΕΖ, ΑΓ ἐκβαλλόμεναι συμπεσοῦνται ὁπουδήποτε· προσαυξῆσαι γὰρ δύναμαι τὸν κῶνον. ἔσται οὖν ἐν τῇ ἐπιφανείᾳ τομή, ἥτις καλεῖται ἔλλειψις, οὕτω κληθεῖσα ἤτοι διὰ τὸ ἐλλείπειν δύο ὀρθαῖς τὰς προειρημένας γωνίας ἢ διὰ τὸ τὴν ἔλλειψιν κύκλον εἶναι ἐλλιπῆ.

οὕτως μὲν οὖν οἱ παλαιοὶ ὑποθέμενοι τὸ τέμνον ἐπίπεδον τὸ διὰ τῆς ΔΕΖ πρὸς ὀρθὰς τῇ ΑΒ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τοῦ κώνου τριγώνου καὶ ἔτι διαφόρους τοὺς κώνους ἐθεώρησαν καὶ ἐπὶ ἑκάστου ἰδίαν τομήν· ὁ δὲ Ἀπολλώνιος ὑποθέμενος τὸν κῶνον καὶ ὀρθὸν καὶ σκαληνὸν τῇ διαφόρῳ τοῦ ἐπιπέδου κλίσει διαφόρους ἐποίησε τὰς τομάς.

ἔστω γὰρ πάλιν ὡς ἐπὶ τῶν αὐτῶν καταγραφῶν τὸ τέμνον ἐπίπεδον τὸ ΚΕΛ, κοινὴ δὲ αὐτοῦ τομὴ καὶ τῆς βάσεως τοῦ κώνου ἡ ΚΖΛ, κοινὴ δὲ πάλιν αὐτοῦ τοῦ ΚΕΛ ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου ἡ ΕΖ, ἥτις καὶ διάμετρος καλεῖται τῆς τομῆς. ἐπὶ πασῶν οὖν τῶν τομῶν ὑποτίθεται τὴν ΚΛ πρὸς ὀρθὰς τῇ ΒΓ βάσει τοῦ ΑΒΓ τριγώνου, λοιπὸν δέ, εἰ μὲν
176 ἡ ΕΖ παράλληλος εἴη τῇ ΑΓ, παραβολὴν γίνεσθαι τὴν ΚΕΛ ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τομήν, εἰ δὲ συμπίπτει τῇ ΑΓ πλευρᾷ ἡ ΕΖ ἐκτὸς τῆς κορυφῆς τοῦ κώνου ὡς κατὰ τὸ Δ, γίνεσθαι τὴν ΚΕΛ τομὴν ὑπερβολήν, εἰ δὲ ἐντὸς συμπίπτει τῇ ΑΓ ἡ ΕΖ, γίνεσθαι τὴν τομὴν ἔλλειψιν, ἣν καὶ θυρεὸν καλοῦσιν. καθόλου οὖν τῆς μὲν παραβολῆς ἡ διάμετρος παράλληλός ἐστι τῇ μιᾷ πλευρᾷ τοῦ τριγώνου, τῆς δὲ

ὑπερβολῆς ἡ διάμετρος συμπίπτει τῇ πλευρᾷ τοῦ τριγώνου ὡς ἐπὶ τὰ πρὸς τῇ κορυφῇ τοῦ κώνου μέρη, τῆς δὲ ἐλλείψεως ἡ διάμετρος συμπίπτει τῇ πλευρᾷ τοῦ τριγώνου ὡς ἐπὶ τὰ πρὸς τῇ βάσει μέρη. κἀκεῖνο δὲ χρὴ εἰδέναι, ὅτι ἡ μὲν παραβολὴ καὶ ἡ ὑπερβολὴ τῶν εἰς ἄπειρόν εἰσιν αὐξανομένων, ἡ δὲ ἔλλειψις οὐκέτι· πᾶσα γὰρ εἰς αὑτὴν συννεύει ὁμοίως τῷ κύκλῳ.

πλειόνων δὲ οὐσῶν ἐκδόσεων, ὡς καὶ αὐτός φησιν ἐν τῇ ἐπιστολῇ, ἄμεινον ἡγησάμην συναγαγεῖν αὐτὰς ἐκ τῶν ἐμπιπτόντων τὰ σαφέστερα παρατιθέμενος ἐν τῷ ῥητῷ διὰ τὴν τῶν εἰσαγομένων εὐμάρειαν, ἔξωθεν δὲ ἐν τοῖς συντεταγμένοις σχολίοις ἐπισημαίνεσθαι τοὺς διαφόρους ὡς εἰκὸς τρόπους τῶν ἀποδείξεων.

φησὶ τοίνυν ἐν τῇ ἐπιστολῇ τὰ πρῶτα τέσσαρα βιβλία περιέχειν ἀγωγὴν στοιχειώδη· ὧν τὸ μὲν πρῶτον περιέχειν τὰς γενέσεις τῶν τριῶν τοῦ κώνου τομῶν καὶ τῶν καλουμένων ἀντικειμένων καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα. ταῦτα δέ ἐστιν, ὅσα συμβαίνει παρὰ τὴν πρώτην αὐτῶν γένεσιν· ἔχουσι γὰρ καὶ ἕτερά τινα παρακολουθήματα. τὸ δὲ δεύτερον

178 τὰ παρὰ τὰς διαμέτρους καὶ τοὺς ἄξονας τῶν τομῶν συμβαίνοντα καὶ τὰς ἀσυμπτώτους καὶ ἄλλα γενικὴν καὶ ἀναγκαίαν χρείαν παρεχόμενα πρὸς τοὺς διορισμούς. ὁ δὲ διορισμὸς ὅτι διπλοῦς ἐστι, παντί που δῆλον, ὁ μὲν μετὰ τὴν ἔκθεσιν ἐφιστάνων, τί ἔστι τὸ ζητούμενον, ὁ δὲ τὴν

πρότασιν οὐ συγχωρῶν καθολικὴν εἶναι, λέγων δέ, πότε καὶ πῶς καὶ ποσαχῶς δυνατὸν συστῆναι τὸ προτιθέμενον, οἷός ἐστιν ὁ ἐν τῷ εἰκοστῷ δευτέρῳ θεωρήματι τοῦ πρώτου βιβλίου τῆς Εὐκλείδου στοιχειώσεως· ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις, τρίγωνον συστήσασθαι· δεῖ δὴ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας, ἐπειδὴ δέδεικται, ὅτι παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι. τὸ δὲ τρίτον τῶν κωνικῶν περιέχειν φησὶ πολλὰ καὶ παράδοξα θεωρήματα χρήσιμα πρὸς τὰς συνθέσεις τῶν στερεῶν τόπων. ἐπιπέδους τόπους ἔθος τοῖς παλαιοῖς γεωμέτραις λέγειν, ὅταν ἐπὶ τῶν προβλημάτων οὐκ ἀφ’ ἑνὸς σημείου μόνον, ἀλλ’ ἀπὸ πλειόνων γίνεται τὸ πρόβλημα, οἷον εἰ ἐπιτάξει τις εὐθείας δοθείσης πεπερασμένης εὑρεῖν τι σημεῖον, ἀφ’ οὗ ἡ ἀχθεῖσα κάθετος ἐπὶ τὴν δοθεῖσαν μέση ἀνάλογον γίνεται τῶν τμημάτων, τόπον καλοῦσι τὸ τοιοῦτον· οὐ μόνον γὰρ ἓν σημεῖόν ἐστι τὸ ποιοῦν τὸ πρόβλημα, ἀλλὰ τόπος ὅλος, ὃν ἔχει ἡ περιφέρεια τοῦ περὶ διάμετρον τὴν δοθεῖσαν εὐθεῖαν κύκλου. ἐὰν γὰρ ἐπὶ τῆς δοθείσης εὐθείας ἡμικύκλιον γραφῇ, ὅπερ ἂν ἐπὶ τῆς περιφερείας λάβῃς σημεῖον καὶ ἀπ’ αὐτοῦ κάθετον

180 ἀγάγῃς ἐπὶ τὴν διάμετρον, ποιήσει τὸ προβληθέν. ὁμοίως δὲ δοθείσης εὐθείας ἐάν τις ἐπιτάξῃ εὑρεῖν ἐκτὸς αὐτῆς σημεῖον, ἀφ’ οὗ αἱ ἐπιζευγνύμεναι ἐπὶ τὰ πέρατα τῆς εὐθείας ἴσαι ἔσονται ἀλλήλαις, καὶ ἐπὶ

τούτου οὐ μόνον ἓν σημεῖόν ἐστι τὸ ποιοῦν τὸ πρόβλημα, ἀλλὰ τόπος, ὃν ἐπέχει ἡ ἀπὸ τῆς διχοτομίας πρὸς ὀρθὰς ἀγομένη· ἐὰν γὰρ τὴν δοθεῖσαν εὐθεῖαν δίχα τεμὼν καὶ ἀπὸ τῆς διχοτομίας πρὸς ὀρθὰς ἀγάγῃς, ὃ ἂν ἐπ' αὐτῆς λάβῃς σημεῖον, ποιήσει τὸ ἐπιταχθέν.

ὅμοιον γράφει καὶ αὐτὸς Ἀπολλώνιος ἐν τῷ Ἀναλυομένῳ τόπῳ ἐπὶ τοῦ ὑποκειμένου.

δύο δοθέντων [εὐθειῶν] ἐν ἐπιπέδῳ [καὶ] σημείων καὶ λόγου δοθέντος ἀνίσων εὐθειῶν δυνατόν ἐστιν ἐν τῷ ἐπιπέδῳ γράψαι κύκλον ὥστε τὰς ἀπὸ τῶν δοθέντων σημείων ἐπὶ τὴν περιφέρειαν τοῦ κύκλου κλωμένας εὐθείας λόγον ἔχειν τὸν αὐτὸν τῷ δοθέντι.

ἔστω τὰ μὲν δοθέντα σημεῖα τὰ Α, Β, λόγος δὲ ὁ τῆς Γ πρὸς τὴν Δ μείζονος οὔσης τῆς Γ· δεῖ δὴ ποιῆσαι τὸ ἐπιταχθέν. ἐπεζεύχθω ἡ ΑΒ καὶ ἐκβεβλήσθω ἐπὶ τὰ πρὸς τῷ Β μέρη, καὶ γεγονέτω, ὡς ἡ Δ πρὸς τὴν Γ, ἡ Γ πρὸς ἄλλην τινὰ μείζονα δηλονότι τῆς Δ, καὶ ἔστω, εἰ τύχοι, πρὸς τὴν ΕΔ, καὶ πάλιν γεγονέτω, ὡς ἡ Ε πρὸς τὴν ΑΒ, ἡ Δ πρὸς τὴν ΒΖ καὶ ἡ Γ πρὸς τὴν Η. φανερὸν δή, ὅτι ἥ τε Γ μέση ἀνάλογόν ἐστι τῆς ΕΔ καὶ τῆς Δ καὶ ἡ

182 Η τῶν ΑΖ, ΖΒ. καὶ κέντρῳ μὲν τῷ Ζ διαστήματι δὲ τῇ Η κύκλος γεγράφθω ὁ ΚΘ. φανερὸν δή, ὅτι τέμνει ἡ ΚΘ περιφέρεια τὴν ΑΒ εὐθεῖαν· ἡ γὰρ Η εὐθεῖα μέση ἀνάλογόν ἐστι τῶν ΑΖ, ΖΒ. εἰλήφθω δὴ ἐπὶ τῆς περιφερείας τυχὸν σημεῖον τὸ Θ, καὶ ἐπε-

ζεύχθωσαν αἱ ΘΑ, ΘΒ, ΘΖ. ἴση ἄρα ἐστὶν ἡ ΘΖ τῇ Η, καὶ διὰ τοῦτό ἐστιν, ὡς ἡ ΑΖ πρὸς τὴν ΖΘ, ἡ ΖΘ πρὸς ΖΒ. καὶ περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ ΘΖΒ ἀνάλογόν εἰσιν· ὅμοιον ἄρα ἐστὶ τὸ ΑΖΘ τῷ ΘΒΖ τριγώνῳ, καὶ ἴση ἡ ὑπὸ ΖΘΒ γωνία τῇ ὑπὸ ΘΑΒ. ἤχθω δὴ διὰ τοῦ Β τῇ ΑΘ παράλληλος ἡ ΒΛ. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΑΖ πρὸς ΖΘ, ἡ ΘΖ πρὸς ΖΒ, καὶ ὡς ἄρα πρώτη ἡ ΑΖ πρὸς τρίτην τὴν ΖΒ, τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΘ. ἀλλ’ ὡς ἡ ΑΖ πρὸς ΖΒ, ἡ ΑΘ πρὸς ΒΛ· καὶ ὡς ἄρα τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΘ, ἡ ΑΘ πρὸς ΒΛ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΒΘΖ τῇ ὑπὸ ΘΑΒ, ἔστι δὲ καὶ ἡ ὑπὸ ΑΘΒ τῇ ὑπὸ ΘΒΛ ἴση· ἐναλλὰξ γάρ· καὶ ἡ λοιπὴ ἄρα τῇ λοιπῇ ἴση ἐστίν, καὶ ὅμοιόν ἐστι τὸ ΑΘΒ τῷ ΒΘΛ, καὶ ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ὡς ἡ ΑΘ πρὸς ΘΒ, ἡ ΘΒ πρὸς ΒΛ, καὶ ὡς τὸ ἀπὸ ΑΘ πρὸς τὸ ἀπὸ ΘΒ, ἡ ΑΘ πρὸς ΒΛ. ἦν δὲ καί, ὡς ἡ ΑΘ πρὸς ΒΛ, τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΘ· ὡς ἄρα τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΘ, τὸ ἀπὸ ΑΘ πρὸς τὸ ἀπὸ ΘΒ, καὶ διὰ τοῦτο, ὡς ἡ ΑΖ πρὸς ΖΘ, ἡ ΑΘ πρὸς ΘΒ. ἀλλ’ ὡς ἡ ΑΖ πρὸς ΖΘ, ἡ ΕΔ πρὸς Γ καὶ ἡ Γ πρὸς Δ· καὶ ὡς ἄρα ἡ Γ πρὸς Δ, ἡ ΑΘ πρὸς ΘΒ. ὁμοίως δὴ δειχθήσονται πᾶσαι αἱ ἀπὸ τῶν Α, Β σημείων ἐπὶ τὴν
184 περιφέρειαν τοῦ κύκλου κλώμεναι τὸν αὐτὸν ἔχουσαιλόγον ταῖς Γ, Δ.

λέγω δή, ὅτι πρὸς ἄλλῳ σημείῳ μὴ ὄντι ἐπὶ τῆς περιφερείας οὐ γίνεται λόγος τῶν ἀπὸ τῶν Α, Β σημείων ἐπ’ αὐτὸ ἐπιζευγνυμένων εὐθειῶν ὁ αὐτὸς τῷ τῆς Γ πρὸς Δ.

εἰ γὰρ δυνατόν, γεγονέτω πρὸς τῷ Μ ἐκτὸς τῆς περιφερείας· καὶ γὰρ εἰ ἐντὸς ληφθείη, τὸ αὐτὸ ἄτοπον συμβήσεται καθ’ ἑτέραν τῶν ὑποθέσεων· καὶ ἐπεζεύχθωσαν αἱ ΜΑ, ΜΒ, ΜΖ, καὶ ὑποκείσθω, ὡς ἡ Γ πρὸς Δ, οὕτως ἡ ΑΜ πρὸς ΜΒ. ἔστιν ἄρα, ὡς ἡ ΕΔ πρὸς Δ, οὕτως τὸ ἀπὸ ΕΔ πρὸς τὸ ἀπὸ Γ καὶ τὸ ἀπὸ ΑΜ πρὸς τὸ ἀπὸ ΜΒ. ἀλλ’ ὡς ἡ ΕΔ πρὸς Δ, οὕτως ὑπόκειται ἡ ΑΖ πρὸς ΖΒ· καὶ ὡς ἄρα ἡ ΑΖ πρὸς ΖΒ, τὸ ἀπὸ ΑΜ πρὸς τὸ ἀπὸ ΜΒ. καὶ διὰ τὰ προδειχθέντα, ἐὰν ἀπὸ τοῦ Β τῇ ΑΜ παράλληλον ἀγάγωμεν, δειχθήσεται, ὡς ἡ ΑΖ πρὸς ΖΒ, τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΜ. ἐδείχθη δὲ καί, ὡς ἡ ΑΖ πρὸς ΖΒ, τὸ ἀπὸ ΑΖ πρὸς τὸ ἀπὸ ΖΘ. ἴση ἄρα ἡ ΖΘ τῇ ΖΜ· ὅπερ ἀδύνατον.

τόποι οὖν ἐπίπεδοι λέγονται τὰ τοιαῦτα· οἱ δὲ λεγόμενοι στερεοὶ τόποι τὴν προσωνυμίαν ἐσχήκασιν ἀπὸ τοῦ τὰς γραμμάς, δι’ ὧν γράφονται τὰ κατ’ αὐτοὺς προβλήματα, ἐκ τῆς τομῆς τῶν στερεῶν τὴν γένεσιν ἔχειν, οἷαί εἰσιν αἱ τοῦ κώνου τομαὶ καὶ ἕτεραι πλείους. εἰσὶ δὲ καὶ ἄλλοι τόποι πρὸς ἐπιφάνειαν λεγόμενοι, οἳ τὴν ἐπωνυμίαν ἔχουσιν ἀπὸ τῆς περὶ αὐτοὺς ἰδιότητος.

186 μέμφεται δὲ ἑξῆς τῷ Εὐκλείδῃ οὐχ, ὡς οἴεται Πάππος καὶ ἕτεροί τινες, διὰ τὸ μὴ εὑρηκέναι δύο

μέσας ἀνάλογον· ὅ τε γὰρ Εὐκλείδης ὑγιῶς εὗρε τὴνμίαν μέσην ἀνάλογον, ἀλλ’ οὐχ ὡς αὐτός φησιν οὐκ εὐτυχῶς, καὶ περὶ τῶν δύο μέσων οὐδὲ ὅλως ἐπεχείρησε ζητῆσαι ἐν τῇ στοιχειώσει, αὐτὸς ὅ τε Ἀπολλώνιος οὐδὲν περὶ τῶν δύο μέσων ἀνάλογον φαίνεται ζητῆσαι ἐν τῷ τρίτῳ βιβλίῳ· ἀλλ’, ὡς ἔοικεν, ἑτέρῳ βιβλίῳ περὶ τόπων γεγραμμένῳ τῷ Εὐκλείδῃ ἐπισκήπτει, ὅπερ εἰς ἡμᾶς οὐ φέρεται.

τὰ δὲ ἐφεξῆς περὶ τοῦ τετάρτου βιβλίου λεγόμενα σαφῆ ἐστιν. τὸ δὲ πέμπτον φησὶ περιέχειν τὰ περὶ τῶν ἐλαχίστων καὶ μεγίστων. ὥσπερ γὰρ ἐπὶ τοῦ κύκλου ἐμάθομεν ἐν τῇ στοιχειώσει, ὅτι ἔστι τι σημεῖον ἐκτός, ἀφ’ οὗ τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν μεγίστη ἐστὶν ἡ διὰ τοῦ κέντρου, τῶν δὲ πρὸς τὴν κυρτὴν ἐλαχίστη ἐστὶν ἡ μεταξὺ τοῦ σημείου καὶ τῆς διαμέτρου, οὕτως καὶ ἐπὶ τῶν τοῦ κώνου τομῶν ζητεῖ ἐν τῷ πέμπτῳ βιβλίῳ. τοῦ δὲ ἕκτου καὶ ἑβδόμου καὶ ὀγδόου σαφῶς ἡ πρόθεσις ὑπ’ αὐτοῦ εἴρηται. καὶ ταῦτα μὲν περὶ τῆς ἐπιστολῆς.

Ἀρχόμενος δὲ τῶν ὅρων γένεσιν ὑπογράφει κωνικῆς ἐπιφανείας, ἀλλ’ οὐ τὸν τί ἔστι διορισμὸν παραδέδωκεν· ἔξεστι δὲ τοῖς βουλομένοις ἐκ τῆς γενέσεως αὐτῆς τὸν ὅρον λαμβάνειν. τὸ δὲ λεγόμενον ὑπ’ αὐτοῦ διὰ καταγραφῆς σαφὲς ποιήσομεν·

ἐὰν ἀπό τινος σημείου πρὸς κύκλου περιφέρειαν καὶ τὰ ἑξῆς. ἔστω κύκλος ὁ ΑΒ, οὗ κέν-
188 τρον τὸ Γ, καὶ σημεῖόν τι μετέωρον τὸ Δ, καὶ ἐπι-

ζευχθεῖσα ἡ ΔΒ ἐκβεβλήσθω εἰς ἄπειρον ἐφ’ ἑκάτερα μέρη ὡς ἐπὶ τὰ Ε, Ζ. ἐὰν δὴ μένοντος τοῦ Δ ἡ ΔΒ φέρηται, ἕως ἂν τὸ Β ἐνεχθὲν κατὰ τῆς τοῦ ΑΒ κύκλου περιφερείας ἐπὶ τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, γεννήσει ἐπιφάνειάν τινα, ἥτις σύγκειται ἐκ δύο ἐπιφανειῶν ἁπτομένων ἀλλήλων κατὰ τὸ Δ, ἣν καὶ καλεῖ κωνικὴν ἐπιφάνειαν. φησὶ δέ, ὅτι καὶ εἰς ἄπειρον αὔξεται διὰ τὸ καὶ τὴν γράφουσαν αὐτὴν εὐθεῖαν οἷον τὴν ΔΒ εἰς ἄπειρον ἐκβάλλεσθαι. κορυφὴν δὲ τῆς ἐπιφανείας λέγει τὸ Δ, ἄξονα δὲ τὴν ΔΓ.

κῶνον δὲ λέγει τὸ περιεχόμενον σχῆμα ὑπό τε τοῦ ΑΒ κύκλου καὶ τῆς ἐπιφανείας, ἣν μόνη γράφει ἡ ΔΒ εὐθεῖα, κορυφὴν δὲ τοῦ κώνου τὸ Δ, ἄξονα δὲ τὴν ΔΓ, βάσιν δὲ τὸν ΑΒ κύκλον.

καὶ ἐὰν μὲν ἡ ΔΓ πρὸς ὀρθὰς ᾖ τῷ ΑΒ κύκλῳ, ὀρθὸν καλεῖ τὸν κῶνον, ἐὰν δὲ μὴ πρὸς ὀρθάς, σκαληνόν· γενήσεται δὲ κῶνος σκαληνός, ὅταν λαβόντες κύκλον ἀπὸ τοῦ κέντρου αὐτοῦ ἀναστήσωμεν εὐθεῖαν μὴ πρὸς ὀρθὰς τῷ ἐπιπέδῳ τοῦ κύκλου, ἀπὸ δὲ τοῦ μετεώρου σημείου τῆς ἀναταθείσης εὐθείας ἐπὶ τὸν κύκλον ἐπιζεύξωμεν εὐθεῖαν καὶ περιαγάγωμεν τὴν ἐπιζευχθεῖσαν εὐθεῖαν περὶ τὸν κύκλον τοῦ πρὸς τῷ μετεώρῳ σημείῳ τῆς ἀναταθείσης μένοντος· τὸ γὰρ προσληφθὲν σχῆμα κῶνος ἔσται σκαληνός.

190 δῆλον δέ, ὅτι ἡ περιαγομένη εὐθεῖα ἐν τῇ περιαγωγῇ μείζων καὶ ἐλάττων γίνεται, κατὰ δέ τινας θέσεις

καὶ ἴση πρὸς ἄλλο καὶ ἄλλο σημεῖον τοῦ κύκλου. ἀποδείκνυται δὲ τοῦτο οὕτως· ἐὰν κώνου σκαληνοῦ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν ἀχθῶσιν εὐθεῖαι, πασῶν τῶν ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν ἀχθεισῶν εὐθειῶν μία μέν ἐστιν ἐλαχίστη μία δὲ μεγίστη, δύο δὲ μόναι ἴσαι παρ' ἑκάτερα τῆς ἐλαχίστης καὶ τῆς μεγίστης, ἀεὶ δὲ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάσσων. ἔστω κῶνος σκαληνός, οὗ βάσις μὲν ὁ ΑΒΓ κύκλος, κορυφὴ δὲ τὸ Δ σημεῖον. καὶ ἐπεὶ ἡ ἀπὸ τῆς κορυφῆς τοῦ σκαληνοῦ κώνου ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ἀγομένη ἤτοι ἐπὶ τῆς περιφερείας τοῦ ΑΒΓΖΗ κύκλου πεσεῖται ἢ ἐκτὸς ἢ ἐντός, ἐμπιπτέτω πρότερον ἐπὶ τῆς περιφερείας ὡς ἐπὶ τῆς πρώτης καταγραφῆς ἡ ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου καὶ ἔστω τὸ Κ, καὶ ἀπὸ τοῦ Ε ἐπὶ τὸ Κ ἐπεζεύχθω ἡ ΕΚ καὶ ἐκβεβλήσθω ἐπὶ τὸ Β, καὶ ἐπεζεύχθω ἡ ΒΔ, καὶ εἰλήφθωσαν δύο ἴσαι περιφέρειαι παρ' ἑκάτερα τοῦ Ε αἱ ΕΖ, ΕΗ, καὶ παρ' ἑκάτερα τοῦ Β αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΕΗ, ΔΖ, ΔΗ, ΕΑ, ΕΓ, ΑΒ, ΒΓ, ΔΑ, ΔΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΕΖ εὐθεῖα τῇ ΕΗ εὐθείᾳ· ἴσας γὰρ περιφερείας ὑποτείνουσιν· κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΕ, βάσις ἄρα ἡ ΔΖ τῇ ΔΗ ἐστιν ἴση. πάλιν ἐπεὶ ἡ ΑΒ περιφέρεια τῇ ΒΓ ἐστιν ἴση, καὶ διάμετρος

192 ἡ ΒΕ, λοιπὴ ἄρα ἡ ΕΖΓ τῇ ΕΗΑ ἐστιν ἴση· ὥστε καὶ ἡ ΑΕ τῇ ΕΓ. κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ· βάσις ἄρα ἡ ΔΑ τῇ ΔΓ ἐστιν ἴση. ὁμοίως δὴ καὶ

πᾶσαι δειχθήσονται αἱ ἴσον ἀπέχουσαι τῆς ΔΕ ἢ τῆς ΔΒ ἴσαι. πάλιν ἐπεὶ τριγώνου τοῦ ΔΕΖ ὀρθή ἐστι γωνία ἡ ὑπὸ ΔΕΖ, μείζων ἐστὶν ἡ ΔΖ τῆς ΔΕ. καὶ πάλιν ἐπεὶ μείζων ἐστὶν ἡ ΕΑ εὐθεῖα τῆς ΕΖ, ἐπεὶ καὶ περιφέρεια ἡ ΕΖΑ τῆς ΕΖ περιφερείας, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΔΕ, ἡ ΔΖ ἄρα τῆς ΔΑ ἐλάσσων ἐστίν. διὰ τὰ αὐτὰ καὶ ἡ ΔΑ τῆς ΔΒ ἐλάσσων ἐστίν. ἐπεὶ οὖν ἡ ΔΕ τῆς ΔΖ ἐλάσσων ἐδείχθη, ἡ δὲ ΔΖ τῆς ΔΑ, ἡ δὲ ΔΑ τῆς ΔΒ, ἐλαχίστη μέν ἐστιν ἡ ΔΕ, μεγίστη δὲ ἡ ΔΒ, ἀεὶ δὲ ἡἔγγιον τῆς ΔΕ τῆς ἀπώτερον ἐλάσσων ἐστίν.

ἀλλὰ δὴ ἡ κάθετος πιπτέτω ἐκτὸς τοῦ ΑΒΓΗΖ κύκλου ὡς ἐπὶ τῆς δευτέρας καταγραφῆς ἡ ΔΕ, καὶ εἰλήφθω πάλιν τὸ κέντρον τοῦ κύκλου τὸ Κ, καὶ ἐπεζεύχθω ἡ ΕΚ καὶ ἐκβεβλήσθω ἐπὶ τὸ Β, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΔΘ, καὶ εἰλήφθωσαν δύο ἴσαι περιφέρειαι παρ᾽ ἑκάτερα τοῦ Θ αἱ ΘΖ, ΘΗ καὶ παρ᾽ ἑκάτερα τοῦ Β αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΕΗ, ΖΚ, ΗΚ, ΔΖ, ΔΗ, ΑΒ, ΒΓ, ΚΑ, ΚΓ, ΔΚ, ΔΑ, ΔΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ΘΖ περιφέρεια τῇ ΘΗ, καὶ γωνία ἄρα ἡ ὑπὸ ΘΚΖ τῇ ὑπὸ ΘΚΗ ἐστιν ἴση. ἐπεὶ οὖν ἡ ΖΚ εὐθεῖα τῇ ΚΗ ἐστιν ἴση· ἐκ κέντρου γάρ· κοινὴ δὲ ἡ ΚΕ, καὶ γωνία ἡ ὑπὸ ΖΚΕ

194 τῇ ὑπὸ ΗΚΕ ἴση, καὶ βάσις ἡ ΖΕ τῇ ΗΕ ἴση. ἐπεὶ οὖν ἡ ΖΕ εὐθεῖα τῇ ΗΕ ἐστιν ἴση, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ, βάσις ἄρα ἡ ΔΖ τῇ ΔΗ ἐστιν ἴση. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΒΑ περιφέρεια τῇ ΒΓ,

καὶ γωνία ἄρα ἡ ὑπὸ ΑΚΒ τῇ ὑπὸ ΓΚΒ ἐστιν ἴση· ὥστε καὶ λοιπὴ εἰς τὰς δύο ὀρθὰς ἡ ὑπὸ ΑΚΕ λοιπῇ εἰς τὰς δύο ὀρθὰς τῇ ὑπὸ ΓΚΕ ἐστιν ἴση. ἐπεὶ οὖν ἡ ΑΚ εὐθεῖα τῇ ΓΚ ἐστιν ἴση· ἐκ κέντρου γάρ· κοινὴ δὲ ἡ ΚΕ, δύο δυσὶν ἴσαι, καὶ γωνία ἡ ὑπὸ ΑΚΕ τῇ ὑπὸ ΓΚΕ· καὶ βάσις ἄρα ἡ ΑΕ τῇ ΓΕ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἡ ΑΕ εὐθεῖα τῇ ΓΕ, κοινὴ δὲ ἡ ΕΔ καὶ πρὸς ὀρθάς, βάσις ἄρα ἡ ΔΑ τῇ ΔΓ ἴση. ὁμοί ως δὲ καὶ πᾶσαι δειχθήσονται αἱ ἴσον ἀπέχουσαι τῆς ΔΒ ἢ τῆς ΔΘ ἴσαι. καὶ ἐπεὶ ἡ ΕΘ τῆς ΕΖ ἐστιν ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ, βάσις ἄρα ἡ ΔΘ βάσεως τῆς ΔΖ ἐστιν ἐλάσσων. πάλιν ἐπεὶ ἡ ἀπὸ τοῦ Ε ἐφαπτομένη τοῦ κύκλου πασῶν τῶν πρὸςτὴν κυρτὴν περιφέρειαν προσπιπτουσῶν μείζων ἐστίν, ἐδείχθη δὲ ἐν τῷ γ′ τῆς στοιχειώσεως τὸ ὑπὸ ΑΕ, ΕΛ ἴσον τῷ ἀπὸ τῆς ΕΖ, ὅταν ἡ ΕΖ ἐφάπτηται, ἔστιν ἄρα, ὡς ἡ ΑΕ πρὸς ΕΖ, ἡ ΕΖ πρὸς ΕΛ. μεί- ζων δέ ἐστιν ἡ ΕΖ τῆς ΕΛ· ἀεὶ γὰρ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάσσων· μείζων ἄρα καὶ ἡ ΑΕ τῆς ΕΖ. ἐπεὶ οὖν ἡ ΕΖ τῆς ΕΑ ἐστιν ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ, βάσις ἄρα ἡ ΔΖ τῆς ΔΑ ἐστιν ἐλάσσων. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΚ τῇ ΚΒ, κοινὴ δὲ ἡ ΚΕ, δύο ἄρα αἱ ΑΚ, ΚΕ ταῖς ΕΚ, ΚΒ, τουτέστιν ὅλῃ τῇ ΕΚΒ, εἰσιν ἴσαι. ἀλλ’ αἱ ΑΚ, ΚΕ τῆς ΑΕ μείζονές εἰσιν· καὶ ἡ ΒΕ
196 ἄρα τῆς ΑΕ μείζων ἐστίν. πάλιν ἐπεὶ ἡ ΑΕ τῆς ΕΒ ἐστιν ἐλάσσων, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΕΔ,

βάσις ἄρα ἡ ΔΑ τῆς ΒΔ ἐστιν ἐλάσσων. ἐπεὶ οὖν ἡ ΔΘ τῆς ΔΖ ἐστιν ἐλάσσων, ἡ δὲ ΔΖ τῆς ΔΑ, ἡ δὲ ΔΑ τῆς ΔΒ, ἐλαχίστη μέν ἐστιν ἡ ΔΘ, μεγίστη δὲ ἡ ΔΒ, ἀεὶ δὲ ἡ ἔγγιον καὶ τὰ ἑξῆς.

ἀλλὰ δὴ ἡ κάθετος πιπτέτω ἐντὸς τοῦ ΑΒΓΗΖ κύκλου ὡς ἐπὶ τῆς τρίτης καταγραφῆς ἡ ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Κ, καὶ ἐπεζεύχθω ἡ ΕΚ καὶ ἐκβεβλήσθω ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Β, Θ, καὶ ἐπεζεύχθωσαν αἱ ΔΘ, ΔΒ, καὶ εἰλήφθωσαν δύο ἴσαι περιφέρειαι παρ' ἑκάτερα τοῦ Θ αἱ ΘΖ, ΘΗ καὶ παρ' ἑκάτερα τοῦ Β αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΕΗ, ΖΚ, ΗΚ, ΔΖ, ΔΗ, ΚΑ, ΚΓ, ΕΑ, ΕΓ, ΔΑ, ΔΓ, ΑΒ, ΒΓ. ἐπεὶ οὖν ἴση ἡ ΘΖ περιφέρεια τῇ ΘΗ, καὶ γωνία ἄρα ἡ ὑπὸ ΘΚΖ γωνίᾳ τῇ ὑπὸ ΘΚΗ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΖ τῇ ΗΚ, κοινὴ δὲ ἡ ΚΕ, καὶ γωνία ἡ ὑπὸ ΖΚΕ γωνίᾳ τῇ ὑπὸ ΗΚΕ ἐστιν ἴση, βάσις ἄρα ἡ ΖΕ τῇ ΗΕ ἐστιν ἴση. ἐπεὶ οὖν ἡ ΖΕ τῇ ΗΕ ἐστιν ἴση, κοινὴ δὲ ἡ ΔΕ, καὶ γωνία ἡ ὑπὸ ΖΕΔ γωνίᾳ τῇ ὑπὸ ΗΕΔ ἐστιν ἴση, βάσις ἄρα ἡ ΔΖ τῇ ΔΗ ἐστιν ἴση. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΒ περιφέρεια τῇ ΒΓ, καὶ γωνία ἄρα ἡ ὑπὸ ΑΚΒ γωνίᾳ τῇ ὑπὸ ΓΚΒ ἐστιν ἴση· ὥστε καὶ λοιπὴ εἰς τὰς δύο ὀρθὰς ἡ ὑπὸ ΑΚΕ λοιπῇ εἰς τὰς δύο ὀρθὰς τῇ ὑπὸ ΓΚΕ ἐστιν ἴση. ἐπεὶ οὖν ἡ ΑΚ τῇ ΚΓ ἐστιν ἴση, κοινὴ δὲ ἡ ΕΚ, καὶ γωνία ἡ ὑπὸ ΑΚΕ γωνίᾳ τῇ ὑπὸ ΓΚΕ

198 ἐστιν ἴση, βάσις ἄρα ἡ ΑΕ τῇ ΓΕ ἐστιν ἴση. ἐπεὶ

οὖν ἡ ΑΕ τῇ ΓΕ ἐστιν ἴση, κοινὴ δὲ ἡ ΕΔ, καὶ γωνία ἡ ὑπὸ ΑΕΔ τῇ ὑπὸ ΓΕΔ ἴση, βάσις ἄρα ἡ ΔΑ τῇ ΔΓ ἐστιν ἴση. ὁμοίως δὴ καὶ πᾶσαι δειχθήσονται αἱ ἴσον ἀπέχουσαι ἢ τῆς ΔΒ ἢ τῆς ΔΘ ἴσαι. καὶ ἐπεὶ ἐν κύκλῳ τῷ ΑΒΓ ἐπὶ τῆς διαμέτρου εἴληπται σημεῖον τὸ Ε μὴ ὂν κέντρον τοῦ κύκλου, μεγίστη μὲν ἡ ΕΒ, ἐλαχίστη δὲ ἡ ΕΘ, ἀεὶ δὲ ἡ ἔγγιον τῆς ΕΘ τῆς ἀπώτερόν ἐστιν ἐλάσσων· ὥστε ἡ ΕΘ τῆς ΕΖ ἐστιν ἐλάσσων. καὶ ἐπεὶ ἡ ΘΕ τῆς ΖΕ ἐλάσσων ἐστίν, κοινὴ δὲ καὶ πρὸς ὀρθὰς αὐταῖς ἡ ΕΔ, βάσις ἄρα ἡ ΔΘ βάσεως τῆς ΔΖ ἐλάσσων ἐστίν. πάλιν ἐπεὶ ἡ μὲν ΕΖ ἔγγιόν ἐστι τῆς ΕΘ, ἡ δὲ ΑΕ πορρωτέρω, ἐλάσσων ἐστὶν ἡ ΕΖ τῆς ΑΕ. ἐπεὶ οὖν ἐλάσσων ἡ ΕΖ τῆς ΕΑ, κοινὴ δὲ καὶ πρὸς ὀρθάς ἐστιν αὐταῖς ἡ ΕΔ, βάσις ἄρα ἡ ΔΖ βάσεως τῆς ΔΑ ἐστιν ἐλάσσων. πάλιν ἐπεὶ ἴση ἡ ΑΚ τῇ ΚΒ, κοινὴ δὲ ἡ ΚΕ, δύο αἱ ΑΚ, ΚΕ δύο ταῖς ΒΚ, ΚΕ, τουτέστιν ὅλῃ τῇ ΒΚΕ, εἰσιν ἴσαι. ἀλλ' αἱ ΑΚ, ΚΕ τῆς ΑΕ μείζονές εἰσιν· καὶ ἡ ΕΒ ἄρα τῆς ΕΑ μείζων ἐστίν. πάλιν ἐπεὶ ἡ ΕΑ τῆς ΕΒ ἐλάσσων ἐστίν, κοινὴ δὲ καὶ πρὸς ὀρθὰς αὐταῖς ἡ ΕΔ, βάσις ἄρα ἡ ΔΑ βάσεως τῆς ΔΒ ἐστιν ἐλάσσων. ἐπεὶ οὖν ἡ ΔΘ τῆς ΔΖ ἐλάσσων, ἡ δὲ ΔΖ τῆς ΔΑ, ἡ δὲ ΔΑ τῆς ΔΒ, ἐλαχίστη μέν ἐστιν ἡ ΔΘ καὶ τὰ ἑξῆς.

Πάσης καμπύλης γραμμῆς, ἥτις ἐστὶν ἐν ἑνὶ ἐπιπέδῳ, διάμετρον καλῶ καὶ τὰ ἑξῆς. τὸ ἐν ἑνὶ ἐπιπέδῳ εἶπε διὰ τὴν ἕλικα τοῦ κυλίνδρου καὶ

200 τῆς σφαίρας· αὗται γὰρ οὐκ εἰσὶν ἐν ἑνὶ ἐπιπέδῳ. ὃ δὲ λέγει, τοιοῦτόν ἐστιν· ἔστω καμπύλη γραμμὴ ἡ ΑΒΓ καὶ ἐν αὐτῇ εὐθεῖαί τινες παράλληλοι αἱ ΑΓ, ΔΕ, ΖΗ, ΘΚ, καὶ διήχθω ἀπὸ τοῦ Β εὐθεῖα ἡ ΒΛ δίχα αὐτὰς τέμνουσα. φησὶν οὖν, ὅτι τῆς ΑΒΓ γραμμῆς διάμετρον μὲν καλῶ τὴν ΒΛ, κορυφὴν δὲ τὸ Β, τεταγμένως δὲ ἐπὶ τὴν ΒΛ κατῆχθαι ἑκάστην τῶν ΑΓ, ΔΕ, ΖΗ, ΘΚ. εἰ δὲ ἡ ΒΛ δίχα καὶ πρὸς ὀρθὰς τέμνει τὰς παραλλήλους, ἄξων καλεῖται.

Ὁμοίως δὲ καὶ δύο καμπύλων γραμμῶν καὶ τὰ ἑξῆς. ἐὰν γὰρ νοήσωμεν τὰς Α, Β γραμμὰς καὶ ἐν αὐταῖς τὰς ΓΔ, ΕΖ, ΗΘ, ΚΛ, ΜΝ, ΞΟ παραλλήλους καὶ τὴν ΑΒ διηγμένην ἐφ’ ἑκάτερα καὶ τέμνουσαν τὰς παραλλήλους δίχα, τὴν μὲν ΑΒ καλῶ, φησίν, πλαγίαν διάμετρον, κορυφὰς δὲ τῶν γραμμῶν τὰ Α, Β σημεῖα, τεταγμένως δὲ ἐπὶ τὴν ΑΒ τὰς ΓΔ, ΕΖ, ΗΘ, ΚΛ, ΜΝ, ΞΟ. εἰ δὲ δίχα καὶ πρὸς ὀρθὰς αὐτὰς τέμνει, ἄξων καλεῖται. ἐὰν δὲ διαχθεῖσά τις εὐθεῖα ὡς ἡ ΠΡ τὰς ΓΞ, ΕΜ, ΗΚ παραλλήλους τῇ ΑΒ δίχα τέμνει, ὀρθία μὲν διάμετρος καλεῖται ἡ ΠΡ, τεταγμένως δὲ κατῆχθαι ἐπὶ τὴν ΠΡ διάμετρον ἑκάστη τῶν ΓΞ, ΕΜ, ΗΚ. εἰ δὲ δίχα καὶ πρὸς ὀρθὰς αὐτὴν τέμνει, ἄξων ὀρθός, ἐὰν δὲ αἱ ΑΒ, ΠΡ

202 δίχα τέμνουσι τὰς ἀλλήλων παραλλήλους, λέγονται συζυγεῖς διάμετροι, ἐὰν δὲ δίχα καὶ πρὸς ὀρθάς, συζυγεῖς ἄξονες ὀνομάζονται.

Εἰς τὸ α′.

Περὶ τῶν διαφόρων καταγραφῶν ἤτοι πτώσεων τῶν θεωρημάτων τοσοῦτον ἰστέον, ὅτι πτῶσις μέν ἐστιν, ὅταν τὰ ἐν τῇ προτάσει δεδομένα τῇ θέσει ᾖ δοθέντα· ἡ γὰρ διάφορος αὐτῶν μετάληψις τοῦ αὐτοῦ συμπεράσματος ὄντος ποιεῖ τὴν πτῶσιν. ὁμοίως δὲ καὶ ἀπὸ τῆς κατασκευῆς μετατιθεμένης γίνεται πτῶσις. πολλὰς δὲ ἐχόντων τῶν θεωρημάτων πάσαις ἡ αὐτὴ ἀπόδειξις ἁρμόζει καὶ ἐπὶ τῶν αὐτῶν στοιχείων πλὴν βραχέων, ὡς ἑξῆς εἰσόμεθα· εὐθὺς γὰρ τὸ πρῶτον θεώρημα τρεῖς πρώσεις ἔχει διὰ τὸ τὸ λαμβανόμενον σημεῖον ἐπὶ τῆς ἐπιφανείας, τουτέστι τὸ Β, ποτὲ μὲν εἰς τὴν κατωτέρω ἐπιφάνειαν εἶναι καὶ τοῦτο διχῶς ἢ ἀνωτέρω τοῦ κύκλου ἢ κατωτέρω, ποτὲ δὲ ἐπὶ τῆς κατὰ κορυφὴν αὐτῇ ἐπικειμένης. τοῦτο δὲ τὸ θεώρημα προέθετο ζητῆσαι, ὅτι οὐκ ἐπὶ πάντα δύο σημεῖα ἐπὶ τῆς ἐπιφανείας λαμβανόμενα ἐπιζευγνυμένη εὐθεῖα ἐπὶ τῆς ἐπιφανείας ἐστίν, ἀλλ' ἡ νεύουσα μόνον ἐπὶ τὴν κορυφήν, διὰ τὸ καὶ ὑπὸ εὐθείας τὸ πέρας ἐχούσης μένον γεγενῆσθαι τὴν κωνικὴν ἐπιφάνειαν. ὅτι δὲ τοῦτο ἀληθές, τὸ δεύτερον θεώρημα δηλοῖ.

Εἰς τὸ β′.

Τὸ δεύτερον θεώρημα τρεῖς ἔχει πτώσεις διὰ τὸ

204 τὰ λαμβανόμενα σημεῖα τὰ Δ, Ε ἢ ἐπὶ τῆς κατὰ κο-

ρυφὴν εἶναι ἐπιφανείας ἢ ἐπὶ τῆς κάτω διχῶς ἢ ἐσωτέρω τοῦ κύκλου ἢ ἐξωτέρω. δεῖ δὲ ἐφιστάνειν, ὅτι τοῦτο τὸ θεώρημα εὑρίσκεται ἔν τισιν ἀντιγράφοις ὅλον διὰ τῆς εἰς ἀδύνατον ἀπαγωγῆς δεδειγμένον.

Εἰς τὸ γʹ.

Τὸ γʹ θεώρημα πτῶσιν οὐκ ἔχει. δεῖ δὲ ἐν αὐτῷ ἐπιστῆσαι, ὅτι ἡ ΑΒ εὐθεῖά ἐστι διὰ τὸ κοινὴ τομὴ εἶναι τοῦ τέμνοντος ἐπιπέδου καὶ τῆς ἐπιφανείας τοῦ κώνου, ἥτις ὑπὸ εὐθείας ἐγράφη τὸ πέρας ἐχούσης μένον πρὸς τῇ κορυφῇ τῆς ἐπιφανείας. οὐ γὰρ πᾶσα ἐπιφάνεια ὑπὸ ἐπιπέδου τεμνομένη τὴν τομὴν ποιεῖ εὐθεῖαν, οὐδὲ αὐτὸς ὁ κῶνος, εἰ μὴ διὰ τῆς κορυφῆς ἔλθῃ τὸ τέμνον ἐπίπεδον.

Εἰς τὸ δʹ.

Αἱ πτώσεις τούτου τοῦ θεωρήματος τρεῖς εἰσιν ὥσπερ καὶ τοῦ πρώτου καὶ δευτέρου.

Εἰς τὸ εʹ.

Τὸ πέμπτον θεώρημα πτῶσιν οὐκ ἔχει. ἀρχόμενος δὲ τῆς ἐκθέσεώς φησιν· τετμήσθω ὁ κῶνος ἐπιπέδῳ διὰ τοῦ ἄξονος ὀρθῷ πρὸς τὴν βάσιν.
ἐπειδὴ δὲ ἐν τῷ σκαληνῷ κώνῳ κατὰ μίαν μόνον θέσιν τὸ διὰ τοῦ ἄξονος τρίγωνον ὀρθόν ἐστι πρὸς τὴν βάσιν, τοῦτο ποιήσομεν οὕτως· λαβόντες τὸ κέντρον τῆς βάσεως ἀναστήσομεν ἀπ’ αὐτοῦ τῷ ἐπιπέδῳ τῆς βάσεως πρὸς ὀρθὰς καὶ δι’ αὐτῆς καὶ τοῦ ἄξονος ἐκβάλλοντες ἐπίπεδον ἕξομεν τὸ ζητούμενον· δέδεικται

206 γὰρ ἐν τῷ ια′ τῆς Εὐκλείδου στοιχειώσεως, ὅτι, ἐὰν εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι’ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται. τὸν δὲ κῶνον σκαληνὸν ὑπέθετο, ἐπειδὴ ἐν τῷ ἰσοσκελεῖ τὸ παράλληλον τῇ βάσει ἐπίπεδον τῷ ὑπεναντίως ἠγμένῳ τὸ αὐτό ἐστιν.

ἔτι φησίν· τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ πρὸς ὀρθὰς μὲν τῷ διὰ τοῦ ἄξονος τριγώνῳ, ἀφαιροῦντι δὲ πρὸς τῇ κορυφῇ τρίγωνον ὅμοιον μὲν τῷ ΑΒΓ τριγώνῳ, ὑπεναντίως δὲ κείμενον. τοῦτο δὲ γίνεται οὕτως· ἔστω τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ, καὶ εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Η, καὶ συνεστάτω πρὸς τῇ ΑΗ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Η τῇ ὑπὸ ΑΓΒ γωνίᾳ ἴση ἡ ὑπὸ ΑΗΚ· τὸ ΑΗΚ ἄρα τρίγωνον τῷ ΑΒΓ ὅμοιον μέν ἐστιν, ὑπεναντίως δὲ κείμενον. εἰλήφθω δὴ ἐπὶ τῆς ΗΚ τυχὸν σημεῖον τὸ Ζ, καὶ ἀπὸ τοῦ Ζ τῷ τοῦ ΑΒΓ τριγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΖΘ, καὶ ἐκβεβλήσθω τὸ διὰ τῶν ΗΚ, ΘΖ ἐπίπεδον. τοῦτο δὴ ὀρθόν ἐστι πρὸς τὸ ΑΒΓ τρίγωνον διὰ τὴν ΖΘ καὶ ποιοῦν τὸ προκείμενον.

ἐν τῷ συμπεράσματί φησιν, ὅτι διὰ τὴν ὁμοιότητα τῶν ΔΖΗ, ΕΖΚ τριγώνων ἴσον ἐστὶ τὸ ὑπὸ ΔΖΕ τῷ ὑπὸ ΗΖΚ. δυνατὸν δέ ἐστι τοῦτο δεῖξαι καὶ δίχα τῆς τῶν τριγώνων ὁμοιότητος λέγοντα, ὅτι, ἐπειδὴ
208 ἑκατέρα τῶν ὑπὸ ΑΚΗ, ΑΔΕ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ Β, ἐν τῷ αὐτῷ τμήματί εἰσι τοῦ περιλαμ-

βάνοντος κύκλου τὰ Δ, Η, Ε, Κ σημεῖα. καὶ ἐπειδὴ ἐν κύκλῳ δύο εὐθεῖαι αἱ ΔΕ, ΗΚ τέμνουσιν ἀλλήλας κατὰ τὸ Ζ, τὸ ὑπὸ ΔΖΕ ἴσον ἐστὶ τῷ ὑπὸ ΗΖΚ.

ὁμοίως δὴ δειχθήσεται, ὅτι καὶ πᾶσαι αἱ ἀπὸ τῆς ΗΘ γραμμῆς ἐπὶ τὴν ΗΚ κάθετοι ἀγόμεναι ἴσον δύνανται τῷ ὑπὸ τῶν τμημάτων. κύκλος ἄρα ἐστὶν ἡ τομή, διάμετρος δὲ αὐτοῦ ἡ ΗΚ. καὶ δυνατὸν μέν ἐστιν ἐπιλογίσασθαι τοῦτο διὰ τῆς εἰς ἀδύνατον ἀπαγωγῆς. εἰ γὰρ ὁ περὶ τὴν ΚΗ γραφόμενος κύκλος οὐχ ἥξει διὰ τοῦ Θ σημείου, ἔσται τὸ ὑπὸ τῶν ΚΖ, ΖΗ ἴσον ἤτοι τῷ ἀπὸ μείζονος τῆς ΖΘ ἢ τῷ ἀπὸ ἐλάσσονος· ὅπερ οὐχ ὑπόκειται. δείξομεν δὲ αὐτὸ καὶ ἐπ' εὐθείας.

ἔστω τις γραμμὴ ἡ ΗΘ, καὶ ὑποτεινέτω αὐτὴν ἡ ΗΚ, εἰλήφθω δὲ καὶ ἐπὶ τῆς γραμμῆς τυχόντα σημεῖα τὰ Θ, Ο, καὶ ἀπ' αὐτῶν ἐπὶ τὴν ΗΚ κάθετοι ἤχθωσαν αἱ ΘΖ, ΟΠ, καὶ ἔστω τὸ μὲν ἀπὸ ΖΘ ἴσον τῷ ὑπὸ ΗΖΚ, τὸ δὲ ἀπὸ ΟΠ τῷ ὑπὸ ΗΠΚ ἴσον. λέγω, ὅτι κύκλος ἐστὶν ἡ ΗΘΟΚ γραμμή. τετμήσθω γὰρ ἡ ΗΚ δίχα κατὰ τὸ Ν, καὶ ἐπεζεύχθωσαν αἱ ΝΘ, ΝΟ. ἐπεὶ οὖν εὐθεῖα ἡ ΗΚ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Ν, εἰς δὲ ἄνισα κατὰ τὸ Ζ, τὸ ὑπὸ ΗΖΚ (25) μετὰ τοῦ ἀπὸ ΝΖ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. τὸ δὲ

210 ὑπὸ ΗΖΚ ἴσον ὑπόκειται τῷ ἀπὸ ΘΖ· τὸ ἄρα ἀπὸ ΘΖ μετὰ τοῦ ἀπὸ ΝΖ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. ἴσα δέ ἐστι τὰ ἀπὸ ΘΖ, ΖΝ τῷ ἀπὸ ΝΘ· ὀρθὴ γάρ ἐστιν

ἡ πρὸς τῷ Ζ· τὸ ἄρα ἀπὸ ΝΘ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ ἀπὸ ΝΟ ἴσον ἐστὶ τῷ ἀπὸ ΝΚ. κύκλος ἄρα ἐστὶν ἡ ΗΘΚ γραμμή, διάμετρος δὲ αὐτοῦ ἡ ΗΚ.

δυνατὸν δέ ἐστι τὰς ΔΕ, ΗΚ διαμέτρους ποτὲ μὲν ἴσας, ποτὲ δὲ ἀνίσους εἶναι, οὐδέποτε μέντοι δίχα τέμνουσιν ἀλλήλας. ἤχθω γὰρ διὰ τοῦ Κ τῇ ΒΓ παράλληλος ἡ ΝΚ. ἐπεὶ οὖν μείζων ἐστὶν ἡ ΒΑ τῆς ΑΓ, μείζων ἄρα καὶ ἡ ΝΑ τῆς ΑΚ. ὁμοίως δὲ καὶ ἡ ΚΑ τῆς ΑΗ διὰ τὴν ὑπεναντίαν τομήν. ὥστε ἡ τῇ ΑΚ ἀπὸ τῆς ΑΝ ἴση λαμβανομένη μεταξὺ πίπτει τῶν Η, Ν σημείων. πιπτέτω ὡς ἡ ΑΞ· ἡ ἄρα διὰ τοῦ Ξ τῇ ΒΓ παράλληλος ἀγομένη τέμνει τὴν ΗΚ. τεμνέτω ὡς ἡ ΞΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΞΑ τῇ ΑΚ, ὡς δὲ ἡ ΞΑ πρὸς ΑΠ, ἡ ΚΑ πρὸς ΑΗ διὰ τὴν ὁμοιότητα τῶν ΗΚΑ, ΞΑΠ τριγώνων, ἡ ΑΗ τῇ ΑΠ ἐστιν ἴση καὶ λοιπὴ ἡ ΗΞ τῇ ΠΚ. καὶ ἐπεὶ αἱ πρὸς τοῖς Ξ, Κ γωνίαι ἴσαι εἰσίν· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ Β· εἰσὶ δὲ καὶ αἱ πρὸς τῷ Ο ἴσαι· κατὰ κορυφὴν γάρ· ὅμοιον ἄρα ἐστὶ τὸ ΞΗΟ τρίγωνον τῷ ΠΟΚ τριγώνῳ. καὶ ἴση ἐστὶν ἡ ΗΞ τῇ ΠΚ· ὥστε καὶ ἡ ΞΟ τῇ ΟΚ καὶ ἡ ΗΟ τῇ ΟΠ καὶ
212 ὅλη ἡ ΗΚ τῇ ΞΠ. καὶ φανερόν, ὅτι, ἐὰν μεταξὺ τῶν Ν, Ξ ληφθῇ τι σημεῖον ὡς τὸ Ρ, καὶ διὰ τοῦ Ρ τῇ ΝΚ παράλληλος ἀχθῇ ἡ ΡΣ, μείζων ἔσται τῆς ΞΠ καὶ διὰ τοῦτο καὶ τῆς ΗΚ, ἐὰν δὲ μεταξὺ τῶν Η, Ξ ληφθῇ τι σημεῖον οἷον τὸ Τ, καὶ δι’ αὐτοῦ παράλλη-

λος ἀχθῇ ἡ ΤΠ, ἐλάττων ἔσται τῆς ΞΠ καὶ τῆς ΚΗ. καὶ ἐπεὶ ἡ ὑπὸ ΞΠΚ γωνία μείζων ἐστὶ τῆς ὑπὸ ΑΞΠ, ἴση δὲ ἡ ὑπὸ ΟΠΚ τῇ ὑπὸ ΟΗΞ, μείζων ἄρα καὶ ἡ ὑπὸ ΟΗΞ τῆς ὑπὸ ΗΞΟ. ἡ ΞΟ ἄρα τῆς ΟΗ μείζων καὶ διὰ τοῦτο καὶ ἡ ΚΟ τῆς ΟΠ. ἐὰν δέ ποτε ἡ ἑτέρα αὐτῶν δίχα διαιρεθῇ, ἡ λοιπὴ εἰς ἄνισα τμηθήσεται.

Εἰς τὸ ϛʹ.

Προσέχειν χρή, ὅτι οὐ μάτην πρόσκειται ἐν τῇ προτάσει τὸ δεῖν τὴν ἀγομένην εὐθεῖαν ἀπὸ τοῦ ἐν τῇ ἐπιφανείᾳ σημείου παράλληλον μιᾷ τινι τῶν ἐν τῇ βάσει εὐθειῶν πρὸς ὀρθὰς οὔσῃ πάντως τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἄγεσθαι παράλληλον· τούτου γὰρ μὴ ὄντος οὐ δυνατόν ἐστιν αὐτὴν δίχα τέμνεσθαι ὑπὸ τοῦ διὰ τοῦ ἄξονος τριγώνου· ὅπερ ἐστὶ φανερὸν ἐκ τῆς ἐν τῷ ῥητῷ καταγραφῆς. εἰ γὰρ ἡ ΜΝ, ᾗτινι παράλληλός ἐστιν ἡ ΔΖΗ, μὴ πρὸς ὀρθὰς εἴη τῇ ΒΓ, δῆλον, ὅτι οὐδὲ δίχα τέμνεται οὐδὲ ἡ ΚΛ. καὶ διὰ τῶν αὐτῶν λόγων συνάγεται, ὅτι ἐστίν, ὡς ἡ ΚΘ πρὸς ΘΛ, οὕτως ἡ ΔΖ πρὸς ΖΗ· καὶ ἡ ΔΗ ἄρα εἰς ἄνισα τμηθήσεται κατὰ τὸ Ζ.

δυνατὸν δὲ κατωτέρω τοῦ κύκλου καὶ ἐπὶ τῆς κατὰ κορυφὴν ἐπιφανείας τὰ αὐτὰ δείκνυσθαι.

214 **Εἰς τὸ ζʹ.**

Τὸ ζʹ θεώρημα πτώσεις ἔχει τέσσαρας· ἢ γὰρ οὐ συμβάλλει ἡ ΖΗ τῇ ΑΓ ἢ συμβάλλει τριχῶς ἢ ἐκτὸς τοῦ κύκλου ἢ ἐντὸς ἢ ἐπὶ τοῦ Γ σημείου.

Μετὰ τὸ ι΄.

Χρὴ ἐπιστῆσαι, ὅτι τὰ ι ταῦτα θεωρήματα ἀλλήλων ἔχονται. ἀλλὰ τὸ πρῶτον ἔχει, ὅτι αἱ ἐν τῇ ἐπιφανείᾳ εὐθεῖαι νεύουσαι ἐπὶ τὴν κορυφὴν ἐν ταύτῃ μένουσιν, τὸ δὲ δεύτερον τὸ ἀνάπαλιν, τὸ δὲ τρίτον ἔχει τὴν διὰ τῆς κορυφῆς τοῦ κώνου τομήν, τὸ δὲ τέταρτον τὴν παράλληλον τῇ βάσει, τὸ πέμπτον τὴν ὑπεναντίαν, τὸ ἕκτον ὡσανεὶ προλαμβάνεται τοῦ ἑβδόμου δεικνύον, ὅτι καὶ πρὸς ὀρθὰς ὀφείλει πάντως εἶναι τῇ διαμέτρῳ τοῦ κύκλου ἡ κοινὴ τομὴ αὐτοῦ καὶ τοῦ τέμνοντος ἐπιπέδου, καὶ ὅτι τούτου οὕτως ἔχοντος αἱ παράλληλοι αὐτῇ διχοτομοῦνται ὑπὸ τοῦ τριγώνου, τὸ δὲ ἕβδομον τὰς ἄλλας τρεῖς τομὰς ἔδειξε καὶ τὴν διάμετρον καὶ τὰς ἐπ’ αὐτὴν καταγομένας παραλλήλους τῇ ἐν τῇ βάσει εὐθείᾳ. ἐν δὲ τῷ ὀγδόῳ δείκνυσιν, ὅπερ ἐν τοῖς προλεγομένοις εἴπομεν, ὅτι ἡ παραβολὴ καὶ ἡ ὑπερβολὴ τῶν εἰς ἄπειρόν εἰσιν αὐξομένων, ἐν δὲ τῷ ἐνάτῳ, ὅτι ἡ ἔλλειψις συννεύουσα εἰς ἑαυτὴν ὁμοίως τῷ κύκλῳ διὰ τὸ τὸ τέμνον ἐπίπεδον συμπίπτειν ἀμφοτέραις ταῖς πλευραῖς τοῦ τριγώνου οὐκ ἔστι κύκλος· κύκλους γὰρ ἐποίουν ἥ τε ὑπεναντία τομὴ καὶ ἡ παράλληλος· καὶ δεῖ ἐπιστῆσαι, ὅτι ἡ διάμετρος τῆς τομῆς ἐπὶ μὲν τῆς παραβολῆς
216 τὴν μίαν πλευρὰν τοῦ τριγώνου τέμνει καὶ τὴν βάσιν, ἐπὶ δὲ τῆς ὑπερβολῆς τήν τε πλευρὰν καὶ τὴν ἐπ’ εὐθείας τῇ λοιπῇ πλευρᾷ ἐκβαλλομένην πρὸς τῇ κορυφῇ, ἐπὶ δὲ τῆς ἐλλείψεως καὶ ἑκατέραν τῶν πλευ-

ρῶν καὶ τὴν βάσιν. τὸ δὲ δέκατον ἁπλούστερον μέν τις ἐπιβάλλων ἴσως ἂν οἰηθείη ταὐτὸν εἶναι τῷ δευτέρῳ, τοῦτο μέντοι οὐχ ὧς ἔχει· ἐκεῖ μὲν γὰρ ἐπὶ πάσης τῆς ἐπιφανείας ἔλεγε λαμβάνεσθαι τὰ δύο σημεῖα, ἐνταῦθα δὲ ἐπὶ τῆς γενομένης γραμμῆς. ἐν δὲ τοῖς ἑξῆς τρισὶν ἀκριβέστερον ἑκάστην τῶν τομῶν τούτων διακρίνει μετὰ τοῦ λέγειν καὶ τὰ ἰδιώματα αὐτῶν τὰ ἀρχικά.

Εἰς τὸ ια′.

Πεποιήσθω, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ, οὕτως ἡ ΘΖ πρὸς ΖΑ· σαφὲς μέν ἐστι τὸ λεγόμενον, πλὴν εἴ τις καὶ ὑπομνησθῆναι βούλεται. ἔστω τῷ ὑπὸ ΒΑΓ ἴσον τὸ ὑπὸ ΟΠΡ, τῷ δὲ ἀπὸ ΒΓ ἴσον παρὰ τὴν ΠΡ παραβληθὲν πλάτος ποιείτω τὴν ΠΣ, καὶ γεγονέτω, ὡς ἡ ΟΠ πρὸς ΠΣ, ἡ ΑΖ πρὸς ΖΘ· γέγονεν ἄρα τὸ ζητούμενον. ἐπεὶ γάρ ἐστιν, ὡς ἡ ΟΠ πρὸς ΠΣ, ἡ ΑΖ πρὸς ΖΘ, ἀνάπαλιν ὡς ἡ ΣΠ πρὸς ΠΟ, ἡ ΘΖ πρὸς ΖΑ. ὡς δὲ ἡ ΣΠ πρὸς ΠΟ, τὸ ΣΡ πρὸς ΡΟ, τουτέστι τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ. τοῦτο χρησιμεύει καὶ τοῖς ἑξῆς δύο θεωρήμασιν.

218 Τὸ δὲ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ λόγον ἔχει τὸν συγκείμενον ἐκ τοῦ ὃν ἔχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ· δέδεικται μὲν ἐν τῷ ἕκτῳ βιβλίῳ τῆς στοιχειώσεως ἐν τῷ εἰκοστῷ τρίτῳ θεωρήματι, ὅτι τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ἐπεὶ δὲ

ἐπακτικώτερον μᾶλλον καὶ οὐ κατὰ τὸν ἀναγκαῖον τρόπον ὑπὸ τῶν ὑπομνηματιστῶν ἐλέγετο, ἐζητήσαμεν αὐτὸ καὶ γέγραπται ἐν τοῖς ἐκδεδομένοις ἡμῖν εἰς τὸ τέταρτον θεώρημα τοῦ δευτέρου βιβλίου τῶν Ἀρχιμήδους περὶ σφαίρας καὶ κυλίνδρου καὶ ἐν τοῖς σχολίοις τοῦ πρώτου βιβλίου τῆς Πτολεμαίου συντάξεως· οὐ χεῖρον δὲ καὶ ἐνταῦθα τοῦτο γραφῆναι διὰ τὸ μὴ πάντως τοὺς ἀναγινώσκοντας κἀκείνοις ἐντυγχάνειν, καὶ ὅτι σχεδὸν τὸ ὅλον σύνταγμα τῶν κωνικῶν κέχρηται αὐτῷ.

λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ’ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθμοῦ, οὗ παρώνυμός ἐστιν ὁ λόγος. ἐπὶ μὲν οὖν τῶν πολλαπλασίων δυνατόν ἐστιν ἀριθμὸν ὁλόκληρον εἶναι τὴν πηλικότητα, ἐπὶ δὲ τῶν λοιπῶν σχέσεων ἀνάγκη τὴν πηλικότητα ἀριθμὸν εἶναι καὶ μόριον ἢ μόρια, εἰ μὴ ἄρα τις ἐθέλοι καὶ ἀρρήτους εἶναι σχέσεις, οἷαί εἰσιν αἱ κατὰ τὰ ἄλογα μεγέθη. ἐπὶ πασῶν δὲ τῶν σχέσεων δῆλον, ὅτι αὐτὴ ἡ πηλικότης πολλαπλασιαζομένη ἐπὶ τὸν ἑπόμενον ὅρον τοῦ λόγου ποιεῖ τὸν ἡγούμενον.

ἔστω τοίνυν λόγος ὁ τοῦ Α πρὸς τὸν Β, καὶ εἰ-

220 λήφθω τις αὐτῶν μέσος, ὡς ἔτυχεν, ὁ Γ, καὶ ἔστω τοῦ Α, Γ λόγου πηλικότης ὁ Δ, τοῦ δὲ Γ, Β ὁ Ε, καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Ζ ποιείτω. λέγω, ὅτι τοῦ λόγου τῶν Α, Β πηλικότης ἐστὶν ὁ Ζ, τουτέστιν ὅτι ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α ποιεῖ. ὁ δὴ Ζ τὸν Β πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ

οὖν ὁ Δ τὸν μὲν Ε πολλαπλασιάσας τὸν Ζ πεποίηκεν, τὸν δὲ Γ πολλαπλασιάσας τὸν Α πεποίηκεν, ἔστιν ἄρα, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α. πάλιν ἐπεὶ ὁ Β τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν ἄρα, ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Γ πρὸς τὸν Η. ἐναλλάξ, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Η. ἦν δέ, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α· ἴσος ἄρα ὁ Η τῷ Α. ὥστε ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α πεποίηκεν.

μὴ ταραττέτω δὲ τοὺς ἐντυγχάνοντας τὸ διὰ τῶν ἀριθμητικῶν δεδεῖχθαι τοῦτο· οἵ τε γὰρ παλαιοὶ κέχρηνται ταῖς τοιαύταις ἀποδείξεσι μαθηματικαῖς μᾶλλον οὔσαις ἢ ἀριθμητικαῖς διὰ τὰς ἀναλογίας, καὶ ὅτι τὸ ζητούμενον ἀριθμητικόν ἐστιν. λόγοι γὰρ καὶ πηλικότητες λόγων καὶ πολλαπλασιασμοὶ τοῖς ἀριθμοῖς πρώτως ὑπάρχουσι καὶ δι' αὐτῶν τοῖς μεγέθεσι, κατὰ τὸν εἰπόντα· ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά.

222 **Εἰς τὸ ιγ′.** Δεῖ σημειώσασθαι, ὅτι τοῦτο τὸ θεώρημα τρεῖς ἔχει καταγραφάς, ὡς καὶ πολλάκις εἴρηται ἐπὶ τῆς ἐλλείψεως· ἡ γὰρ ΔΕ ἢ ἀνωτέρω τοῦ Γ συμπίπτει τῇ ΑΓ ἢ κατ' αὐτοῦ τοῦ Γ ἢ ἐξωτέρω ἐκβαλλομένη τῇ ΑΓ συμπίπτει.

Εἰς τὸ ιδ′.

Δυνατὸν ἦν καὶ οὕτως δεῖξαι, ὅτι, ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ

ΞΤΟ.

ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΒΓ τῇ ΞΟ, ἔστιν, ὡς ἡ ΓΣ πρὸς ΣΑ, ἡ ΞΤ πρὸς ΤΑ, καὶ διὰ τὰ αὐτά, ὡς ἡ ΑΣ πρὸς ΣΒ, ἡ ΑΤ πρὸς ΤΟ· δι' ἴσου ἄρα, ὡς ἡ ΓΣ πρὸς ΣΒ, ἡ ΞΤ πρὸς ΤΟ. καὶ ὡς ἄρα τὸ ἀπὸ ΓΣ πρὸς τὸ ὑπὸ ΓΣΒ, τὸ ἀπὸ ΞΤ πρὸς τὸ ὑπὸ ΞΤΟ. ἔστι δὲ διὰ τὴν ὁμοιότητα τῶν τριγώνων, ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ἀπὸ ΣΓ, τὸ ἀπὸ ΑΤ πρὸς τὸ ἀπὸ ΞΤ· δι' ἴσου ἄρα, ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ. καί ἐστιν, ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΞΤΟ, ἡ ΘΕ πρὸς ΘΡ· καὶ ὡς ἄρα ἡ ΘΕ πρὸς ΕΠ, ἡ ΕΘ πρὸς ΘΡ. ἴση ἄρα ἐστὶν ἡ ΕΠ τῇ ΘΡ. πτῶσιν μὲν οὖν οὐκ ἔχει, φανερὸς δέ ἐστιν ὁ σκοπὸς συνεχὴς ὢν τοῖς πρὸ αὐτοῦ τρισίν· ὁμοίως γὰρ ἐκείνοις τὴν διάμετρον τῶν ἀντικειμένων ζητεῖ τὴν ἀρχικὴν καὶ τὰς παρ' ἃς δύνανται.

224 **Εἰς τὸ ιϛ΄.**

Ἴσον ἄρα τὸ ὑπὸ ΒΚΑ τῷ ὑπὸ ΑΛΒ· ἴση ἄρα ἐστὶν ἡ ΚΑ τῇ ΒΛ· ἐπεὶ γὰρ τὸ ὑπὸ ΒΚΑ τῷ ὑπὸ ΑΛΒ ἐστιν ἴσον, ἀνάλογον ἔσται, ὡς ἡ ΚΒ πρὸς ΑΛ, ἡ ΛΒ πρὸς ΑΚ. καὶ ἐναλλάξ, ὡς ἡ ΚΒ πρὸς ΒΛ, ἡ ΛΑ πρὸς ΑΚ· καὶ συνθέντι, ὡς ἡ ΚΛ πρὸς ΛΒ, ἡ ΛΚ πρὸς ΚΑ· ἴση ἄρα ἡ ΚΑ τῇ ΒΛ.

δεῖ ἐπιστῆσαι, ὅτι ἐν τῷ πεντεκαιδεκάτῳ καὶ ἑκκαιδεκάτῳ θεωρήματι σκοπὸν ἔσχε ζητῆσαι τὰς καλουμένας δευτέρας καὶ συζυγεῖς διαμέτρους τῆς ἐλλείψεως καὶ τῆς ὑπερβολῆς ἤτοι τῶν ἀντικειμένων· ἡ γὰρ παραβολὴ οὐκ ἔχει τοιαύτην διάμετρον. παρατηρητέον δέ, ὅτι αἱ μὲν τῆς ἐλλείψεως διάμετροι ἐντὸς ἀπολαμβάνονται, αἱ δὲ τῆς ὑπερβολῆς καὶ τῶν ἀντικειμένων ἐκτός. καταγράφοντας δὲ δεῖ τὰς μὲν παρ' ἃς δύνανται ἤτοι τὰς ὀρθίας πλευρὰς πρὸς ὀρθὰς τάττειν καὶ δηλονότι καὶ τὰς παραλλήλους αὐταῖς, τὰς δὲ τεταγμένως καταγομένας καὶ τὰς δευτέρας διαμέτρους οὐ πάντως· μάλιστα γὰρ ἐν ὀξείᾳ γωνίᾳ δεῖ κατάγειν αὐτάς, ἵνα σαφεῖς ὦσιν τοῖς ἐντυγχάνουσιν ἕτεραι οὖσαι τῶν παραλλήλων τῇ ὀρθίᾳ πλευρᾷ.

Μετὰ τὸ ἑκκαιδέκατον θεώρημα ὅρους ἐκτίθεται περὶ τῆς καλουμένης δευτέρας διαμέτρου τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, οὓς διὰ καταγραφῆς σαφεῖς ποιήσομεν.

ἔστω ὑπερβολὴ ἡ ΑΒ, διάμετρος δὲ αὐτῆς ἔστω ἡ ΓΒΔ, παρ' ἣν δὲ δύνανται αἱ ἐπὶ τὴν ΒΓ κατ-

226 αγόμεναι ἡ ΒΕ. φανερὸν οὖν, ὅτι ἡ μὲν ΒΓ εἰς ἄπειρον αὔξεται διὰ τὴν τομήν, ὡς δέδεικται ἐν τῷ ὀγδόῳ θεωρήματι, ἡ δὲ ΒΔ, ἥτις ἐστὶν ἡ ὑποτείνουσα τὴν ἐκτὸς τοῦ διὰ τοῦ ἄξονος τριγώνου γωνίαν πεπέρασται. ταύτην δὴ διχοτομοῦντες κατὰ τὸ Ζ καὶ ἀγαγόντες ἀπὸ τοῦ Α τεταγμένως κατηγμένην τὴν ΑΗ, διὰ δὲ τοῦ Ζ τῇ ΑΗ παράλληλον τὴν ΘΖΚ καὶ ποιήσαντες τὴν ΘΖ τῇ ΖΚ ἴσην, ἔτι μέντοι καὶ τὸ ἀπὸ ΘΚ ἴσον τῷ ὑπὸ ΔΒΕ, ἕξομεν τὴν ΘΚ δευτέραν διά-

μετρον. τοῦτο γὰρ δυνατὸν διὰ τὸ τὴν ΘΚ ἐκτὸς οὖσαν τῆς τομῆς εἰς ἄπειρον ἐκβάλλεσθαι καὶ δυνατὸν εἶναι ἀπὸ τῆς ἀπείρου προτεθείσῃ εὐθείᾳ ἴσην ἀφελεῖν. τὸ δὲ Ζ κέντρον καλεῖ, τὴν δὲ ΖΒ καὶ τὰς ὁμοίως αὐτῇ ἀπὸ τοῦ Ζ πρὸς τὴν τομὴν φερομένας ἐκ τοῦ κέντρου.

ταῦτα μὲν ἐπὶ τῆς ὑπερβολῆς καὶ τῶν ἀντικειμένων· καὶ φανερόν, ὅτι πεπερασμένη ἐστὶν ἑκατέρα τῶν διαμέτρων, ἡ μὲν πρώτη αὐτόθεν ἐκ τῆς γενέσεως τῆς τομῆς, ἡ δὲ δευτέρα, διότι μέση ἀνάλογόν ἐστι πεπερασμένων εὐθειῶν τῆς τε πρώτης διαμέτρου καὶ τῆς παρ᾽ ἣν δύνανται αἱ καταγόμεναι ἐπ᾽ αὐτὴν τεταγμένως.

ἐπὶ δὲ τῆς ἐλλείψεως οὔπω δῆλον τὸ λεγόμενον. ἐπειδὴ γὰρ εἰς ἑαυτὴν συννεύει, καθάπερ ὁ κύκλος, καὶ ἐντὸς ἀπολαμβάνει πάσας τὰς διαμέτρους καὶ ὡρισμένας αὐτὰς ἀπεργάζεται· ὥστε οὐ πάντως ἐπὶ τῆς ἐλλείψεως ἡ μέση ἀνάλογον τῶν τοῦ εἴδους πλευρῶν καὶ διὰ τοῦ κέντρου τῆς τομῆς ἀγομένη καὶ ὑπὸ τῆς διαμέτρου διχοτομουμένη ὑπὸ τῆς τομῆς περατοῦται·

228 δυνατὸν δὲ αὐτὴν συλλογίζεσθαι δι᾽ αὐτῶν τῶν εἰρημένων ἐν τῷ πεντεκαιδεκάτῳ θεωρήματι. ἐπεὶ γάρ, ὡς ἐκεῖ δέδεικται, αἱ ἐπὶ τὴν ΔΕ καταγόμεναι παράλληλοι τῇ ΑΒ δύνανται τὰ παρακείμενα παρὰ τὴν τρίτην αὐταῖς ἀνάλογον γινομένην, τουτέστι τὴν ΖΔ, ἔστιν, ὡς ἡ ΔΕ πρὸς τὴν ΑΒ, ἡ ΑΒ πρὸς ΔΖ· ὥστε μέση ἀνάλογόν ἐστιν ἡ ΑΒ τῶν ΕΔ, ΔΖ. καὶ διὰ τοῦτο καὶ αἱ κατα-

γόμεναι ἐπὶ τὴν ΑΒ παράλληλοι τῇ ΔΕ δυνήσονται τὰ παρὰ τὴν τρίτην ἀνάλογον παρακείμενα τῶν ΔΕ, ΑΒ, τουτέστι τὴν ΑΝ. διὰ δὴ τοῦτο μέση ἀνάλογον γίνεται ἡ ΔΕ δευτέρα διάμετρος τῶν ΒΑ, ΑΝ τοῦ εἴδους πλευρῶν.

δεῖ δὲ εἰδέναι καὶ τοῦτο διὰ τὸ εὔχρηστον τῶν καταγραφῶν· ἐπεὶ γὰρ ἄνισοί εἰσιν αἱ ΑΒ, ΔΕ διάμετροι· ἐν μόνῳ γὰρ τῷ κύκλῳ ἴσαι εἰσίν· δῆλον, ὅτι ἡ μὲν πρὸς ὀρθὰς ἀγομένη τῇ ἐλάσσονι αὐτῶν ὡς ἐνταῦθα ἡ ΔΖ ἅτε τρίτη ἀνάλογον οὖσα τῶν ΔΕ, ΑΒ μείζων ἐστὶν ἀμφοῖν, ἡ δὲ πρὸς ὀρθὰς ἀγομένη τῇ μείζονι ὡς ἐνταῦθα ἡ ΑΝ διὰ τὸ τρίτην ἀνάλογον εἶναι τῶν ΑΒ, ΔΕ ἐλάσσων ἐστὶν ἀμφοῖν· ὥστε καὶ συνεχῶς εἶναι τὰς τέσσαρας ἀνάλογον· ὡς γὰρ ἡ ΑΝ πρὸς ΔΕ, ἡ ΔΕ πρὸς ΑΒ καὶ ἡ ΑΒ πρὸς ΔΖ.

Εἰς τὸ ιζʹ.

Ὁ μὲν Εὐκλείδης ἐν τῷ πεντεκαιδεκάτῳ θεωρήματι τοῦ τρίτου βιβλίου τῆς στοιχειώσεως ἔδειξεν, ὅτι ἡ

230 πρὸς ὀρθὰς ἀγομένη ἀπ’ ἄκρας τῆς διαμέτρου ἐκτός τε πίπτει καὶ ἐφάπτεται τοῦ κύκλου, ὁ δὲ Ἀπολλώνιος ἐν τούτῳ καθολικόν τι δείκνυσι δυνάμενον ἐφαρμόσαι ταῖς τρισὶ τοῦ κώνου καὶ τῷ κύκλῳ.

τοσοῦτον διαφέρει ὁ κύκλος τῶν τοῦ κώνου τομῶν, ὅτι ἐπ’ ἐκείνου μὲν αἱ τεταγμένως κατηγμέναι πρὸς ὀρθὰς ἄγονται τῇ διαμέτρῳ· οὐδὲ γὰρ ἄλλαι εὐθεῖαι παράλληλοι ἑαυταῖς ὑπὸ τῆς διαμέτρου τοῦ κύκλου διχοτομοῦνται· ἐπὶ δὲ τῶν τριῶν τομῶν οὐ πάντως πρὸς ὀρθὰς ἄγονται, εἰ μὴ ἐπὶ μόνους τοὺς ἄξονας.

Εἰς τὸ ιη′.

Ἔν τισιν ἀντιγράφοις τὸ θεώρημα τοῦτο ἐπὶ μόνης παραβολῆς καὶ ὑπερβολῆς ἐστιν, κάλλιον δὲ καθολικώτερον ἔχειν τὴν πρότασιν, εἰ μὴ ὅτι τὸ ἐπὶ τῆς ἐλλείψεως ἐκείνοις ὡς ἀναμφίβολον παραλέλειπται· ἡ γὰρ ΓΔ ἐντὸς οὖσα τῆς τομῆς πεπερασμένης οὔσης καὶ αὐτὴ κατ’ ἀμφότερα τέμνει τὴν τομήν.

δεῖ δὲ ἐπιστῆσαι, ὅτι, κἂν ἡ ΑΖΒ τέμνῃ τὴν τομήν, ἡ αὐτὴ ἀπόδειξις ἁρμόζει.

Εἰς τὸ κ′.

Ἀπὸ τούτου τοῦ θεωρήματος ἀρχόμενος ἐφεξῆς ἐν πᾶσι τὰ συμπτώματα τῆς παραβολῆς αὐτῇ δείκνυσιν ὑπάρχοντα καὶ οὐκ ἄλλῃ τινί, ὡς ἐπὶ τὸ πολὺ δὲ τῇ ὑπερβολῇ καὶ τῇ ἐλλείψει καὶ τῷ κύκλῳ τὰ αὐτὰ δείκνυσιν ὑπάρχοντα.

ἐπειδὴ δὲ οὐκ ἄχρηστον φαίνεται τοῖς τὰ μηχα-
232 νικὰ γράφουσι διὰ τὴν ἀπορίαν τῶν ὀργάνων καὶ πολλάκις διὰ συνεχῶν σημείων γράφειν τὰς τοῦ κώνου τομὰς ἐν ἐπιπέδῳ, διὰ τούτου τοῦ θεωρήματος ἔστι πορίσασθαι συνεχῆ σημεῖα, δι’ ὧν γραφήσεται ἡ παραβολὴ κανόνος παραθέσει. ἐὰν γὰρ ἐκθῶμαι εὐ-

θεῖαν ὡς τὴν ΑΒ καὶ ἐπ’ αὐτῆς λάβω συνεχῆ σημεῖα ὡς τὰ Ε, Ζ καὶ ἀπ’ αὐτῶν πρὸς ὀρθὰς τῇ ΑΒ καὶ ποιήσω ὡς τὰς ΕΓ, ΖΔ λαβὼν ἐπὶ τῆς ΕΓ τυχὸν σημεῖον τὸ Γ, εἰ μὲν εὐρυτέραν βουληθείην ποιῆσαι παραβολήν, πόρρω τοῦ Ε, εἰ δὲ στενωτέραν, ἐγγύτερον, καὶ ποιήσω, ὡς τὴν ΑΕ πρὸς ΑΖ, τὸ ἀπὸ ΕΓ πρὸς τὸ ἀπὸ ΖΔ, τὰ Γ, Δ σημεῖα ἐπὶ τῆς τομῆς ἔσται. ὁμοίως δὲ καὶ ἄλλα ληψόμεθα, δι’ ὧν γραφήσεται ἡ παραβολή.

Εἰς τὸ κα΄.

Τὸ θεώρημα σαφῶς ἔκκειται καὶ πτῶσιν οὐκ ἔχει· δεῖ μέντοι ἐπιστῆσαι, ὅτι ἡ παρ’ ἣν δύνανται, τουτέστιν ἡ ὀρθία πλευρά, ἐπὶ τοῦ κύκλου ἴση ἐστὶ τῇ διαμέτρῳ. εἰ γάρ ἐστιν, ὡς τὸ ἀπὸ ΔΕ πρὸς τὸ ὑπὸ ΑΕΒ, ἡ ΓΑ πρὸς ΑΒ, ἴσον δὲ τὸ ἀπὸ ΔΕ τῷ ὑπὸ ΑΕΒ ἐπὶ τοῦ κύκλου μόνου, ἴση ἄρα καὶ ἡ ΓΑ τῇ ΑΒ.

δεῖ δὲ καὶ τοῦτο εἰδέναι, ὅτι αἱ καταγόμεναι ἐν τῇ τοῦ κύκλου περιφερείᾳ πρὸς ὀρθάς εἰσι πάντως τῇ διαμέτρῳ καὶ ἐπ’ εὐθείας γίνονται ταῖς παραλλήλοις τῇ ΑΓ.

διὰ δὲ τούτου τοῦ θεωρήματος τῷ αὐτῷ τρόπῳ τοῖς ἐπὶ τῆς παραβολῆς εἰρημένοις προσέχοντες γρά-

234 φομεν ὑπερβολὴν καὶ ἔλλειψιν κανόνος παραθέσει. ἐκκείσθω γὰρ εὐθεῖα ἡ ΑΒ καὶ προσεκβεβλήσθω ἐπ’ ἄπειρον ἐπὶ τὸ Η, καὶ ἀπὸ τοῦ Α ταύτῃ πρὸς ὀρθὰς ἤχθω ἡ ΑΓ, καὶ ἐπεζεύχθω ἡ ΒΓ καὶ ἐκβεβλήσθω,

καὶ εἰλήφθω τινὰ σημεῖα ἐπὶ τῆς ΑΗ τὰ Ε, Η, καὶ ἀπὸ τῶν Ε, Η τῇ ΑΓ παράλληλοι ἤχθωσαν αἱ ΕΘ, ΗΚ, καὶ γινέσθω τῷ μὲν ὑπὸ ΑΗΚ ἴσον τὸ ἀπὸ ΖΗ, τῷ δ’ ὑπὸ ΑΕΘ ἴσον τὸ ἀπὸ ΔΕ· διὰ γὰρ τῶν Α, Δ, Ζ ἥξει ἡ ὑπερβολή. ὁμοίως δὲ κατασκευάσομεν καὶ τὰ ἐπὶ τῆς ἐλλείψεως.

Εἰς τὸ κγ′.

Δεῖ ἐπιστῆσαι, ὅτι ἐν τῇ προτάσει δύο διαμέτρους λέγει οὐχ ἁπλῶς τὰς τυχούσας, ἀλλὰ τὰς καλουμένας συζυγεῖς, ὧν ἑκατέρα παρὰ τεταγμένως κατηγμένην ἧκται καὶ μέσον λόγον ἔχει τῶν τοῦ εἴδους πλευρῶν τῆς ἑτέρας διαμέτρου, καὶ διὰ τοῦτο δίχα τέμνουσι τὰς ἀλλήλων παραλλήλους, ὡς δέδεικται ἐν τῷ ιε′ θεωρήματι. εἰ γὰρ μὴ οὕτως ληφθῇ, συμβήσεται τὴν μεταξὺ εὐθεῖαν τῶν δύο διαμέτρων τῇ ἑτέρᾳ αὐτῶν παράλληλον εἶναι· ὅπερ οὐχ ὑπόκειται.

ἐπειδὴ δὲ τὸ Η ἔγγιόν ἐστι τῆς διχοτομίας τῆς ΑΒ ἤπερ τὸ Θ, καί ἐστι τὸ μὲν ὑπὸ ΒΗΑ μετὰ τοῦ ἀπὸ ΗΜ ἴσον τῷ ἀπὸ ΑΜ, τὸ δὲ ὑπὸ ΑΘΒ μετὰ

236 τοῦ ἀπὸ ΘΜ ἴσον τῷ αὐτῷ, τὸ δὲ ἀπὸ ΘΜ τοῦ ἀπὸ ΗΜ μεῖζον, τὸ ἄρα ὑπὸ ΒΗΑ μεῖζον τοῦ ὑπὸ ΒΘΑ.

Εἰς τὸ κε′.

Ἔν τισι φέρεται καὶ αὕτη ἡ ἀπόδειξις·

εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τὸ Θ, καὶ ἐπεζεύχθω ἡ ΖΘ· ἡ ΖΘ ἄρα ἐκβαλλομένη συμπίπτει τῇ ΔΓ· ὥστε καὶ ἡ ΖΕ. πάλιν δὴ εἰλήφθω, καὶ ἐπεζεύχθω ἡ ΚΖ καὶ ἐκβεβλήσθω· συμπεσεῖται δὴ τῇ ΒΑ ἐκβαλλομένῃ· ὥστε καὶ ἡ ΖΗ.

Εἰς τὸ κϛ΄.

Τὸ θεώρημα τοῦτο πτώσεις ἔχει πλείους, πρῶτον μέν, ὅτι ἡ ΕΖ ἢ ἐπὶ τὰ κυρτὰ μέρη τῆς τομῆς λαμβάνεται ὡς ἐνταῦθα ἢ ἐπὶ τὰ κοῖλα, ἔπειτα, ὅτι ἡ ἀπὸ τοῦ Ε παρὰ τεταγμένως κατηγμένην ἔσω μὲν καθ' ἓν σημεῖον συμβάλλει ἀδιαφόρως τῇ διαμέτρῳ ἀπείρῳ οὔσῃ, ἔξω δὲ οὖσα καὶ μάλιστα ἐπὶ τῆς ὑπερβολῆς ἔχει θέσιν ἢ ἐξωτέρω τοῦ Β ἢ ἐπὶ τοῦ Β ἢ μεταξὺ τῶν Α, Β.

Εἰς τὸ κζ΄.

Ἔν τισιν ἀντιγράφοις τοῦ κζ΄ θεωρήματος φέρεται τοιαύτη ἀπόδειξις·

ἔστω παραβολή, ἧς διάμετρος ἡ ΑΒ, καὶ ταύτην τεμνέτω εὐθεῖά τις ἡ ΗΔ ἐντὸς τῆς τομῆς. λέγω,

238 ὅτι ἡ ΗΔ ἐκβαλλομένη ἐφ' ἑκάτερα τὰ μέρη συμπεσεῖται τῇ τομῇ.

ἤχθω γάρ τις διὰ τοῦ Α παρατεταγμένως ἡ ΑΕ· ἡ ΑΕ ἄρα ἐκτὸς πεσεῖται τῆς τομῆς.

ἤτοι δὴ ἡ ΗΔ τῇ ΑΕ παράλληλός ἐστιν ἢ οὔ.

εἰ μὲν οὖν παράλληλός ἐστιν, αὐτὴ τεταγμένως κατῆκται· ὥστε ἐκβαλλομένη ἐφ' ἑκάτερα, ἐπεὶ δίχα τέμνεται ὑπὸ τῆς· διαμέτρου, συμπεσεῖται τῇ τομῇ. μὴ ἔστω δὴ παράλληλος τῇ ΑΕ, ἀλλὰ ἐκβαλλομένη συμπιπτέτω τῇ ΑΕ κατὰ τὸ Ε ὡς ἡ ΗΔΕ.

ὅτι μὲν οὖν τῇ τομῇ ἐπὶ τὰ ἕτερα μέρη συμπίπτει, ἐφ' ἅ ἐστι τὸ Ε, δῆλον· εἰ γὰρ τῇ ΑΕ συμβάλλει, πολὺ πρότερον τεμεῖ τὴν τομήν.

λέγω, ὅτι καὶ ἐπὶ τὰ ἕτερα μέρη ἐκβαλλομένη συμπίπτει τῇ τομῇ.

ἔστω γὰρ παρ’ ἣν δύνανται ἡ ΜΑ, καὶ ἐκβεβλήσθω ἐπ’ εὐθείας αὐτῇ ἡ ΑΖ· ἡ ΜΑ ἄρα τῇ ΑΒ πρὸς ὀρθάς ἐστιν. πεποιήσθω, ὡς τὸ ἀπὸ ΑΕ πρὸς τὸ ΑΕΔ τρίγωνον, οὕτως ἡ ΜΑ πρὸς ΑΖ, καὶ διὰ τῶν Μ, Ζ τῇ ΑΒ παράλληλοι ἤχθωσαν αἱ ΖΚ, ΜΝ· τετραπλεύρου οὖν ὄντος τοῦ ΛΑΔΗ καὶ θέσει οὔσης τῆς ΛΑ ἤχθω τῇ ΛΑ παράλληλος ἡ ΓΚΒ ἀποτέμνουσα τὸ ΓΚΗ τρίγωνον τῷ ΛΑΔΗ τετραπλεύρῳ ἴσον, καὶ διὰ τοῦ Β τῇ ΖΑΜ παράλληλος ἤχθω ἡ ΞΒΝ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ ΑΕ πρὸς τὸ ΑΕΔ τρίγωνον, ἡ ΜΑ πρὸς ΑΖ, ἀλλ’ ὡς μὲν τὸ ἀπὸ ΑΕ πρὸς τὸ ΑΕΔ τρίγωνον, τὸ ἀπὸ ΓΒ πρὸς τὸ ΔΓΒ τρίγωνον· παράλληλος γάρ ἐστιν ἡ ΑΕ τῇ ΓΒ, καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΓΕ, ΑΒ· ὡς δὲ ἡ ΜΑ πρὸς

240 ΑΖ, τὸ ΑΜΝΒ παραλληλόγραμμον πρὸς τὸ ΑΞ παραλληλόγραμμον, ὡς ἄρα τὸ ἀπὸ ΓΒ πρὸς τὸ ΓΔΒ τρίγωνον, οὕτως τὸ ΑΜΝΒ παραλληλόγραμμον πρὸς τὸ ΑΖΞΒ παραλληλόγραμμον· ἐναλλάξ, ὡς τὸ ἀπὸ ΓΒ πρὸς τὸ ΑΜΝΒ παραλληλόγραμμον, οὕτως τὸ ΓΔΒ τρίγωνον πρὸς τὸ ΑΖΞΒ παραλληλόγραμμον. ἴσον δέ ἐστι τὸ ΖΑΒΞ παραλληλόγραμμον τῷ ΓΒΔ τριγώνῳ· ἐπεὶ γὰρ τὸ ΓΗΚ τρίγωνον τῷ ΑΛΗΔ τετραπλεύρῳ ἐστὶν ἴσον, κοινὸν δὲ τὸ ΗΔΒΚ τετράπλευρον, τὸ ΛΑΒΚ παραλληλόγραμμον τῷ ΓΔΒ τριγώνῳ ἐστὶν ἴσον· τὸ δὲ ΛΑΒΚ

παραλληλόγραμμον τῷ ΖΑΒΞ παραλληλογράμμῳ ἐστὶν ἴσον· ἐπὶ γὰρ τῆς αὐτῆς βάσεώς ἐστι τῆς ΑΒ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΒ, ΖΚ. ἴσον ἄρα ἐστὶ τὸ ΓΔΒ τρίγωνον τῷ ΞΖΑΒ παραλληλογράμμῳ· ὥστε καὶ τὸ ἀπὸ ΓΒ τῷ ΑΜΝΒ παραλληλογράμμῳ ἐστὶν ἴσον. τὸ δὲ ΜΑΒΝ παραλληλόγραμμον ἴσον ἐστὶ τῷ ὑπὸ ΜΑΒ· ἡ γὰρ ΜΑ πρὸς ὀρθάς ἐστι τῇ ΑΒ· τὸ ἄρα ὑπὸ ΜΑΒ ἴσον ἐστὶ τῷ ἀπὸ ΓΒ καί ἐστιν ἡ ΜΑ ὀρθία τοῦ εἴδους πλευρά, ἡ δὲ ΑΒ διάμετρος, καὶ ἡ ΓΒ τεταγμένως· παράλληλος γάρ ἐστι τῇ ΑΕ· τὸ Γ ἄρα πρὸς τῇ τομῇ ἐστιν. ἡ ΔΗΓ ἄρα συμβάλλει τῇ τομῇ κατὰ τὸ Γ· ὅπερ ἔδει δεῖξαι.

σχόλια εἰς τὸ προτεθὲν θεώρημα.

[πεποιήσθω δή, ὡς τὸ ἀπὸ ΑΕ πρὸς τὸ ΑΕΔ τρίγωνον, ἡ ΜΑ πρὸς ΑΖ] τοῦτο δέδεικται ἐν σχολίῳ τοῦ ια´ θεωρήματος. ἀναγράψας γὰρ τὸ ἀπὸ ΑΕ καὶ παρὰ τὴν πλευρὰν αὐτοῦ τῷ ΑΕΔ τριγώνῳ ἴσον παραβαλὼν ἕξω τὸ ζητούμενον.

242 **εἰς τὸ αὐτό.**

[τετραπλεύρου ὄντος τοῦ ΛΑΔΗ ἤχθω τῇ ΛΑ παράλληλος ἡ ΓΚΒ ἀποτέμνουσα τὸ ΓΗΚ τρίγωνον τῷ ΛΑΔΗ τετραπλεύρῳ ἴσον] τοῦτο δὲ ποιήσομεν οὕτως· ἐὰν γάρ, ὡς ἐν τοῖς στοιχείοις ἐμάθομεν, τῷ δοθέντι εὐθυγράμμῳ τῷ ΛΑΔΗ τετραπλεύρῳ ἴσον καὶ ἄλλῳ τῷ δοθέντι τῷ ΑΕΔ τριγώνῳ ὅμοιον τὸ αὐτὸ συστησώμεθα τὸ ΣΤΥ, ὥστε ὁμόλογον εἶναι τὴν ΣΥ τῇ ΑΔ, καὶ ἀπολάβωμεν τῇ μὲν ΣΥ ἴσην τὴν ΗΚ, τῇ δὲ ΤΥ ἴσην τὴν ΗΓ, καὶ ἐπιζεύξωμεν τὴν ΓΚ, ἔσται τὸ ζητούμενον. ἐπεὶ γὰρ ἡ πρὸς τῷ Υ γωνία ἴση ἐστὶ τῇ Δ, τουτέστι τῇ Η, διὰ τοῦτο ἴσον καὶ ὅμοιον τὸ ΓΗΚ τῷ ΣΤΥ. καὶ ἴση ἡ Γ γωνία τῇ Ε, καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΓΚ τῇ ΑΕ.

φανερὸν δή, ὅτι, ὅταν ἡ ΑΒ ἄξων ἐστίν, ἡ ΜΑ ἐφάπτεται τῆς τομῆς, ὅταν δὲ μὴ ἄξων, τέμνει, εἰ πρὸς ὀρθὰς ἄγεται πάντως τῇ διαμέτρῳ.

Εἰς τὸ κη΄.

Ὅτι, κἂν ἡ ΓΔ τέμνῃ τὴν ὑπερβολήν, τὰ αὐτὰ συμβήσεται, ὥσπερ ἐπὶ τοῦ ὀκτωκαιδεκάτου.

Εἰς τὸ λ΄.

[Καὶ ὡς ἄρα ἐπὶ μὲν τῆς ἐλλείψεως συνθέντι, ἐπὶ δὲ τῶν ἀντικειμένων ἀνάπαλιν καὶ ἀνα-
244 στρέψαντι] ἐπὶ μὲν οὖν τῆς ἐλλείψεως ἐροῦμεν· ἐπειδή ἐστιν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΔΖ, τὸ ὑπὸ ΑΗΒ πρὸς τὸ ἀπὸ ΗΕ, ὡς δὲ τὸ ἀπὸ ΔΖ πρὸς τὸ ἀπὸ ΖΓ, τὸ ἀπὸ ΕΗ πρὸς τὸ ἀπὸ ΗΓ, δι’ ἴσου, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΖΓ, τὸ ὑπὸ ΑΗΒ πρὸς τὸ ἀπὸ ΗΓ· συνθέντι, ὡς τὸ ὑπὸ ΑΖΒ μετὰ τοῦ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΖΓ, τουτέστι τὸ ἀπὸ ΑΓ πρὸς τὸ ἀπὸ ΓΖ· ἡ γὰρ ΑΒ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Ζ· οὕτως τὸ ἀπὸ ΓΒ πρὸς τὸ ἀπὸ ΓΗ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΑΓ πρὸς τὸ ἀπὸ ΓΒ, τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΗ. ἐπὶ δὲ τῶν ἀντικειμένων· ἐπεί ἐστιν, ὡς τὸ ὑπὸ ΒΖΑ

πρὸς τὸ ἀπὸ ΖΓ, τὸ ὑπὸ ΑΗΒ πρὸς τὸ ἀπὸ ΓΗ, διότι δι’ ἴσου, ἀνάπαλιν, ὡς τὸ ἀπὸ ΖΓ πρὸς τὸ ὑπὸ ΒΖΑ, τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΑΗΒ· ἀναστρέψαντι, ὡς τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΑ, τὸ ἀπὸ ΗΓ πρὸς τὸ ἀπὸ ΓΒ· εὐθεῖα γάρ τις ἡ ΑΒ τέτμηται δίχα κατὰ τὸ Γ, καὶ πρόσκειται ἡ ΖΑ, καὶ τὸ ὑπὸ ΒΖΑ μετὰ τοῦ ἀπὸ ΑΓ ἴσον ἐστὶ τῷ ἀπὸ ΓΖ, ὥστε τὸ ἀπὸ ΓΖ τοῦ ὑπὸ ΒΖΑ ὑπερέχει τῷ ἀπὸ ΑΓ, καὶ καλῶς εἴρηται τὸ ἀναστρέψαντι.

Εἰς τὸ λα′.

[Διελόντι τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ ΑΗΒ μείζονα λόγον ἔχει ἤπερ τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ ΑΘΒ] ἐπεὶ γὰρ εὐθεῖα ἡ ΑΒ τέτμηται δίχα κατὰ τὸ Γ, καὶ πρόσκειται αὐτῇ ἡ ΒΗ, τὸ ὑπὸ ΑΗΒ μετὰ τοῦ ἀπὸ ΓΒ ἴσον ἐστὶ τῷ ἀπὸ ΓΗ· ὥστε τὸ ἀπὸ ΓΗ τοῦ ὑπὸ ΑΗΒ ὑπερέχει τῷ ἀπὸ ΓΒ. διὰ δὲ τὴν

246 αὐτὴν αἰτίαν καὶ τὸ ἀπὸ ΓΘ τοῦ ὑπὸ ΑΘΒ ὑπερέχει τῷ ἀπὸ ΓΒ· ὥστε ὀρθῶς εἴρηται τὸ διελόντι.

Εἰς τὸ λβ′.

Ἐν τῷ ἑπτακαιδεκάτῳ θεωρήματι ἁπλούστερον ἔδειξεν, ὅτι ἡ διὰ τῆς κορυφῆς παρὰ τὴν κατηγμένην τεταγμένως ἀγομένη ἐφάπτεται, ἐνταῦθα δὲ τὸ ἐν τοῖς στοιχείοις ἐπὶ τοῦ κύκλου μόνου δεδειγμένον καθολικώτερον ἐπὶ πάσης κώνου τομῆς ὑπάρχον ἐπιδείκνυσι.

δεῖ μέντοι ἐπιστῆσαι, ὅπερ κἀκεῖ ἐδείχθη, ὅτι καμπύλην μὲν ἴσως γραμμὴν οὐδὲν ἄτοπόν ἐστιν ἐμπίπτειν μεταξὺ τῆς εὐθείας καὶ τῆς τομῆς, εὐθεῖαν δὲ ἀμήχανον· τεμεῖ γὰρ αὕτη τὴν τομὴν καὶ οὐκ ἐφάψεται· δύο γὰρ ἐφαπτομένας εὐθείας κατὰ τοῦ αὐτοῦ σημείου εἶναι ἀδύνατον.

πολυτρόπως δεδειγμένου τούτου τοῦ θεωρήματος ἐν διαφόροις ἐκδόσεσιν ἡμεῖς τὴν ἀπόδειξιν ἁπλουστέραν καὶ σαφεστέραν ἐποιήσαμεν.

Εἰς τὸ λδʹ. Δεῖ ἐπιστῆσαι, ὅτι ἡ ΓΔ κατηγμένη ἐπὶ τὴν διάμετρον ἐπὶ μὲν τῆς ὑπερβολῆς τὰς ΔΒ, ΔΑ ὁρίζουσα τὴν ΒΑ καταλιμπάνει ὀφείλουσαν τμηθῆναι εἰς τὸν τῶν ΒΔΑ λόγον, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἀνάπαλιν τὴν ΒΑ τέμνουσα εἰς ὡρισμένον λόγον τὸν τῶν ΒΔΑ ἐπιζητεῖν ἡμᾶς ποιεῖ τὸν τῶν ΒΕ, ΕΑ· οὐδὲν γὰρ δυσχερὲς λόγου δοθέντος ἴσον αὐτῷ πορίσασθαι.

248 δεῖ μέντοι εἰδέναι, ὅτι καθ' ἑκάστην τομὴν καταγραφαί εἰσι δύο τοῦ Ζ σημείου ἢ ἐσωτέρω τοῦ Γ λαμβανομένου ἢ ἐξωτέρω· ὥστε εἶναι τὰς πάσας πτώσεις ἕξ.

χρῆται δὲ καὶ δύο λήμμασιν, ἅπερ ἑξῆς γράψομεν.

[μεῖζον ἄρα τὸ ὑπὸ ΑΝΞ τοῦ ὑπὸ ΑΟΞ· ἡ ΝΟ ἄρα πρὸς ΞΟ μείζονα λόγον ἔχει ἤπερ ἡ ΟΑ πρὸς ΑΝ] ἐπεὶ γὰρ τὸ ὑπὸ ΑΝ, ΝΞ μεῖζόν ἐστι τοῦ ὑπὸ ΑΟ, ΟΞ, γινέσθω τῷ ὑπὸ ΑΝ, ΝΞ ἴσον τὸ ὑπὸ τῆς ΑΟ καὶ ἄλλης τινὸς τῆς ΞΠ, ἥτις μείζων ἔσται τῆς ΞΟ· ἔστιν ἄρα, ὡς ἡ ΟΑ πρὸς ΑΝ, ἡ ΝΞ πρὸς ΞΠ. ἡ δὲ ΝΞ πρὸς ΞΟ μείζονα λόγον ἔχει ἤπερ πρὸς τὴν ΞΠ· καὶ ἡ ΟΑ ἄρα πρὸς ΑΝ ἐλάττονα λόγον ἔχει ἤπερ ἡ ΝΞ πρὸς ΞΟ.

φανερὸν δὴ καὶ τὸ ἀνάπαλιν, ὅτι, κἂν ἡ ΝΞ πρὸς ΞΟ μείζονα λόγον ἔχῃ ἤπερ ἡ ΟΑ πρὸς ΑΝ, τὸ ὑπὸ ΞΝ, ΝΑ μεῖζόν ἐστι τοῦ ὑπὸ ΑΟ, ΟΞ.

γινέσθω γάρ, ὡς ἡ ΟΑ πρὸς ΑΝ, οὕτως ἡ ΝΞ πρὸς μείζονα δηλονότι τῆς ΞΟ ὡς τὴν ΞΠ· τὸ ἄρα ὑπὸ ΞΝ, ΝΑ ἴσον ἐστὶ τῷ ὑπὸ ΑΟ, ΞΠ· ὥστε μεῖ- (20) ζόν ἐστι τὸ ὑπὸ ΞΝ, ΝΑ τοῦ ὑπὸ ΑΟ, ΟΞ.

εἰς τὸ αὐτό.

[ἀλλ’ ὡς μὲν τὸ ὑπὸ ΒΚ, ΑΝ πρὸς τὸ ἀπὸ ΓΕ, τὸ ὑπὸ ΒΔΑ πρὸς τὸ ἀπὸ ΕΔ] ἐπεὶ οὖν διὰ

250 τὸ παραλλήλους εἶναι τὰς ΑΝ, ΕΓ, ΚΒ ἐστιν, ὡς ἡ ΑΝ πρὸς ΕΓ, ἡ ΑΔ πρὸς ΔΕ, ὡς δὲ ἡ ΕΓ πρὸς ΚΒ, ἡ ΕΔ πρὸς ΔΒ, δι’ ἴσου ἄρα, ὡς ἡ ΑΝ πρὸς ΚΒ, ἡ ΑΔ πρὸς ΔΒ· καὶ ὡς ἄρα τὸ ἀπὸ ΑΝ πρὸς τὸ ὑπὸ ΑΝ, ΚΒ, τὸ ἀπὸ ΑΔ πρὸς τὸ ὑπὸ ΑΔΒ. ὡς δὲ τὸ ἀπὸ ΕΓ πρὸς τὸ ἀπὸ ΑΝ, τὸ ἀπὸ ΕΔ πρὸς τὸ ἀπὸ ΔΑ· δι’ ἴσου ἄρα, ὡς τὸ ἀπὸ ΕΓ πρὸς τὸ ὑπὸ ΑΝ, ΚΒ, τὸ ἀπὸ ΕΔ πρὸς τὸ ὑπὸ ΑΔΒ· καὶ ἀνάπαλιν, ὡς τὸ ὑπὸ ΚΒ, ΑΝ πρὸς τὸ ἀπὸ ΕΓ, τὸ ὑπὸ ΒΔΑ πρὸς τὸ ἀπὸ ΕΔ.

Εἰς τὸ λζ′.

Διὰ τούτων τῶν θεωρημάτων φανερόν, ὅπως ἐστὶ δυνατὸν διὰ τοῦ δοθέντος σημείου ἐπὶ τῆς διαμέτρου καὶ τῆς κορυφῆς τῆς τομῆς ἐφαπτομένην ἀγαγεῖν.

Εἰς τὸ λη′.

Ἔν τισιν ἀντιγράφοις τὸ θεώρημα τοῦτο ἐπὶ μόνης τῆς ὑπερβολῆς εὑρίσκεται δεδειγμένον, καθολικῶς δὲ ἐνταῦθα δέδεικται· τὰ γὰρ αὐτὰ συμβαίνει καὶ ἐπὶ τῶν ἄλλων τομῶν. καὶ τῷ Ἀπολλωνίῳ δὲ δοκεῖ μὴ μόνον τὴν ὑπερβολήν, ἀλλὰ καὶ τὴν ἔλλειψιν ἔχειν δευτέραν διάμετρον, ὡς πολλάκις αὐτοῦ ἠκούσαμεν ἐν τοῖς προλαβοῦσιν.

καὶ ἐπὶ μὲν τῆς ἐλλείψεως πτῶσιν οὐκ ἔχει, ἐπὶ δὲ τῆς ὑπερβολῆς τρεῖς· τὸ γὰρ Ζ σημεῖον, καθ' ὃ συμβάλλει ἡ ἐφαπτομένη τῇ δευτέρᾳ διαμέτρῳ, ἢ κατω-

252 τέρω τοῦ Δ ἐστιν ἢ ἐπὶ τοῦ Δ ἢ ἀνωτέρω τοῦ Δ, καὶ διὰ τοῦτο τὸ Θ ὁμοίως αὐτῷ τρεῖς ἕξει τόπους, καὶ προσεκτέον, ὅτι, εἴτε κατωτέρω πέσῃ τὸ Ζ τοῦ Δ, καὶ τὸ Θ τοῦ Γ ἔσται κατωτέρω, εἴτε τὸ Ζ ἐπὶ τὸ Δ, καὶ τὸ Θ ἐπὶ τὸ Γ, εἴτε ἀνωτέρω τὸ Ζ τοῦ Δ, καὶ τὸ Θ τοῦ Γ ἔσται ἀνωτέρω.

Εἰς τὸ μα′.

Τὸ θεώρημα τοῦτο ἐπὶ μὲν τῆς ὑπερβολῆς πτῶσιν οὐκ ἔχει, ἐπὶ δὲ τῆς ἐλλείψεως, ἐὰν ἡ καταγομένη ἐπὶ τὸ κέντρον πίπτῃ, τὰ δὲ λοιπὰ γένηται τὰ αὐτά, τὸ ἀπὸ τῆς κατηγμένης εἶδος ἴσον ἔσται τῷ ἀπὸ τῆς ἐκ τοῦ κέντρου εἴδει.

ἔστω γὰρ ἔλλειψις, ἧς διάμετρος ἡ ΑΒ, κέντρον τὸ Δ, καὶ κατήχθω τεταγμένως ἡ ΓΔ, καὶ ἀναγεγράφθω ἀπό τε τῆς ΓΔ καὶ τῆς ΑΔ εἴδη ἰσογώνια τὰ ΑΖ, ΔΗ, ἐχέτω δὲ ἡ ΔΓ πρὸς ΓΗ τὸν συγκείμενον λόγον ἔκ τε τοῦ ὃν ἔχει ἡ ΑΔ πρὸς ΔΖ καὶ τοῦ ὃν ἔχει ἡ ὀρθία πρὸς τὴν πλαγίαν.

λέγω, ὅτι τὸ ΑΖ ἴσον ἐστὶ τῷ ΔΗ.

ἐπεὶ γὰρ ἐν τῷ ῥητῷ δέδεικται, ὡς τὸ ἀπὸ ΑΔ πρὸς τὸ ΑΖ, οὕτως τὸ ὑπὸ ΑΔΒ πρὸς τὸ ΔΗ, φημί, ὅτι καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΑΔ πρὸς τὸ ὑπὸ ΑΔΒ, οὕτως τὸ ΑΖ πρὸς τὸ ΔΗ. ἴσον δὲ τὸ ἀπὸ ΑΔ τῷ ὑπὸ ΑΔΒ· ἴσον ἄρα καὶ τὸ ΑΖ τῷ ΔΗ.

Εἰς τὸ μβʹ.

Τὸ θεώρημα τοῦτο ἔχει πτώσεις ια, μίαν μὲν, εἰ ἐσωτέρω λαμβάνοιτο τὸ Δ τοῦ Γ· δῆλον γάρ, ὅτι καὶ

254 αἱ παράλληλοι ἐσωτέρω πεσοῦνται τῶν ΑΓΘ. ἑτέρας δὲ πέντε οὕτως· ἐὰν τὸ Δ ἐξωτέρω ληφθῇ τοῦ Γ, ἡ μὲν ΔΖ παράλληλος δηλονότι ἐξωτέρω πεσεῖται τῆς ΘΓ, ἡ δὲ ΔΕ ἢ μεταξὺ τῶν Α, Β ἢ ἐπὶ τὸ Β ἢ μεταξὺ τῶν Β, Θ ἢ ἐπὶ τὸ Θ ἢ ἐξωτέρω τοῦ Θ· τοῦ γὰρ Α ἐξωτέρω πεσεῖν αὐτὴν ἀδύνατον, ἐπειδὴ τὸ Δ ἐξωτέρω ἐστὶ τοῦ Γ καὶ δηλονότι καὶ ἡ δι’ αὐτοῦ παράλληλος ἀγομένη τῇ ΑΓ. ἐὰν δὲ τὸ Δ ἐπὶ τὰ ἕτερα μέρη ληφθῇ τῆς τομῆς, ἢ ἀμφότεραι αἱ παράλληλοι μεταξὺ τῶν Θ, Β περατωθήσονται, ἢ ἡ μὲν ΔΖ ἐσωτέρω τοῦ Θ, τὸ δὲ Ε ἐπὶ τὸ Θ, ἢ τῆς ΔΖ ὡσαύτως μενούσης τὸ Ε ἐξωτέρω τοῦ Θ ἐλεύσεται· τοῦ δὲ Ε πάλιν ἐξωτέρω πίπτοντος τὸ Ζ ἢ ἐπὶ τὸ Θ πεσεῖται, ὡς εἶναι τὴν ΓΘΔ μίαν εὐθεῖαν, εἰ καὶ μὴ σώζεται κυρίως τότε τὸ τῆς παραλλήλου ἰδίωμα, ἢ ἐξωτέρω τοῦ Θ. δεῖ δὲ ἐπὶ τῆς ἀποδείξεως τῶν τελευταίων πέντε πτώσεων τὴν ΔΖ ἐκβάλλειν ἕως τῆς τομῆς καὶ τῆς ΗΓ παραλλήλου καὶ οὕτως ποιεῖσθαι τὴν ἀπό-

δειξιν.

δυνατὸν δὲ καὶ ἄλλην μίαν καταγραφὴν ἐπινοεῖν ἐκ τούτων, ὅταν δὴ λαμβανομένου ἑτέρου σημείου αἱ ἐξ ἀρχῆς εὐθεῖαι ποιῶσι τὸ λεγόμενον, ἀλλὰ τοῦτο θεώρημα μᾶλλόν ἐστιν ἢ πτῶσις.

Εἰς τὸ μγ′.

Ἔν τισι φέρεται ἀπόδειξις τοῦ θεωρήματος τούτου τοιαύτη·

256 ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ ΖΓΔ τῷ ἀπὸ ΓΒ, ἔστιν ἄρα, ὡς ἡ ΖΓ πρὸς ΓΒ, ἡ ΓΒ πρὸς ΓΔ· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΓΖ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΒ εἶδος, οὕτως ἡ ΖΓ πρὸς τὴν ΓΔ. ἀλλ' ὡς μὲν τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΖΓ τρίγωνον πρὸς τὸ ΛΓΒ τρίγωνον, ὡς δὲ ἡ ΖΓ πρὸς ΓΔ, τὸ ΕΖΓ τρίγωνον πρὸς τὸ ΕΓΔ τρίγωνον· ὡς ἄρα τὸ ΕΓΖ τρίγωνον πρὸς τὸ ΒΛΓ τρίγωνον, τὸ ΕΓΖ πρὸς τὸ ΕΓΔ τρίγωνον. ἴσον ἄρα τὸ ΕΓΔ τρίγωνον τῷ ΒΓΛ. καὶ ὡς ἄρα ἐπὶ μὲν τῆς ὑπερβολῆς ἀναστρέψαντι, ἐπὶ δὲ τῆς ἐλλείψεως ἀνάπαλιν καὶ διελόντι, [ὡς] τὸ ΕΖΓ τρίγωνον πρὸς τὸ ΕΛΒΖ τετράπλευρον, οὕτως τὸ ΕΓΖ πρὸς τὸ ΕΔΖ τρίγωνον· ἴσον ἄρα τὸ ΕΔΖ τρίγωνον τῷ ΕΛΒΖ τετραπλεύρῳ. καὶ ἐπεί ἐστιν, ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΓΖ πρὸς τὸ ΛΓΒ τρίγωνον, ἐπὶ μὲν τῆς ὑπερβολῆς διελόντι, ἐπὶ δὲ τῆς ἐλλείψεως ἀνάπαλιν καὶ ἀναστρέψαντι καὶ ἀνάπαλίν ἐστιν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΒΓ, τὸ ΕΛΒΖ τετράπλευρον πρὸς τὸ ΒΛΓ τρίγωνον. ὁμοίως

δὲ καί, ὡς τὸ ἀπὸ ΓΒ πρὸς τὸ ὑπὸ ΑΚΒ, οὕτως τὸ ΛΓΒ τρίγωνον πρὸς τὸ ΜΛΒΚ τετράπλευρον· δι' ἴσου ἄρα, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ὑπὸ ΑΚΒ, τὸ ΕΛΒΖ τετράπλευρον πρὸς τὸ ΛΒΚΜ. ὡς δὲ τὸ ὑπὸ ΑΖΒ πρὸς τὸ ὑπὸ ΑΚΒ, τὸ ἀπὸ ΕΖ πρὸς τὸ ἀπὸ ΗΚ, ὡς δὲ τὸ ἀπὸ ΕΖ πρὸς τὸ ἀπὸ ΗΚ, τὸ ΕΔΖ τρίγωνον πρὸς τὸ ΗΘΚ τρίγωνον· καὶ ὡς ἄρα τὸ ΕΔΖ πρὸς τὸ ΗΘΚ, τὸ ΕΛΒΖ τετράπλευρον πρὸς τὸ ΜΛΒΚ. ἐναλλάξ, ὡς τὸ ΕΔΖ πρὸς τὸ ΕΛΒΖ, οὕτως τὸ ΗΘΚ πρὸς τὸ ΜΛΒΚ. ἴσον δὲ
258 τὸ ΕΔΖ τῷ ΕΛΒΖ ἐδείχθη· ἴσον ἄρα καὶ τὸ ΗΘΚ τῷ ΜΛΒΚ τετραπλεύρῳ. τὸ ἄρα ΜΓΚ τρίγωνον τοῦ ΗΘΚ διαφέρει τῷ ΛΒΓ.

ἐπιστῆσαι δεῖ ταύτῃ τῇ δείξει· ὀλίγην γὰρ ἀσάφειαν ἔχει ἐν ταῖς ἀναλογίαις τῆς ἐλλείψεως· ἵνα τὰ διὰ τὴν συντομίαν τοῦ ῥητοῦ ὁμοῦ λεγόμενα διῃρημένως ποιήσωμεν, οἷον—φησὶ γάρ· ἐπεί ἐστιν, ὡς τὸ ἀπὸ ΖΓ πρὸς τὸ ἀπὸ ΓΒ, τὸ ΕΓΖ τρίγωνον πρὸς τὸ ΛΒΓ, ἀνάπαλιν καὶ ἀναστρέψαντι καὶ ἀνάπαλιν —ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ἀπὸ ΓΖ, τὸ ΛΒΓ πρὸς τὸ ΕΖΓ· ἀναστρέψαντι, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΑΖΒ, τουτέστιν ἡ ὑπεροχὴ τοῦ ἀπὸ ΓΒ πρὸς τὸ ἀπὸ ΓΖ διὰ τὸ διχοτομίαν εἶναι τὸ Γ τῆς ΑΒ, οὕτως τὸ ΛΒΓ τρίγωνον πρὸς τὸ ΛΒΖΕ τετράπλευρον· ἀνάπαλιν, ὡς τὸ ὑπὸ ΑΖΒ πρὸς τὸ ἀπὸ ΒΓ, τὸ ΕΛΒΖ τετράπλευρον πρὸς τὸ ΛΒΓ τρίγωνον.

ἔχει δὲ πτώσεις ἐπὶ μὲν τῆς ὑπερβολῆς $\overline{\iota\alpha}$, ὅσας εἶχε καὶ τὸ πρὸ αὐτοῦ ἐπὶ τῆς παραβολῆς, καὶ ἄλλην μίαν, ὅταν τὸ ἐπὶ τοῦ Η λαμβανόμενον σημεῖον ταὐτὸν ᾖ τῷ Ε· τότε γὰρ συμβαίνει τὸ ΕΔΖ τρίγωνον μετὰ τοῦ ΛΒΓ ἴσον εἶναι τῷ ΓΕΖ· δέδεικται μὲν γὰρ τὸ ΕΔΖ τρίγωνον ἴσον τῷ ΛΒΖΕ τετραπλεύρῳ, τὸ δὲ ΛΒΖΕ τοῦ ΓΖΕ τριγώνου διαφέρει τῷ ΛΒΓ. ἐπὶ δὲ τῆς ἐλλείψεως ἢ ταὐτόν ἐστι τὸ Η τῷ Ε ἢ ἐσωτέρω λαμβάνεται τοῦ Ε· καὶ δῆλον, ὅτι ἀμφότεραι αἱ παράλληλοι μεταξὺ πεσοῦνται τῶν Δ, Ζ, ὡς ἔχει

260 ἐν τῷ ῥητῷ. εἰ δὲ ἐξωτέρω ληφθῇ τὸ Η τοῦ Ε, καὶ ἡ ἀπ' αὐτοῦ τῇ ΕΖ παράλληλος μεταξὺ πέσῃ τῶν Ζ, Γ, τὸ Θ σημεῖον ποιεῖ πτώσεις πέντε· ἢ γὰρ μεταξὺ τῶν Δ, Β πίπτει ἢ ἐπὶ τὸ Β ἢ μεταξὺ τῶν Β, Ζ ἢ ἐπὶ τὸ Ζ ἢ μεταξὺ τῶν Ζ, Γ. ἐὰν δὲ ἡ διὰ τοῦ Η τῇ κατηγμένῃ παράλληλος ἐπὶ τὸ Γ κέντρον πίπτῃ, τὸ Θ πάλιν σημεῖον ποιήσει ἄλλας πέντε πτώσεις ὡσαύτως· καὶ δεῖ ἐπὶ τούτῳ σημειώσασθαι, ὅτι τὸ ὑπὸ τῶν παραλλήλων ταῖς ΕΔ, ΕΖ γιγνόμενον τρίγωνον ἴσον γίνεται τῷ ΛΒΓ τριγώνῳ· ἐπεὶ γάρ ἐστιν, ὡς τὸ ἀπὸ ΕΖ πρὸς τὸ ἀπὸ ΗΓ, τὸ ΕΔΖ τρίγωνον πρὸς τὸ ΗΘΓ· ὅμοια γάρ· ὡς δὲ τὸ ἀπὸ ΕΖ πρὸς τὸ ἀπὸ ΗΓ, τὸ ὑπὸ ΒΖΑ πρὸς τὸ ὑπὸ ΒΓΑ, τουτέστι τὸ ἀπὸ ΒΓ, ὡς ἄρα τὸ ΕΔΖ τρίγωνον πρὸς τὸ ΗΘΓ, τὸ ὑπὸ ΒΖΑ πρὸς τὸ ἀπὸ ΒΓ. ὡς δὲ τὸ ὑπὸ ΒΖΑ πρὸς τὸ ἀπὸ ΒΓ, οὕτως ἐδείχθη ἔχον τὸ ΛΒΖΕ τετράπλευρον πρὸς τὸ ΛΒΓ τρίγωνον· καὶ ὡς ἄρα τὸ ΕΔΖ τρίγωνον πρὸς τὸ ΗΘΓ, τὸ ΛΒΖΕ τετράπλευ-

ρον πρὸς τὸ ΛΒΓ τρίγωνον. καὶ ἐναλλάξ. καὶ ἄλλως δὲ ταύτας δυνατὸν δεῖξαι λέγοντας, ὅτι ἐπὶ τῶν διπλασίων αὐτῶν παραλληλογράμμων ταῦτα δέδεικται ἐν τῷ σχολίῳ τοῦ μα´ θεωρήματος.

ἐὰν δὲ ἡ διὰ τοῦ Η τῇ ΕΖ παράλληλος ἀγομένη μεταξὺ πέσῃ τῶν Γ, Α, ἐκβληθήσεται μέν, ἕως ὅτε ἡ ΓΕ αὐτῇ συμπέσῃ, τὸ δὲ Θ σημεῖον ποιήσει πτώσεις

262 ζ̄· ἢ γὰρ μεταξὺ τῶν Β, Δ ἢ ἐπὶ τὸ Β πίπτει ἢ μεταξὺ τῶν Β, Ζ ἢ ἐπὶ τὸ Ζ ἢ μεταξὺ τῶν Ζ, Γ ἢ ἐπὶ τὸ Γ ἢ μεταξὺ τῶν Γ, Α· καὶ ἐπὶ τούτων τῶν πτώσεων συμβαίνει τὴν διαφορὰν τῶν ΛΒΓ, ΗΘΚ τριγώνων κατωτέρω συνίστασθαι τῆς ΑΒ εὐθείας ὑπὸ τῆς ΛΓ ἐκβαλλομένης.

ἐὰν δὲ τὸ Η ἐπὶ τὰ ἕτερα μέρη ληφθῇ τῆς τομῆς, καὶ ἡ ἀπὸ τοῦ Η τῇ ΕΖ παράλληλος μεταξὺ πίπτῃ τῶν Β, Ζ, ἐκβληθήσεται μὲν διὰ τὴν ἀπόδειξιν, ἕως οὗ τέμῃ τὴν ΛΓ, τὸ δὲ Θ σημεῖον ποιήσει πτώσεις ζ̄ ἢ μεταξὺ ὂν τῶν Β, Ζ ἢ ἐπὶ τὸ Ζ πῖπτον ἢ μεταξὺ τῶν Ζ, Γ ἢ ἐπὶ τὸ Γ ἢ μεταξὺ τῶν Γ, Α ἢ ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α. ἐὰν δὲ ἡ ἀπὸ τοῦ Η τῇ ΕΖ παράλληλος ἐπὶ τὸ Ζ πίπτῃ, ὥστε μίαν εὐθεῖαν εἶναι τὴν ΕΖΗ, τὸ Θ σημεῖον ποιήσει πτώσεις ε̄· ἢ γὰρ μεταξὺ τῶν Ζ, Γ πεσεῖται ἢ ἐπὶ τὸ Γ ἢ μεταξὺ τῶν Γ, Α ἢ ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α. ἐὰν δὲ ἡ ΗΚ μεταξὺ πίπτῃ τῶν Ζ, Γ, τὸ Θ ποιήσει πτώσεις ε̄· ἢ γὰρ μεταξὺ τῶν Ζ, Γ πεσεῖται ἢ ἐπὶ τὸ Γ ἢ μεταξὺ τῶν Γ, Α ἢ ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α. ἐὰν δὲ ἡ

ΗΚ ἐπὶ τὸ Γ κέντρον πίπτῃ, τὸ Θ σημεῖον ποιήσει πτώσεις τρεῖς ἢ μεταξὺ πῖπτον τῶν Γ, Α ἢ ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α· καὶ ἐπὶ τούτων τῶν πτώσεων συμβήσεται πάλιν τὸ ΗΘΚ τρίγωνον ἴσον γίνεσθαι τῷ ΛΒΓ τριγώνῳ. ἐὰν δὲ ἡ ΗΚ μεταξὺ πίπτῃ τῶν Γ, Α, τὸ Θ σημεῖον ἢ μεταξὺ τῶν Γ, Α πεσεῖται ἢ ἐπὶ τὸ Α ἢ ἐξωτέρω τοῦ Α.

συμβαίνει οὖν ἐπί τινος ἐλλείψεως τὰς πάσας πτώσεις εἶναι μ̅β̅ καὶ ἐπὶ τῆς τοῦ κύκλου δὲ περιφερείας
264 τοσαύτας, ὡς εἶναι τὰς πάσας πτώσεις τούτου τοῦ θεωρήματος ϟ̅ϛ̅.

Εἰς τὸ μδʹ.

[Ἐπεὶ οὖν ἀντικείμεναί εἰσιν αἱ ΖΑ, ΒΕ, ὧν διάμετρος ἡ ΑΒ, ἡ δὲ διὰ τοῦ κέντρου ἡ ΖΓΕ καὶ ἐφαπτόμεναι τῶν τομῶν αἱ ΖΗ, ΔΕ, παράλληλός ἐστιν ἡ ΖΗ τῇ ΕΔ] ἐπεὶ γὰρ ὑπερβολή ἐστιν ἡ ΑΖ καὶ ἐφαπτομένη ἡ ΖΗ καὶ κατηγμένη ἡ ΖΟ, ἴσον ἐστὶ τὸ ὑπὸ ΟΓΗ τῷ ἀπὸ ΓΑ διὰ τὸ λζʹ θεώρημα· ὁμοίως δὴ καὶ τὸ ὑπὸ ΞΓΔ τῷ ὑπὸ ΓΒ ἐστιν ἴσον. ἔστιν ἄρα, ὡς τὸ ὑπὸ ΟΓΗ πρὸς τὸ ἀπὸ ΑΓ, οὕτως τὸ ὑπὸ ΞΓΔ πρὸς τὸ ἀπὸ ΒΓ, καὶ ἐναλλάξ, ὡς τὸ ὑπὸ ΟΓΗ πρὸς τὸ ὑπὸ ΞΓΔ, τὸ ἀπὸ ΑΓ πρὸς τὸ ἀπὸ ΓΒ. ἴσον δὲ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΒ· ἴσον ἄρα καὶ τὸ ὑπὸ ΟΓΗ τῷ ὑπὸ ΞΓΔ. καί ἐστιν ἡ ΟΓ τῇ ΓΞ ἴση· καὶ ἡ ΗΓ ἄρα τῇ ΓΔ ἐστιν ἴση· ἔστι δὲ καὶ ἡ ΖΓ τῇ ΓΕ διὰ τὸ λʹ· αἱ ἄρα ΖΓΗ ἴσαι εἰσὶ ταῖς ΕΓΔ. καὶ γωνίας ἴσας περιέχουσι τὰς πρὸς τῷ Γ·

κατὰ κορυφὴν γάρ. ὥστε καὶ ἡ ΖΗ τῇ ΕΔ ἐστιν ἴση καὶ ἡ ὑπὸ ΓΖΗ γωνία τῇ ὑπὸ ΓΕΔ. καί εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ ΖΗ τῇ ΕΔ.

αἱ πτώσεις αὐτοῦ $\overline{\text{ιβ}}$ εἰσιν, καθάπερ ἐπὶ τῆς ὑπερβολῆς ἐν τῷ μγ′ ἔχει, καὶ ἡ ἀπόδειξις ἡ αὐτή.

Εἰς τὸ με′.

Ἐπιστῆσαι χρὴ τῷ θεωρήματι τούτῳ πλείους ἔχοντι πτώσεις. ἐπὶ μὲν γὰρ τῆς ὑπερβολῆς ἔχει $\overline{\text{κ}}$· τὸ γὰρ

266 ἀντὶ τοῦ Β λαμβανόμενον σημεῖον ἢ ταὐτόν ἐστι τῷ Α ἢ ταὐτὸν τῷ Γ· τότε γὰρ συμβαίνει τὸ ἀπὸ τῆς ΑΘ τρίγωνον ὅμοιον τῷ ΓΔΛ ταὐτὸν εἶναι τῷ ἀποτεμνομένῳ τριγώνῳ ὑπὸ τῶν παραλλήλων ταῖς ΔΛΓ. ἐὰν δὲ μεταξὺ ληφθῇ τὸ Β σημεῖον τῶν Α, Γ, καὶ τὰ Δ, Λ ἀνωτέρω ὦσι τῶν περάτων τῆς δευτέρας διαμέτρου, γίνονται πτώσεις τρεῖς· τὰ γὰρ Ζ, Ε ἢ ἀνωτέρω τῶν περάτων φέρονται ἢ ἐπ’ αὐτὰ ἢ κατωτέρω. ἐὰν δὲ τὰ Δ, Λ ἐπὶ τὰ πέρατα ὦσι τῆς δευτέρας διαμέτρου, τὰ Ζ, Ε κατωτέρω ἐνεχθήσονται. ὁμοίως δὲ καὶ † ἐὰν ἐξωτέρω ληφθῇ τοῦ Γ τὸ Β, [καὶ] ἡ ΘΓ ἐπὶ τὸ Γ ἐκβληθήσεται, συμβαίνει δὲ οὕτως γίνεσθαι ἄλλας πτώσεις τρεῖς· τοῦ γὰρ Δ σημείου ἢ ἀνωτέρω φερομένου τοῦ πέρατος τῆς δευτέρας διαμέτρου ἢ ἐπ’ αὐτὸ ἢ κατωτέρω καὶ τὸ Ζ ὁμοίως φερόμενον ποιήσει τὰς τρεῖς πτώσεις. ἐὰν δὲ ἐπὶ τὰ ἕτερα μέρη τῆς τομῆς ληφθῇ τὸ Β σημεῖον, ἡ μὲν ΓΘ ἐκβληθήσεται ἐπὶ τὸ Θ διὰ τὴν ἀπόδειξιν, αἱ δὲ ΒΖ, ΒΕ ποιοῦσι πτώσεις τρεῖς, ἐπειδὴ τὸ Λ ἐπὶ τὸ πέρας

φέρεται τῆς δευτέρας διαμέτρου ἢ ἀνωτέρω ἢ κατωτέρω.

ἐπὶ δὲ τῆς ἐλλείψεως καὶ τῆς τοῦ κύκλου περιφερείας οὐδὲν ποικίλον ἐροῦμεν, ἀλλὰ ὅσα ἐν τῷ προλαβόντι θεωρήματι ἐλέχθη· ὡς εἶναι τὰς πτώσεις τοῦ θεωρήματος τούτου $\overline{\text{ρδ}}$.

268 δύναται δὲ τὰ τῆς προτάσεως δείκνυσθαι καὶ ἐπὶ ἀντικειμένων.

Εἰς τὸ μϛʹ. Τοῦτο τὸ θεώρημα πτώσεις ἔχει πλείους, ἃς δείξομεν προσέχοντες ταῖς πτώσεσι τοῦ μβʹ.

ὑποδείγματος δὲ χάριν, ἐὰν τὸ Ζ ἐπὶ τὸ Β πίπτοιτο, αὐτόθεν ἐροῦμεν· ἐπεὶ τὸ ΒΔΛ ἴσον ἐστὶ τῷ ΘΒΔΜ, κοινὸν ἀφῃρήσθω τὸ ΝΜΔΒ· λοιπὸν ἄρα τὸ ΛΝΜ τῷ ΝΘΒ ἐστιν ἴσον.

ἐπὶ δὲ τῆς λοιπῆς ἐροῦμεν· ἐπειδὴ τὸ ΛΕΔ τῷ ΘΒΔΜ ἐστιν ἴσον, τουτέστι τῷ ΚΗΔΜ καὶ τῷ ΗΖΕ, τουτέστι τῷ ΖΚΝ καὶ τῷ ΝΕΔΜ, κοινὸν ἀφῃρήσθω τὸ ΝΕΔΜ· καὶ λοιπὸν ἄρα τὸ ΛΝΜ τῷ ΚΖΝ ἴσον.

Εἰς τὸ μζʹ.

Τοῦτο τὸ θεώρημα ἐπὶ μὲν τῆς ὑπερβολῆς πτώσεις ἔχει, ὅσας τὸ πρὸ αὐτοῦ ἐπὶ τῆς παραβολῆς εἶχεν, τὰς
270 δὲ ἀποδείξεις αὐτῶν ποιησόμεθα προσέχοντες ταῖς πτώσεσι τοῦ μγʹ θεωρήματος, καὶ ἐπὶ τῆς ἐλλείψεως δὲ τὰς ἀποδείξεις ἐκ τῶν πτώσεων τοῦ μγʹ, οἷον ἐπὶ τῆς ὑποκειμένης καταγραφῆς τοῦ Η σημείου ἐκτὸς εἰλημμένου, ἐπειδὴ ἴσον ἐστὶ τὸ ΛΑΓ τρίγωνον τοῖς ΘΗΩ, ΩΓΜ, τουτέστι τοῖς ΟΘΓ, ΟΗΜ τριγώνοις, τῷ δὲ ΛΑΓ ἴσον ἐστὶ τό τε ΞΠΓ τρίγωνον καὶ τὸ

ΛΑΠΞ τετράπλευρον, τουτέστι τὸ ΝΘΠ τρίγωνον διὰ τὰ δεδειγμένα ἐν τῷ μγ´ θεωρήματι, καὶ τὰ ΞΠΓ, ΝΘΠ ἄρα τρίγωνα ἴσα ἐστὶ τοῖς ΟΘΓ, ΟΜΗ τριγώνοις. κοινὸν ἀφῃρήσθω τὸ ΘΟΓ τρίγωνον· λοιπὸν ἄρα τὸ ΞΟΝ τῷ ΗΟΜ ἴσον ἐστίν. καὶ παράλληλος ἡ ΝΞ τῇ ΜΗ· ἴση ἄρα ἡ ΝΟ τῇ ΟΗ.

Εἰς τὸ μη´.

Καὶ τούτου αἱ πτώσεις ὡσαύτως ἔχουσι τοῖς προειρημένοις ἐπὶ τοῦ μζ´ κατὰ τὴν τῆς ὑπερβολῆς καταγραφήν.

Εἰς τὸ μθ´.

[Λοιπὸν ἄρα τὸ ΚΛΝ τρίγωνον τῷ ΔΛΠΓ παραλληλογράμμῳ ἐστὶν ἴσον. καὶ ἴση ἐστὶν ἡ ὑπὸ ΔΛΠ γωνία τῇ ὑπὸ ΚΛΝ γωνίᾳ· διπλάσιον ἄρα ἐστὶ τὸ ὑπὸ ΚΛΝ τοῦ ὑπὸ ΛΔΓ] ἐκκείσθω γὰρ χωρὶς τὸ ΚΛΝ τρίγωνον καὶ τὸ ΔΛΠΓ παραλ-

272 ληλόγραμμον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΚΛΝ τρίγωνον τῷ ΔΠ παραλληλογράμμῳ, ἤχθω διὰ τοῦ Ν τῇ ΛΚ παράλληλος ἡ ΝΡ, διὰ δὲ τοῦ Κ τῇ ΛΝ ἡ ΚΡ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΛΡ καὶ διπλάσιον τοῦ ΚΛΝ τριγώνου· ὥστε καὶ τοῦ ΔΠ παραλληλογράμμου. ἐκβεβλήσθωσαν δὴ αἱ ΔΓ, ΛΠ ἐπὶ τὰ Σ, Τ, καὶ κείσθω τῇ ΔΓ ἴση ἡ ΓΣ, τῇ δὲ ΛΠ ἡ ΠΤ, καὶ ἐπεζεύχθω ἡ ΣΤ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΔΤ διπλάσιον τοῦ ΔΠ· ὥστε ἴσον τὸ ΛΡ τῷ ΛΣ. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον διὰ τὸ τὰς πρὸς τῷ Λ γωνίας κατὰ κορυφὴν οὔσας ἴσας εἶναι· τῶν δὲ ἴσων καὶ ἰσογωνίων

παραλληλογράμμων ἀντιπεπόνθασιν αἱ περὶ τὰς ἴσας γωνίας πλευραί· ἔστιν ἄρα, ὡς ἡ ΚΛ πρὸς ΛΤ, τουτέστι πρὸς ΔΣ, ἡ ΔΛ πρὸς ΛΝ, καὶ τὸ ὑπὸ ΚΛΝ ἴσον ἐστὶ τῷ ὑπὸ ΛΔΣ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΔΣ τῆς ΔΓ, τὸ ὑπὸ ΚΛΝ διπλάσιόν ἐστι τοῦ ΛΔΓ.

ἐὰν δὲ ἡ μὲν ΔΓ τῇ ΛΠ ἐστι παράλληλος, ἡ δὲ ΓΠ τῇ ΛΔ μή ἐστι παράλληλος, τραπέζιον μὲν δηλονότι ἐστὶ τὸ ΔΓΠΛ, καὶ οὕτως δέ φημι, ὅτι τὸ ὑπὸ ΚΛΝ ἴσον ἐστὶ τῷ ὑπὸ ΔΛ καὶ συναμφοτέρου τῆς ΓΔ, ΛΠ. ἐὰν γὰρ τὸ μὲν ΛΡ ἀναπληρωθῇ, ὡς προείρηται, ἐκβληθῶσι δὲ καὶ αἱ ΔΓ, ΛΠ, καὶ τεθῇ τῇ μὲν ΛΠ ἴση ἡ ΓΣ, τῇ δὲ ΔΓ ἡ ΠΤ, καὶ ἐπιζευχθῇ ἡ ΣΤ, παραλληλόγραμμον ἔσται τὸ ΔΤ διπλάσιον τοῦ ΔΠ, καὶ ἡ ἀπόδειξις ἡ αὐτὴ ἁρμόσει. χρησιμεύσει δὲ τοῦτο εἰς τὸ ἑξῆς.

Εἰς τὸ ν′.

Αἱ πτώσεις τούτου τοῦ θεωρήματος ὡσαύτως ἔχουσι ταῖς τοῦ μγ′, ὁμοίως δὲ καὶ ἐπὶ τοῦ να′.

274 **Εἰς τὸν ἐπίλογον.**

Τὴν ἐκ τῆς γενέσεως διάμετρον λέγει τὴν γεναμένην ἐν τῷ κώνῳ κοινὴν τομὴν τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου· ταύτην δὲ καὶ ἀρχικὴν διάμετρον λέγει. καί φησιν, ὅτι πάντα τὰ δεδειγμένα συμπτώματα τῶν τομῶν ἐν τοῖς προειρημένοις θεωρήμασιν ὑποθεμένων ἡμῶν τὰς ἀρχικὰς διαμέτρους συμβαίνειν δύνανται καὶ τῶν ἄλλων πασῶν διαμέτρων ὑποτιθεμένων.

Εἰς τὸ νδʹ.

[Καὶ ἀνεστάτω ἀπὸ τῆς ΑΒ ἐπίπεδον ὀρθὸν πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ ἐν αὐτῷ περὶ τὴν ΑΒ γεγράφθω κύκλος ὁ ΑΕΒΖ, ὥστε τὸ τμῆμα τῆς διαμέτρου τοῦ κύκλου τὸ ἐν τῷ ΑΕΒ τμήματι πρὸς τὸ τμῆμα τῆς διαμέτρου τὸ ἐν τῷ ΑΖΒ τμήματι μὴ μείζονα λόγον ἔχειν τοῦ ὃν ἔχει ἡ ΑΒ πρὸς ΒΓ] ἔστωσαν δύο εὐθεῖαι αἱ ΑΒ, ΒΓ, καὶ δέον ἔστω περὶ τὴν ΑΒ κύκλον γράψαι, ὥστε τὴν διάμετρον αὐτοῦ τέμνεσθαι ὑπὸ τῆς ΑΒ οὕτως, ὥστε τὸ πρὸς τῷ Γ μέρος αὐτῆς πρὸς τὸ λοιπὸν μὴ μείζονα λόγον ἔχειν τοῦ τῆς ΑΒ πρὸς ΒΓ. ὑποκείσθω μὲν νῦν τὸν αὐτόν, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Δ, καὶ δι' αὐτοῦ πρὸς ὀρθὰς τῇ ΑΒ ἤχθω ἡ ΕΔΖ, καὶ γεγονέτω, ὡς ἡ ΑΒ πρὸς
276 ΒΓ, ἡ ΕΔ πρὸς ΔΖ, καὶ δίχα τετμήσθω ἡ ΕΖ· δῆλον δή, ὅτι, εἰ μὲν ἡ ΑΒ τῇ ΒΓ ἐστιν ἴση καὶ ἡ ΕΔ τῇ ΔΖ, διχοτομία ἔσται τῆς ΕΖ τὸ Δ, εἰ δὲ ἡ ΑΒ τῆς ΒΓ μείζων καὶ ἡ ΕΔ τῆς ΔΖ, ἡ διχοτομία κατωτέρω ἐστὶ τοῦ Δ, εἰ δὲ ἡ ΑΒ τῆς ΒΓ ἐλάσσων, ἀνωτέρω.

ἔστω δὲ νῦν τέως κατωτέρω ὡς τὸ Η, καὶ κέντρῳ τῷ Η διαστήματι τῷ ΗΖ κύκλος γεγράφθω· δεῖ δὴ διὰ τῶν Α, Β σημείων ἥξειν ἢ ἐσωτέρω ἢ ἐξωτέρω. καὶ εἰ μὲν διὰ τῶν Α, Β σημείων ἔρχοιτο, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ὑπερπιπτέτω δὲ τὰ Α, Β, καὶ ἐκβληθεῖσα ἐφ' ἑκάτερα ἡ ΑΒ συμπιπτέτω τῇ περιφερείᾳ

κατὰ τὰ Θ, Κ, καὶ ἐπεζεύχθωσαν αἱ ΖΘ, ΘΕ, ΕΚ, ΚΖ, καὶ ἤχθω διὰ τοῦ Β τῇ μὲν ΖΚ παράλληλος ἡ ΜΒ, τῇ δὲ ΚΕ ἡ ΒΛ, καὶ ἐπεζεύχθωσαν αἱ ΜΑ, ΑΛ· ἔσονται δὴ καὶ αὐταὶ παράλληλοι ταῖς ΖΘ, ΘΕ διὰ τὸ ἴσην εἶναι τὴν μὲν ΑΔ τῇ ΔΒ, τὴν δὲ ΔΘ τῇ ΔΚ καὶ πρὸς ὀρθὰς εἶναι τὴν ΖΔΕ τῇ ΘΚ. καὶ ἐπεὶ ὀρθή ἐστιν ἡ πρὸς τῷ Κ γωνία, καὶ παράλληλοι αἱ ΜΒΛ ταῖς ΖΚΕ, ὀρθὴ ἄρα καὶ ἡ πρὸς τῷ Β· διὰ τὰ αὐτὰ δὴ καὶ ἡ πρὸς τῷ Α. ὥστε ὁ περὶ τὴν ΜΛ κύκλος γραφόμενος ἥξει διὰ τῶν Α, Β. γεγράφθω ὡς ὁ ΜΑΛΒ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΜΒ τῇ ΖΚ, ἔστιν, ὡς ἡ ΖΔ πρὸς ΔΜ, ἡ ΚΔ πρὸς ΔΒ. ὁμοίως δὴ καί, ὡς ἡ ΚΔ πρὸς ΔΒ, ἡ ΕΔ πρὸς ΔΛ. καὶ
278 ἐναλλάξ, ὡς ἡ ΕΔ πρὸς ΔΖ, τουτέστιν ἡ ΑΒ πρὸς ΒΓ, ἡ ΛΔ πρὸς ΔΜ.

ὁμοίως δέ, κἂν ὁ γραφόμενος περὶ τὴν ΖΕ κύκλος τέμνοι τὴν ΑΒ, τὸ αὐτὸ δειχθήσεται.

Εἰς τὸ νε′.

[Καὶ ἐπὶ τῆς ΑΔ γεγράφθω ἡμικύκλιον τὸ ΑΖΔ, καὶ ἤχθω τις εἰς τὸ ἡμικύκλιον παράλληλος τῇ ΑΘ ἡ ΖΗ ποιοῦσα τὸν τοῦ ἀπὸ ΖΗ πρὸς τὸ ὑπὸ ΔΗΑ λόγον τὸν αὐτὸν τῷ τῆς ΓΑ πρὸς τὴν διπλασίαν τῆς ΑΔ] ἔστω ἡμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΓ, ὁ δὲ δοθεὶς λόγος ὁ τῆς ΕΖ πρὸς ΖΗ, καὶ δέον ἔστω ποιῆσαι τὰ προκείμενα.

κείσθω τῇ ΕΖ ἴση ἡ ΖΘ, καὶ τετμήσθω ἡ ΘΗ δίχα κατὰ τὸ Κ, καὶ ἤχθω ἐν τῷ ἡμικυκλίῳ τυχοῦσα εὐθεῖα ἡ ΓΒ ἐν γωνίᾳ τῇ ὑπὸ ΑΓΒ, καὶ ἀπὸ τοῦ Λ κέντρου ἤχθω ἐπ’ αὐτὴν κάθετος ἡ ΛΣ καὶ ἐκβληθεῖσα συμβαλλέτω τῇ περιφερείᾳ κατὰ τὸ Ν, καὶ διὰ τοῦ Ν τῇ ΓΒ παράλληλος ἤχθω ἡ ΝΜ· ἐφάψεται ἄρα τοῦ κύκλου. καὶ πεποιήσθω, ὡς ἡ ΖΘ πρὸς ΘΚ, ἡ ΜΞ πρὸς ΞΝ, καὶ κείσθω τῇ ΞΝ ἴση ἡ ΝΟ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΛΟ τέμνουσαι τὸ ἡμικύκλιον κατὰ τὰ Π, Ρ, καὶ ἐπεζεύχθω ἡ ΠΡΔ.

ἐπεὶ οὖν ἴση ἐστὶν ἡ ΞΝ τῇ ΝΟ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΝΛ, ἴση ἄρα καὶ ἡ ΛΟ τῇ ΛΞ. ἔστι δὲ καὶ ἡ ΛΠ τῇ ΛΡ· καὶ λοιπὴ ἄρα ἡ ΠΟ τῇ ΡΞ
280 ἐστιν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΠΡΔ τῇ ΜΟ. καί ἐστιν, ὡς ἡ ΖΘ πρὸς ΘΚ, ἡ ΜΞ πρὸς ΝΞ· ὡς δὲ ἡ ΘΚ πρὸς ΘΗ, ἡ ΝΞ πρὸς ΞΟ· δι’ ἴσου ἄρα, ὡς ἡ ΘΖ πρὸς ΘΗ, ἡ ΜΞ πρὸς ΞΟ· ἀνάπαλιν, ὡς ἡ ΗΘ πρὸς ΘΖ, ἡ ΟΞ πρὸς ΞΜ· συνθέντι, ὡς ἡ ΗΖ πρὸς ΖΘ, τουτέστι πρὸς ΖΕ, ἡ ΟΜ πρὸς ΜΞ, τουτέστιν ἡ ΠΔ πρὸς ΔΡ. ὡς δὲ ἡ ΠΔ πρὸς ΔΡ, τὸ ὑπὸ ΠΔΡ πρὸς τὸ ἀπὸ ΔΡ, ἴσον δὲ τὸ ὑπὸ ΠΔΡ τῷ ὑπὸ ΑΔΓ· ὡς ἄρα ἡ ΗΖ πρὸς ΖΕ, τὸ ὑπὸ ΑΔΓ πρὸς τὸ ἀπὸ ΔΡ. ἀνάπαλιν ἄρα, ὡς ἡ ΕΖ πρὸς ΖΗ, τὸ ἀπὸ ΔΡ πρὸς τὸ ὑπὸ ΑΔΓ.

Εἰς τὸ νη΄.

[Καὶ ἐπὶ τῆς ΑΕ γεγράφθω ἡμικύκλιον τὸ ΑΕΖ, καὶ τῇ ΑΔ παράλληλος ἤχθω ἐν αὐτῷ ἡ ΖΗ λόγον ποιοῦσα τὸν τοῦ ἀπὸ ΖΗ πρὸς τὸ ὑπὸ ΑΗΕ τὸν τῆς ΓΑ πρὸς τὴν διπλασίαν τῆς

ΑΕ] ἔστω ἡμικύκλιον τὸ ΑΒΓ καὶ ἐν αὐτῷ εὐθεῖά τις ἡ ΑΒ, καὶ κείσθωσαν δύο εὐθεῖαι ἄνισοι αἱ ΔΕ, ΕΖ, καὶ ἐκβεβλήσθω ἡ ΕΖ ἐπὶ τὸ Η, καὶ τῇ ΔΕ ἴση κείσθω ἡ ΖΗ, καὶ τετμήσθω ὅλη ἡ ΕΗ δίχα κατὰ τὸ Θ, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Κ, καὶ ἀπ’ αὐτοῦ κάθετος ἐπὶ τὴν ΑΒ ἤχθω καὶ συμβαλλέτω τῇ περιφερείᾳ κατὰ τὸ Λ, καὶ διὰ τοῦ Λ τῇ ΑΒ παράλληλος ἤχθω ἡ ΛΜ, καὶ ἐκβλη-
282 θεῖσα ἡ ΚΑ συμβαλλέτω τῇ ΛΜ κατὰ τὸ Μ, καὶ πεποιήσθω, ὡς ἡ ΘΖ πρὸς ΖΗ, ἡ ΛΜ πρὸς ΜΝ, καὶ τῇ ΛΝ ἴση ἔστω ἡ ΛΞ, καὶ ἐπεζεύχθωσαν αἱ ΝΚ, ΚΞ καὶ ἐκβεβλήσθωσαν, καὶ ἀναπληρωθεὶς ὁ κύκλος τεμνέτω αὐτὰς κατὰ τὰ Π, Ο, καὶ ἐπεζεύχθω ἡ ΟΡΠ.

ἐπεὶ οὖν ἐστιν, ὡς ἡ ΖΘ πρὸς ΖΗ, ἡ ΛΜ πρὸς ΜΝ, συνθέντι, ὡς ἡ ΘΗ πρὸς ΗΖ, ἡ ΛΝ πρὸς ΝΜ· ἀνάπαλιν, ὡς η ΖΗ πρὸς ΗΘ, ἡ ΝΜ πρὸς ΝΛ, ὡς δὲ ἡ ΖΗ πρὸς ΗΕ, ἡ ΜΝ πρὸς ΝΞ· διελόντι, ὡς ἡ ΖΗ πρὸς ΖΕ, ἡ ΝΜ πρὸς ΜΞ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΝΛ τῇ ΛΞ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΛΚ, ἴση ἄρα καὶ ἡ ΚΝ τῇ ΚΞ. ἔστι δὲ καὶ ἡ ΚΟ τῇ ΚΠ ἴση· παράλληλος ἄρα ἡ ΝΞ τῇ ΟΠ. ὅμοιον ἄρα τὸ ΚΜΝ τρίγωνον τῷ ΟΚΡ τριγώνῳ καὶ τὸ ΚΜΞ τῷ ΠΡΚ. ἔστιν ἄρα, ὡς ἡ ΚΜ πρὸς ΚΡ, ἡ ΜΝ πρὸς ΡΟ. ἀλλὰ καί, ὡς αὐτὴ ἡ ΚΜ πρὸς ΚΡ, ἡ ΜΞ πρὸς ΠΡ· καὶ ὡς ἄρα ἡ ΝΜ πρὸς ΡΟ, ἡ ΜΞ πρὸς ΠΡ· καὶ ἐναλλάξ, ὡς ἡ ΝΜ πρὸς ΜΞ, ἡ ΟΡ πρὸς ΡΠ. ἀλλ’ ὡς μὲν ἡ ΝΜ πρὸς ΜΞ, ἡ ΗΖ πρὸς ΖΕ, τουτέστιν ἡ ΔΕ πρὸς ΕΖ, ὡς δὲ ἡ ΟΡ πρὸς ΡΠ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΟΡΠ· καὶ ὡς ἄρα ἡ ΔΕ πρὸς ΕΖ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΟΡΠ. ἴσον δὲ τὸ ὑπὸ ΟΡΠ τῷ ὑπὸ ΑΡΓ. ὡς ἄρα ἡ ΔΕ πρὸς ΕΖ, τὸ ἀπὸ ΟΡ πρὸς τὸ ὑπὸ ΑΡΓ.

284 Εἴρηται μὲν ἐν τοῖς μετὰ τὸ ι′ θεώρημα σχολίοις ὁ σκοπὸς τῶν $\overline{\iota\gamma}$ πρώτων θεωρημάτων καὶ ἐν τοῖς εἰς τὸ ἑκκαιδέκατον ὁ τῶν ἑξῆς τριῶν, δεῖ δὲ εἰδέναι, ὅτι ἐν μὲν τῷ ιζ′ φησίν, ὅτι ἡ διὰ τῆς κορυφῆς παρὰ τεταγμένως κατηγμένην ἀγομένη ἐκτὸς πίπτει, ἐν δὲ τῷ ιη′ φησίν, ὅτι ἡ παράλληλος τῇ ὁπωσοῦν ἐφαπτομένῃ ἐντὸς τῆς τομῆς ἀγομένη τεμεῖ τὴν τομήν, ἐν τῷ ιθ′, ὅτι ἡ ἀπό τινος σημείου τῆς διαμέτρου παρὰ τεταγμένως κατηγμένην συμπίπτει τῇ τομῇ, ἐν τῷ κ′ καὶ κα′ τὰς καταγομένας ζητεῖ τῶν τομῶν, ὅπως ἔχουσι πρὸς ἀλλήλας καὶ τὰ τῆς διαμέτρου ὑπ’ αὐτῶν γινόμενα τμήματα, ἐν τῷ κβ′ καὶ κγ′ λέγει περὶ τῆς εὐθείας τῆς κατὰ δύο σημεῖα τῇ τομῇ συμπιπτούσης, ἐν τῷ κδ′ καὶ κε′ περὶ τῆς εὐθείας τῆς καθ’ ἓν τῇ τομῇ συμπιπτούσης, τουτέστιν ἐφαπτομένης, ἐν τῷ κϛ′ περὶ τῆς ἀγομένης παραλλήλου τῇ διαμέτρῳ τῆς παραβολῆς καὶ τῆς ὑπερβολῆς, ἐν τῷ κζ′ περὶ τῆς τεμνούσης τὴν διάμετρον τῆς παραβολῆς, ὅτι κατ’ ἀμφότερα μέρη συμπίπτει τῇ τομῇ, ἐν τῷ κη′ περὶ τῆς ἀγομένης παραλλήλου τῇ ἐφαπτομένῃ μιᾶς τῶν ἀντικειμένων, ἐν τῷ κθ′ περὶ τῆς διὰ τοῦ κέντρου τῶν ἀντικειμένων

ἐκβαλλομένης, ἐν τῷ λ′ φησιν, ὅτι διχοτομεῖται ἡ διὰ τοῦ κέντρου ἐκβαλλομένη τῆς ἐλλείψεως καὶ τῶν ἀντικειμένων, ἐν τῷ λα′ φησίν, ὅτι ἐπὶ τῆς ὑπερβολῆς ἡ ἐφαπτομένη τὴν διάμετρον τέμνει μεταξὺ τῆς κορυφῆς καὶ τοῦ κέντρου, ἐν τῷ λβ′ καὶ γ′ καὶ δ′ καὶ ε′ καὶ ϛ′ περὶ τῶν ἐφαπτομένων ποιεῖται τὸν λόγον, ἐν τῷ

286 λζ′ περὶ τῶν ἐφαπτομένων καὶ τῶν ἀπὸ τῆς ἁφῆς κατηγμένων τῆς ἐλλείψεως καὶ τῆς ὑπερβολῆς, ἐν τῷ λη′ περὶ τῶν ἐφαπτομένων τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, ὅπως ἔχουσι πρὸς τὴν δευτέραν διάμετρον, ἐν τῷ λθ′ καὶ μ′ περὶ τῶν αὐτῶν ποιεῖται τὸν λόγον τοὺς συγκειμένους ἐκ τούτων λόγους ἐπιζητῶν, ἐν τῷ μα′ περὶ τῶν ἀναγραφομένων παραλληλογράμμων ἀπὸ τῆς κατηγμένης καὶ τῆς ἐκ τοῦ κέντρου τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, ἐν τῷ μβ′ ἐπὶ τῆς παραβολῆς λέγει ἴσον εἶναι τὸ ὑπὸ τῆς ἐφαπτομένης καὶ τῆς κατηγμένης καταλαμβανόμενον τρίγωνον τῷ ἰσοϋψεῖ αὐτῷ παραλληλογράμμῳ, ἡμίσειαν δ’ ἔχοντι βάσιν, ἐν τῷ μγ′ ἐπὶ τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως ζητεῖ, πῶς ἔχουσι πρὸς ἄλληλα τὰ ὑπὸ τῶν ἐφαπτομένων καὶ τῶν κατηγμένων ἀπολαμβανόμενα τρίγωνα, ἐν τῷ μδ′ τὸ αὐτὸ ἐν ταῖς ἀντικειμέναις, ἐν τῷ με′ τὸ αὐτὸ ἐπὶ τῆς δευτέρας διαμέτρου τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, ἐν τῷ μϛ′ περὶ τῶν μετὰ τὴν ἀρχικὴν διάμετρον τῆς παραβολῆς ἑτέρων, ἐν τῷ μζ′ περὶ τῶν ἑτέρων διαμέτρων τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, ἐν τῷ μη′ περὶ τῶν ἑτέρων διαμέτρων τῶν

ἀντικειμένων, ἐν τῷ μθ´ περὶ τῶν παρ’ ἃς δύνανται αἱ καταγόμεναι ἐπὶ τὰς ἑτέρας διαμέτρους τῆς παραβολῆς, ἐν τῶ ν´ περὶ τοῦ αὐτοῦ τῆς ὑπερβολῆς καὶ τῆς ἐλλείψεως, ἐν τῷ να´ περὶ τοῦ αὐτοῦ τῶν ἀντικειμένων. ταῦτα εἰπὼν καὶ προσθεὶς τοῖς εἰρημένοις

288 ἐπίλογόν τινα ἐν τῷ νβ´ καὶ νγ´ δεικνύει πρόβλημα, ὡς δυνατὸν ἐν ἐπιπέδῳ γράψαι τὴν παραβολήν, ἐν τῷ νδ´ καὶ νε´ λέγει, πῶς δεῖ γράψαι τὴν ὑπερβολήν, ἐν τῷ νϛ´ καὶ νζ´ καὶ νη´, πῶς δεῖ γράψαι τὴν ἔλλειψιν, ἐν τῷ νθ´ λέγει, πῶς δεῖ γράφειν ἀντικειμένας, ἐν τῷ ξ´ περὶ τῶν συζύγων ἀντικειμένων.

APPENDIX B
PASSAGES ON COMPOUND RATIO

B.1 From Eutocius' Commentary on Archimedes' *On the Sphere and the Cylinder*

120 Ἐπεὶ οὖν ὁ τῆς ΡΛ πρὸς ΛΧ λόγος συνῆπται ἔκ τε τοῦ, ὃν ἔχει ἡ ΡΛ πρὸς ΛΔ, καὶ ἡ ΛΔ πρὸς ΛΧ ὅτι μὲν ἡ σύνθεσις τῶν λόγων λαμβάνεται τῆς ΛΔ μέσης λαμβανομένης, ὡς κἀν τῇ Στοιχειώσει ἐλαμβάνετο, φανερόν· ἐπεὶ δὲ τὸ λεγόμενον ἀδιαρθρώτως πως καὶ οὐχ οὕτως, ὥστε τὴν ἔννοιαν ἀποπληρῶσαι, λέλεκται, ὡς ἔστιν εὑρεῖν ἐντυγχάνοντας Πάππῳ τε καὶ Θέωνι καὶ Ἀρκαδίῳ ἐν πολλοῖς συντάγμασιν οὐκ ἀποδεικτικῶς, ἀλλ᾽ ἐπαγωγῇ τὸ λεγόμενον παριστῶσιν, οὐδὲν ἄτοπον πρὸς βραχὺ ἐνδιατρίψαντας τῷ λόγῳ τὸ σαφέστερον παραστῆσαι.

φημὶ τοίνυν, ὅτι, ἐὰν δύο ἀριθμῶν ἤτοι μεγεθῶν μέσος τις ὅρος ληφθῇ, ὁ τῶν ἐξ ἀρχῆς ληφθέντων ἀριθμῶν λόγος σύγκειται ἐκ τοῦ λόγου, ὃν ἔχει ὁ πρῶτος πρὸς τὸν μέσον, καὶ τοῦ, ὃν ἔχει ὁ μέσος πρὸς τὸν τρίτον. ὑπομνηστέον δὴ πρότερον, πῶς ἐλέγετο λόγος ἐκ λόγων συγκεῖσθαι. ὡς γὰρ ἐν τῇ Στοιχειώσει· ὅταν αἱ τῶν λόγων πηλικότητες ἐφ᾽ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσίν τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθμοῦ, οὗ παρώνυμός ἐστιν ὁ διδόμενος λόγος, ὥς φασιν ἄλλοι τε καὶ Νικόμαχος ἐν τῷ πρώτῳ Περὶ μουσικῆς καὶ Ἡρώνας ἐν τῷ ὑπομνήματι τῷ εἰς τὴν Ἀριθμητικὴν εἰσαγωγήν, ταὐτὸν δὲ εἰπεῖν καὶ τοῦ ἀριθμοῦ τοῦ πολλα-

πλασιαζομένου ἐπὶ τὸν ἑπόμενον ὅρον τοῦ λόγου καὶ ποιοῦντος τὸν ἡγούμενον. καὶ κυριώτερον μὲν ἐπὶ τῶν πολλαπλασίων ἡ πηλικότης ἂν λαμβάνοιτο, ἐπὶ δὲ τῶν ἐπιμορίων ἢ ἐπιμερῶν οὐκέτι τὴν πηλικότητα δυνατὸν λαμβάνεσθαι ἀδιαιρέτου μενούσης τῆς μονάδος· ὥστ’ ἐπ’ ἐκείνων διαιρετέον τὴν μονάδα, ὃ εἰ καὶ μὴ κατὰ τὸ προσῆκον τῇ ἀριθμητικῇ ἀλλὰ τῇ λογιστικῇ τυγχάνει. διαιρεῖται δὲ ἡ μονὰς κατὰ τὸ μέρος ἢ τὰ μέρη, ἀφ’ ὧν ὠνό-
122 μασται ὁ λόγος, ὥστε εἶναι ὡς ἐν σαφεστέρῳ τῷ λέγειν τοῦ μὲν ἡμιολίου λόγου πηλικότητα πρὸς τῇ μονάδι καὶ τὸ ἥμισυ τῆς μονάδος, τοῦ δὲ ἐπιτρίτου πρὸς τῇ μονάδι τὸ τρίτον, ὥστε, καθὰ καὶ ἀνωτέρῳ εἴρηται, τὴν πηλικότητα τοῦ λόγου ἐπὶ τὸν ἑπόμενον ὅρον πολλαπλασιαζομένην ποιεῖν τὸν ἡγούμενον. τοῦ γὰρ ἐννέα πρὸς τὰ ἓξ ἡμιολίου πηλικότης οὖσα ἡ μονὰς καὶ τὸ ἥμισυ πολλαπλασιασθεῖσα ἐπὶ τὸν ϛ ποιεῖ τὸν θ, καὶ ἐπὶ τῶν ἄλλων δὲ τὸ αὐτὸ ἔξεστι κατανοεῖν.

τούτων δὴ προσαφηνισθέντων ἐπανακτέον ἐπὶ τὸ προτεθέν. ἔστωσαν γὰρ οἱ δοθέντες δύο ἀριθμοὶ οἱ Α, Β, μέσος δὲ αὐτῶν εἰλήφθω τις ὁ Γ· δεικτέον δή, ὅτι ὁ τοῦ Α πρὸς τὸν Β λόγος συνῆπται ἐκ τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Β.

εἰλήφθω γὰρ τοῦ μὲν Α, Γ λόγου πηλικότης ὁ Δ, τοῦ δὲ Γ, Β ὁ Ε· ὁ ἄρα Γ τὸν Δ πολλαπλασιάσας τὸν Α ποιεῖ, ὁ δὲ Β τὸν Ε πολλαπλασιάσας τὸν Γ. ὁ δὴ Δ τὸν Ε πολλαπλασιάσας τὸν Ζ ποιείτω. λέγω, ὅτι ὁ Ζ πηλικότης ἐστὶ τοῦ τοῦ Α πρὸς τὸν Β λόγου, τουτέστιν,

ὅτι ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α ποιεῖ. ὁ γὰρ Β τὸν Ζ πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ οὖν ὁ Β τὸν μὲν Ζ πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Γ, ἔστιν ἄρα, ὡς ὁ Ζ πρὸς τὸν Ε, ὁ Η πρὸς τὸν Γ. πάλιν, ἐπεὶ ὁ Δ τὸν μὲν Ε πολλαπλασιάσας τὸν Ζ πεποίηκεν, τὸν δὲ Γ πολλαπλασιάσας τὸν Α πεποίηκεν, ἔστιν ἄρα, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α. ἐναλλάξ, ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Γ πρὸς τὸν Α, καὶ ἀνάπαλιν, ὡς ὁ Ζ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Γ. ἀλλ’ ὡς ὁ Ζ πρὸς τὸν Ε, ἐδείχθη ὁ Η πρὸς τὸν Γ· καὶ ὡς ἄρα ὁ Η πρὸς τὸν Γ, ὁ Α πρὸς τὸν Γ· ἴσος ἄρα ὁ Α τῷ Η. ἀλλ’ ὁ Β τὸν Ζ πολλαπλασιάσας τὸν Η πε-
124 ποίηκεν· καὶ ὁ Β ἄρα τὸν Ζ πολλαπλασιάσας τὸν Α ποιεῖ· ὁ Ζ ἄρα πηλικότης ἐστὶ τοῦ τοῦ Α πρὸς τὸν Β λόγου. καὶ ἔστιν ὁ Ζ τοῦ Δ ἐπὶ τὸν Ε πολλαπλασιασθέντος, τουτέστι τῆς πηλικότητος τοῦ Α, Γ λόγου ἐπὶ τὴν πηλικότητα τοῦ Γ, Β λόγου· ὁ ἄρα τοῦ Α πρὸς τὸν Β λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Β· ὅπερ ἔδει δεῖξαι.

ἵνα δὲ καὶ ἐπὶ ὑποδείγματος φανερὸν γένηται τὸ εἰρημένον, παρεμπιπτέτω τοῦ ιβ καὶ τοῦ β μέσος τις ἀριθμὸς ὁ δ. λέγω, ὅτι ὁ τοῦ ιβ πρὸς τὸν β λόγος, τουτέστιν ὁ ἑξαπλάσιος, σύγκειται ἔκ τε τοῦ τριπλασίου τοῦ ιβ πρὸς τὰ δ καὶ τοῦ διπλασίου τοῦ δ πρὸς τὰ β.

ἐὰν γὰρ τὰς πηλικότητας τῶν λόγων πολλαπλασιάσωμεν ἐπ’ ἀλλήλας, τουτέστι τὸν γ ἐπὶ τὸν β, γίνεται ὁ ς πηλικότης ὢν τοῦ ιβ πρὸς τὰ β λόγου, καί ἐστιν ἑξαπλάσιος, ὅνπερ καὶ προέκειτο ὑποδεῖξαι.

εἰ δὲ καὶ ὁ μέσος παρεμπίπτων μὴ ὑπάρχῃ τοῦ μὲν μείζονος ἐλάττων, τοῦ δὲ ἐλάττονος μείζων, ἀλλ' ἢ τὸ ἀνάπαλιν ἢ ἀμφοτέρων μείζων ἢ ἀμφοτέρων ἐλάττων, καὶ οὕτως ἡ σύνθεσις ἡ προειρημένη ἀκολουθήσει. τοῦ θ καὶ τοῦ ϛ μέσος τις παρεμπιπτέτω ἀμφοτέρων μείζων ὁ ιβ. λέγω, ὅτι ἔκ τε τοῦ ὑπεπιτρίτου τοῦ θ πρὸς τὸν ιβ λόγου καὶ τοῦ διπλασίου τοῦ ιβ πρὸς τὸν ϛ σύγκειται ὁ ἡμιόλιος τοῦ θ πρὸς τὰ ϛ.

ἡ γὰρ πηλικότης τοῦ θ πρὸς τὸν ιβ λόγου ἐστὶ τρία τέταρτα, τουτέστιν ἥμισυ καὶ τέταρτον, ἡ δὲ πηλικότης τοῦ ιβ πρὸς τὸν ϛ ἐστιν ὁ β. ἐὰν οὖν πολλαπλασιάσωμεν τὸν β ἐπὶ τὸ ἥμισυ καὶ τέταρτον, γίνεται μονὰς α καὶ ἥμισυ, ἥτις πηλικότης ἐστὶ τοῦ ἡμιολίου λόγου, ὃν ἔχει καὶ ὁ θ πρὸς τὸν ϛ. ὁμοίως δέ, κἂν τοῦ θ καὶ ϛ μέσος ἐμπέσῃ ὁ δ, ἐκ τοῦ θ πρὸς δ διπλασιεπιτετάρτου καὶ τοῦ δ πρὸς ϛ ὑφημιολίου σύγκειται ὁ ἡμιόλιος λόγος. πάλιν γὰρ τὴν πηλικότητα τοῦ διπλασιεπιτετάρτου τὰ β δ´ ἐπὶ τὴν πηλικότητα τοῦ ὑφημιολίου, τουτέστι τὰ δύο τρίτα,
126 πολλαπλασιάσαντες ἕξομεν τὸ ἓν ἥμισυ πηλικότητα τοῦ ἡμιολίου, ὡς εἴρηται, λόγου. καὶ ἐπὶ πάντων δὲ ὁμοίως ὁ αὐτὸς ἁρμόσει λόγος.

συμφανὲς δὲ ἐκ τῶν εἰρημένων, ὡς, ἐὰν δύο δοθέντων ἀριθμῶν ἤτοι μεγεθῶν κἂν μὴ εἷς μέσος, πλείους δέ, παρεμπίπτωσιν ὅροι, ὁ τῶν ἄκρων λόγος σύγκειται ἐκ πάντων τῶν λόγων, ὧν ἔχουσιν οἱ κατὰ τὸ ἑξῆς κείμενοι ὅροι ἀρχόμενοι ἀπὸ πρώτου καὶ λήγοντες εἰς τὸν ἔσχατον τῇ κατὰ τοὺς ἐχομένους τάξει.

δύο γὰρ ὄντων ὅρων τῶν Α, Β παρεμπιπτέτωσαν πλείους ἑνὸς οἱ Γ, Δ. λέγω, ὅτι ὁ τοῦ Α πρὸς τὸν Β λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Δ, καὶ ὁ Δ πρὸς τὸν Β.

ἐπεὶ γὰρ ὁ τοῦ Α πρὸς τὸν Β σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Δ, καὶ ὁ Δ πρὸς τὸν Β, ὡς ἀνωτέρω εἴρηται, ὁ δὲ τοῦ Α πρὸς τὸν Δ λόγος σύγκειται ἔκ τε τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Δ, ὁ ἄρα τοῦ Α πρὸς τὸν Β λόγος συνῆπται ἔκ τε τοῦ, ὃν ἔχει ὁ Α πρὸς τὸν Γ, καὶ ὁ Γ πρὸς τὸν Δ, καὶ ὁ Δ πρὸς τὸν Β. ὁμοίως δὲ καὶ ἐπὶ τῶν λοιπῶν δειχθήσεται.

B.2 From Eutocius' Commentary on Apollonius' *Conics*

218 Τὸ δὲ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ λόγον ἔχει τὸν συγκείμενον ἐκ τοῦ ὃν ἔχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ· δέδεικται μὲν ἐν τῷ ἕκτῳ βιβλίῳ τῆς στοιχειώσεως ἐν τῷ εἰκοστῷ τρίτῳ θεωρήματι, ὅτι τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ἐπεὶ δὲ ἐπακτικώτερον μᾶλλον καὶ οὐ κατὰ τὸν ἀναγκαῖον τρόπον ὑπὸ τῶν ὑπομνηματιστῶν ἐλέγετο, ἐζητήσαμεν αὐτὸ καὶ γέγραπται ἐν τοῖς ἐκδεδομένοις ἡμῖν εἰς τὸ τέταρτον θεώρημα τοῦ δευτέρου βιβλίου τῶν Ἀρχιμήδους περὶ σφαίρας καὶ κυλίνδρου καὶ ἐν τοῖς σχολίοις τοῦ πρώτου βιβλίου τῆς Πτολεμαίου συντάξεως· οὐ χεῖρον δὲ καὶ ἐνταῦθα τοῦτο γραφῆναι διὰ τὸ μὴ πάντως τοὺς ἀναγινώσκοντας κἀκείνοις ἐντυγχάνειν, καὶ ὅτι σχεδὸν

τὸ ὅλον σύνταγμα τῶν κωνικῶν κέχρηται αὐτῷ. λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ᾽ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα, πηλικότητος δηλονότι λεγομένης τοῦ ἀριθμοῦ, οὗ παρώνυμός ἐστιν ὁ λόγος. ἐπὶ μὲν οὖν τῶν πολλαπλασίων δυνατόν ἐστιν ἀριθμὸν ὁλόκληρον εἶναι τὴν πηλικότητα, ἐπὶ δὲ τῶν λοιπῶν σχέσεων ἀνάγκη τὴν πηλικότητα ἀριθμὸν εἶναι καὶ μόριον ἢ μόρια, εἰ μὴ ἄρα τις ἐθέλοι καὶ ἀρρήτους εἶναι σχέσεις, οἷαί εἰσιν αἱ κατὰ τὰ ἄλογα μεγέθη. ἐπὶ πασῶν δὲ τῶν σχέσεων δῆλον, ὅτι αὐτὴ ἡ πηλικότης πολλαπλασιαζομένη ἐπὶ τὸν ἑπόμενον ὅρον τοῦ λόγου ποιεῖ τὸν ἡγούμενον.

ἔστω τοίνυν λόγος ὁ τοῦ Α πρὸς τὸν Β, καὶ εἰ-
220 λήφθω τις αὐτῶν μέσος, ὡς ἔτυχεν, ὁ Γ, καὶ ἔστω τοῦ Α, Γ λόγου πηλικότης ὁ Δ, τοῦ δὲ Γ, Β ὁ Ε, καὶ ὁ Δ τὸν Ε πολλαπλασιάσας τὸν Ζ ποιείτω. λέγω, ὅτι τοῦ λόγου τῶν Α, Β πηλικότης ἐστὶν ὁ Ζ, τουτέστιν ὅτι ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α ποιεῖ. ὁ δὴ Ζ τὸν Β πολλαπλασιάσας τὸν Η ποιείτω. ἐπεὶ οὖν ὁ Δ τὸν μὲν Ε πολλαπλασιάσας τὸν Ζ πεποίηκεν, τὸν δὲ Γ πολλαπλασιάσας τὸν Α πεποίηκεν, ἔστιν ἄρα, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α. πάλιν ἐπεὶ ὁ Β τὸν Ε πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Η πεποίηκεν, ἔστιν ἄρα, ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Γ πρὸς τὸν Η. ἐναλλάξ, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Η. ἦν δέ, ὡς ὁ Ε πρὸς τὸν Γ, ὁ Ζ πρὸς τὸν Α· ἴσος ἄρα ὁ Η

τῷ Α. ὥστε ὁ Ζ τὸν Β πολλαπλασιάσας τὸν Α πεποίηκεν.

μὴ ταραττέτω δὲ τοὺς ἐντυγχάνοντας τὸ διὰ τῶν ἀριθμητικῶν δεδεῖχθαι τοῦτο· οἵ τε γὰρ παλαιοὶ κέχρηνται ταῖς τοιαύταις ἀποδείξεσι μαθηματικαῖς μᾶλλον οὔσαις ἢ ἀριθμητικαῖς διὰ τὰς ἀναλογίας, καὶ ὅτι τὸ ζητούμενον ἀριθμητικόν ἐστιν. λόγοι γὰρ καὶ πηλικότητες λόγων καὶ πολλαπλασιασμοὶ τοῖς ἀριθμοῖς πρώτως ὑπάρχουσι καὶ δι' αὐτῶν τοῖς μεγέθεσι, κατὰ τὸν εἰπόντα· ταῦτα γὰρ τὰ μαθήματα δοκοῦντι εἶμεν ἀδελφά.

B.3 From Theon's Commentary on Ptolemy's *Syntaxis*

532 **Λῆμμα.**

Εἷς λόγος ἐκ δύο λόγων ἢ καὶ πλειόνων συγκεῖσθαι λέγεται, ὅταν αἱ

533 τῶν λόγων πηλικότητες πολλαπλασιασθεῖσαι ποιῶσί τινα πηλικότητα λόγου.

Ἐχέτω γὰρ τὸ ΑΒ πρὸς τὸ ΓΔ λόγον δεδομένον, καὶ τὸ ΓΔ πρὸς τὸ

534 ΕΖ λόγον· λέγω ὅτι ὁ τοῦ ΑΒ πρὸς ΕΖ λόγος σύγκειται ἔκ τε τοῦ ΑΒ πρὸς ΓΔ καὶ τοῦ ΓΔ πρὸς ΕΖ, τουτέστιν ὅτι ἐὰν ἡ τοῦ ΑΒ πρὸς τὸ ΓΔ λόγου πηλικότης πολλαπλασιασθῇ ἐπὶ τὴν τοῦ ΓΔ πρὸς ΕΖ, τοῦ λόγου πηλικότητα ποιεῖ τὴν τοῦ ΑΒ πρὸς τὴν ΕΖ.

Ἔστω γὰρ πρότερον τὸ μὲν ΑΒ τοῦ ΔΓ μεῖζον, τὸ δὲ ΓΔ τοῦ ΕΖ. καὶ ἔστω τὸ μὲν ΑΒ τοῦ ΓΔ διπλάσιον, τὸ δὲ ΓΔ τοῦ ΕΖ τριπλάσιον. ἐπεὶ οὖν τὸ μὲν ΓΔ τοῦ ΕΖ τριπλάσιόν ἐστιν, τὸ δὲ ΑΒ τοῦ ΓΔ διπλάσιον, τὸ ἄρα ΑΒ τοῦ ΕΖ ἐστὶν ἑξαπλάσιον, ἐπεὶ καὶ ἐὰν τὸ τριπλάσιόν τινος διπλασιάσωμεν, γίνεται αὐτοῦ ἑξαπλάσιον. τοῦτο γάρ ἐστιν κυρίως σύνθεσις.

Ἢ οὕτωσί. ἐπεὶ τὸ ΑΒ τοῦ ΓΔ ἐστὶν διπλάσιον, διῃρήσθω τὸ ΑΒ εἰς τὰ τῷ ΓΔ ἴσα τὰ ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ ΓΔ τοῦ ΕΖ ἐστὶν τριπλάσιον, διὰ τὰ αὐτὰ δὴ καὶ τὸ ΗΒ τοῦ ΕΖ ἐστὶν τριπλάσιον. ὅλον ἄρα τὸ ΑΒ τοῦ ΕΖ ἐστὶν ἑξαπλάσιον. ὁ ἄρα τοῦ ΑΒ πρὸς τὸ ΕΖ λόγος συνῆκται διὰ τοῦ ΓΔ μέσου ὅρου συγκείμενος ἔκ τε τοῦ ΑΒ πρὸς ΓΔ λόγου καὶ τοῦ ΓΔ πρὸς ΕΖ.

Ἀλλ’ ὁμοίως δὲ καὶ ἐὰν ἔλαττον ᾖ ἑκατέρου τῶν ΑΒ, ΕΖ τὸ ΓΔ, τὸ αὐτὸ συναχθήσεται. ἔστω γὰρ πάλιν τὸ μὲν ΑΒ τοῦ ΓΔ τριπλάσιον, τὸ δὲ ΓΔ ἥμισυ τοῦ ΕΖ. καὶ ἐπεὶ τὸ ΓΔ ἥμισύ ἐστιν τοῦ ΕΖ, τοῦ δὲ ΓΔ τριπλάσιον τὸ ΑΒ, τὸ ΑΒ ἄρα ἡμιόλιόν ἐστιν τοῦ ΕΖ· ἐὰν γὰρ τὸ ἥμισύ τινος τριπλασιάσωμεν, ἕξει αὐτὸ ἅπαξ καὶ ἡμισάκις.

Ἢ καὶ οὕτως. ἐπεὶ τὸ μὲν ΑΒ τοῦ ΓΔ ἐστὶν τριπλάσιον τὸ δὲ ΓΔ τοῦ ΕΖ ἥμισυ, οἵων ἐστὶν τὸ ΑΒ ἴσων τῷ ΓΔ τριῶν τοιούτων ἐστὶν τὸ ΕΖ δύο. ὥστε ἡμιόλιον ἔσται τὸ ΑΒ τοῦ ΕΖ. ὁ ἄρα τοῦ ΑΒ πρὸς τὸ ΕΖ λόγος συνῆκται διὰ τοῦ ΓΔ μέσου ὅρου συγκείμενος ἔκ τε τοῦ ΑΒ πρὸς ΓΔ λόγου καὶ τοῦ ΓΔ πρὸς ΕΖ.

Ἀλλὰ δὴ πάλιν ἔστω τὸ ΓΔ ἑκατέρου τῶν ΑΒ, ΕΖ μεῖζον· καὶ ἔστω τὸ μὲν ΑΒ τοῦ ΓΔ ἥμισυ μέρος, τὸ δὲ ΓΔ τοῦ ΕΖ ἐπίτριτον. ἐπεὶ οὖν οἵων ἐστὶν τὸ ΑΒ δύο τοιούτων τὸ ΓΔ τεσσάρων, οἵων δὲ τὸ ΓΔ τεσσάρων τοιούτων τὸ ΕΖ τριῶν, καὶ οἵων ἄρα τὸ ΑΒ δύο τοιούτων τὸ ΕΖ τριῶν.

συνῆκται ἄρα πάλιν ὁ τοῦ ΑΒ πρὸς ΕΖ λόγος διὰ τοῦ ΓΔ μέσου ὅρου ὁ τῶν δύο πρὸς τὰ τρία τοῦ τε ὑποδιπλασίου καὶ τοῦ ἐπιτρίτου, ὁ ὑφημιόλιος. ὁμοίως δὴ καὶ ἐπὶ πλειόνων καὶ ἐπὶ τῶν λοιπῶν πτώσεων.

Καὶ δῆλον ὡς ὅτι ἐὰν ἀπὸ τοῦ συγκειμένου λόγου εἷς ὁποιοσοῦν τῶν συντιθέντων ἀφαιρεθῇ, ἑνὸς τῶν ἄκρων ἀφανισθέντος, ὁ λοιπὸς τῶν συντιθέντων καταλειφθήσεται.

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