APOLLONIUS OF PERGACONICS. BOOKS ONE - SEVEN

INTRODUCTION

A. Apollonius at Perga

Apollonius was born at Perga (Περγα) on the Southern coast of Asia Minor, near the modern Turkish city of Bursa. Little is known about his life before he arrived in Alexandria, where he studied. Certain information about Apollonius' life in Asia Minor can be obtained from his preface to Book 2 of Conics.

The name "Apollonius"(Apollonius) means "devoted to Apollo", similarly to "Artemius" or "Demetrius" meaning "devoted to Artemis or Demeter".

In the mentioned preface Apollonius writes to Eudemus of Pergamum that he sends him one of the books of Conics via his son also named Apollonius. The coincidence shows that this name was traditional in the family, and in all probability Apollonius' ancestors were priests of Apollo.

Asia Minor during many centuries was for Indo-European tribes a bridge to Europe from their pre-fatherland south of the Caspian Sea.
The Indo-European nation living in Asia Minor in 2nd and the beginning of the 1st millennia B.C. was usually called Hittites.

Hittites are mentioned in the Bible and in Egyptian papyri. A military leader serving under the Biblical king David was the Hittite Uriah. His wife Bathsheba, after his death, became the wife of king David and the mother of king Solomon.

Hittites had a cuneiform writing analogous to the Babylonian one and hieroglyphs analogous to Egyptian ones.

The Czech historian Bedrich Hrozny (1879-1952) who has deciphered Hittite cuneiform writing had established that the Hittite language belonged to the Western group of Indo-European languages [Hro]. Hence it is clear that such nations of Europe as Greeks and Romans, Galls and Goths, Slavies and Lithuanians were descendants of Hittite tribes. As the masculine words in the most ancient of these languages have the endings of -os, -us, -as, -es, -is, the Hittite masculine words had ending of -ash, -ush, -ish.

The Hittite word "vadar" for water is near to the Russian and Czech "voda", English "water", German "Wasser", and Greek "υδωρ ". The Hittite word "pahhur" for fire is near to the English word "fire", German "Feuer", and Greek "πυρ". The Hittite word "gordion" for town is near to Russian "gorod" and "ograda", Czech "hrad", English "garden", and German "Garten". The Hittite word “eshmi” for “I am” is near to Russian “yesm’ “, Czech “jsem”, Latin “sum”, Greek “ειμι “ and English “I am”.

In the first millennium B.C., after migration of Hittite tribes from the East to the west of Asia Minor and to Europe, the Hittite Empire disintegrated and many separate Hittite kingdoms appeared. The most important of these kingdoms were situated in the Western part of Asia Minor. The most famous cities of these Hittite kingdoms were Ilion in Troy, Pergamum in Moesia, Sardis in Lydia, Gordion in Phrygia, and Myres in Lycia. The king of Lydia Croeses was famous for his richness; with the name of the king of Phrygia Gordias was connected the legend of “Gordias’ knot”. City of Pergamum was the first city where pergament was made.

In the same millennium on the Jonian coast of Asia Minor the Greek cities Miletus, Ephesus and others appeared.

During the Greek - Persian wars all of Asia Minor was occupied by the Persians. After the victory of Greeks all Hittite states of Asia Minor became Greek states. In this period Pergamum was the cultural and scientific center of Asia Minor.

Later all these states were conquered by Romans and became provinces of the Roman Empire. After the division of this empire into Western and Eastern parts, Asia Minor entered into Byzantium. In 14-15th centuries Asia Minor was conquered by Turks and entered into Turkey.

The Greek state where the city Perga was located had the name Pamphylia. This name, as well as its Hittite prototype, meant “belonging to all tribes”. This name shows that Pamphylia played an exclusive role among Hittite states.

It is explained by the fact that main shrines common for all Hittite tribes were situated there. B.Hrozny proved that Greeks borrowed from Hittites the cults of the god of thunder, Zavaya, the god of Sun, Apulunash, and his sister-twin goddess of Moon, Artimu, whom they called Zeus, Apollo and Artemis [Hro, p.147].

The Hittite name “Perga” is near to Greek “πυργος” and German “Burg” and means “tower, castle”; in the original sense of the word “perga”, “rock”, is near to German “Berg” - “mountain”. This word was connected with the words “perunash” and “perginash” meaning “god of thunder, destroyer of rocks”. The word “perga” enters in the name of the city Pergamum.

Hittite Perga was the center of the cults of Zavaya, Apulunash, and Artimu.

When Perga became a Greek town, the main shrines of Zeus and Apollo were moved to Olympia and Delphi, and the main shrine of Artemis was left in

Perga. The other shrine of Artemis, one of the “Seven Wonders of the World”, was also situated in Asia Minor at Ephesus.

Herodotus in his History wrote that kings of some Hittite states sent rich gifts to the Apollo’s shrine in Delphi, where the shrine was situated in his time. No doubt that they in fact sent their gifts into Perga.

It is very probable that Apollonius’ kin comes from priests of Apulunash.

B. Apollonius at Ephesus

In the preface to Book 2 of Conics, Apollonius writes to Eudemus of Pergamum that he sends him his son Apollonius bringing the second book of Conics. He asks Eudemus to acquaint with this book Philonides, the geometer, whom Apollonius introduced to Eudemus in Ephesus, if ever he happens to be about Pergamum.

German historian Cronert [Cro] reports that Philonides was a student of Eudemus, mathematician and philosopher - Epicurean, who later worked at the court of Seleucid kings Antioch IV Epiphanus (183-175 B.C.) and Demetrius I Soter (162-150 B.C.).

Eudemus was the first teacher of Philonides. No doubt that Eudemus was also the teacher of Apollonius at Ephesus, and it is natural that Apollonius sent him his main work.

When Apollonius finished his study at Ephesus, Eudemus recommended that he continue his study at Alexandria.

C. Apollonius at Alexandria

Apollonius’ teachers at Alexandria were pupils of Euclid. In the preface to Book 1 of Conics, Apollonius writes that he composed this work at Alexandria.

Apollonius’ nickname in this scientific capital of the Hellenistic world was “Epsilon”. Since the nickname of Eratosthenes was “Beta”, it is clear that the most great Alexandria mathematicians had as nicknames the first letters of the Greek alphabet: Euclid - “Alpha”, Archimedes - “Gamma”, and Conon of Samos - “Delta”

Apollonius’ first works were on astronomy. Claudius Ptolemy quotes in Chapter 1 of Book 12 of Almagest Apollonius’ non-extant work on equivalence of epicyclic and eccentric hypotheses of motion of planets. This quotation shows that Apollonius was one of the initiators of the theory of motion of planets by means of deferents and epicycles presented in Almagest.

Further works of Apollonius were devoted to mathematics. Since his main work Conics and many treatises were on geometry, Apollonius was called at Alexandria “Great Geometer”.

D. Conic sections before Apollonius

The appearance of conic sections was also connected with the cult of Apollo. There sections were used for solving the so-called Delic problem of duplication of cube.

This problem was connected with following legend: on the island Delos, believed to be the place of birth of Apollo and Artemis, a plague epidemic broke out. The inhabitants of the island appealed to the shrine of Apollo at Delphi for aid. The priests of the shrine told them that they must duplicate the cubic altar of the shrine. The Delians made the second cube equal to the first one and stood over it, but the plague did not cease. Then the priests told that the double altar must be cubic like the old one. If the edge of the old altar was equal to a, the edge of the new altar must be equal to the root of the equation

$$x^3 = 2a^3 . \quad (0.1)$$

It is possibly that the legend on the duplication of Apollo’s cubic altar appeared earlier when the main shrine of Apollo was at Perga.

The problem of duplication of a cube was solved by some Greek mathematicians of the 4th c. B.C. Menaechmus found that this problem can be reduced to the finding two mean proportionals between a and b, that is

$$a : x = x : y = y : b \qquad (0.2)$$

for $b = 2a$.

Menaechmus found that the solution x of equation (0.1) is equal to the abscissa of the point of intersection of two parabolas $x^2 = ay$ and $y^2 = 2ax$ or of one of these parabolas with the hyperbola $xy = 2a^2$.

Menaechmus defined a parabola as the section of the surface of a right circular cone with right angle at its vertex by a plane orthogonal to a rectilinear generator of the cone, and a hyperbola as the analogous section of the surface of a right circular cone with obtuse angle at its vertex. The equations of these conic sections are determined by equalities (0.2).

The works of Menaechmus are lost. The first known titles of works on conic sections are On Solid Loci (Περι στερεοι τοποι) by Aristaeus and Elements of Conics (Κωνικων στοιξεια) by Euclid. Both of these works are also non-extant, but it is known that Aristaeus' work consisted of 5 books and Euclid's work consisted of 4 books.

Ancient mathematicians used the word "locus" for lines and surfaces. Modern mathematicians regard lines and surfaces as sets of points, but this viewpoint was impossible for ancient scientists because they could not conceive that a set of points having no sizes has a non-zero length or a non-zero area. Aristotle wrote in his Physics: "Nothing that is continuous can be composed of indivisible parts: e.g., a line cannot be composed of points, the line being continuous and the point indivisible [Ar, p. 231a]. Therefore ancient mathematicians regarded lines and surfaces only as "loci" (τοποι), that is places for points.

Greek mathematicians called straight lines and circumferences of circles that can be drawn by a ruler and compass "plane loci" and conic sections they called "solid loci".

Conic sections are considered in many works of Archimedes who called a parabola a "section of right-angled cone", single branch of a hyperbola - a "section of obtuse-angled cone", and an ellipse - a "section of acute-angled cone". Archimedes called a paraboloid of revolution a "right-angled conoid" and a single sheet of a hyperboloid of revolution of two sheets an "obtuse-angled conoid". No doubt that Menaechmus, Aristaeus, and Euclid used the same names of conic sections.

The equations of parabolas used by Menaechmus for solving the Delic problem are particular cases of the equation

$$y^2 = 2px \quad (0.3)$$

in the system of rectangular coordinates whose axis 0x is the axis of symmetry of this parabola and whose axis 0y is the tangent to this parabola at its vertex. The magnitude p is now called the parameter of the parabola.

Euclid in Prop. II.14 of Elements proves that if Β is an arbitrary point of the circumference of a circle with the diameter ΑΧ, and Δ is the basis of the perpendicular dropped from Β onto ΑΧ, the line ΒΔ is mean proportional between ΑΔ and ΔΧ, that is ΑΔ:ΒΔ = ΒΔ:ΔΧ. If we denote ΑΔ = x, ΔΧ = x', ΒΔ = y, we obtain the equation

$$y^2 = xx' \qquad (0.4)$$

of the circumference with “two abscissas” in the system of rectangular coordinates whose axis 0x = AX and axes 0y and 0y' are tangents to the circumference at the points A and X.

Archimedes in Prop. I.4 of his treatise On Conoids and Spheroids proves that an ellipse can be obtained from a circumference of a circle by the contraction to its diameter in the direction perpendicular to this diameter

$$x' = x\ ,\ \ y' = ky \qquad (0.5)$$

where $k < 1$. Therefore the equation with two abscissas of an ellipse in the system of rectangular coordinates whose axis 0x is the major axis of the ellipse and axes of ordinates are tangents to the ellipse at the ends of its major axis has the form

$$y^2 = k^2xx'\ . \qquad (0.6)$$

The branch of a hyperbola used by Menaechmus in the system of rectangular coordinates whose axes are asymptotes of the hyperbola is determined by the equation $xy = \text{const}$. In another system of rectangular coordinates, whose axis 0x is the axis of symmetry of the hyperbola, and axes of ordinates are tangents to both branches of the hyperbola at their vertices, this hyperbola is determined by equation (0.4).

An arbitrary hyperbola can be obtained from the equilateral hyperbola used by Menaechmus by transformation (0.5), which is a contraction to the axis of symmetry of this hyperbola for $k <1$ and a dilatation from this axis for $k >1$. Therefore the equation with two abscissas of an arbitrary hyperbola in the system of rectangular coordinates whose axis 0x is the axis of symmetry of the hyperbola and the axes of ordinates are tangents to both branches of the hyperbola at their vertices has form (0.6).

Archimedes determined ellipses and hyperbolas by equations (0.6).

If the major axis of an ellipse and the real axis of a hyperbola are equal to 2a and the minor axis of an ellipse and the imaginary axis of a hyperbola are equal to 2b, the coefficient k in equations (0.6) is equal to b/a. in the case of the ellipse $x'= 2a - x$ and in the case of the hyperbola $x' = 2a + x$. Therefore these equations have the form

$$y^2 = (b^2/a^2)x(2a - x) \qquad (0.7)$$

for the ellipse and

$$y^2 = (b^2/a^2)x(2a + x) . \quad (0.8)$$

for the hyperbola. If we denote $b^2/a = p$, equations (0.7) of an ellipse can be rewritten as

$$y^2 = 2px - (p/a)x^2 , \quad (0.9)$$

equations (0.8) of a hyperbola can be rewritten as

$$y^2 = 2px + (p/a)x^2 . \quad (0.10)$$

Equations (0.9) and (0.10) are given in the systems of the rectangular coordinates whose axis 0x is the major axis of the ellipse and the real axis of the hyperbola, and whose axis 0y is tangent to the ellipse at the left end of its major axis and tangent to the hyperbola at the right end of its real axis. Magnitudes p in these equations are called parameters of the ellipse and hyperbola.

E. Structure of Conics

Apollonius' Conics consisted of 8 books. Books 1-4 are extant in Greek original, Books 5-7 are extant only in medieval Arabic translations by Thabit ibn Qurra edited by his teachers Ahmad and al-Hasan banu Musa ibn Shakir, Book 8 is lost.

The books of Conics consist of prefaces addressed to Eudemus or Attalus of Pergamum, definitions, and propositions.

Apollonius' propositions, like propositions of Euclid's Elements, are theorems or problems.

In the beginning of every proposition, its general statement in italic and its formulation with notations of points and lines are given. The formulations of propositions Apollonius begins with the words Λεγω - "I say".

After that, the proof of a theorem or the solution of a problem follows. In beginning of the solution of every problem its analysis is given, where known points and lines are indicated; next, the synthesis, that is the required construction, is described.

Apollonius' style is very concise, therefore the translators insert in the text explanatory words in brackets and references to Euclid and Apollonius' propositions in parentheses.

F. Editions of Conics

The most important editions of Apollonius' Conics are:

[Ap1] - the first Latin translation of Books 1-4 published by Federigo Commandino (1509-1575).

[Ap 2] - the Greek text of Books 1-4 and the Latin translation of all 7 books published by Edmund Halley (1656-1742).

[Ap 3] - the critical Greek text of Books 1-4 established by Johan Ludvig Heiberg (1854-1928) and published by him with the Latin translation.

[Ap 4] - the English translation of Books 1-3 by Robert Catesby Taliaferro (1907-1987) published by Encyclopedia Britannica in the Great Books of the Western World series. The translation of Book 1 was first published in 1939 by St. John's College at Annapolis in The Classics of the St. John's Program series.

[Ap 5] - the revised edition of the translation [Ap4] published by Dana Densmore and William H. Donahue.

[Ap 6] - the English translation of Book 4 by Michael N. Fried (b. 1960). This translation was first published as Appendix to the book [FU](pp.416 -485).

[Ap 7] - the critical Arabic text of Books 5-7 established by Gerald James Toomer (b. 1934) and published by him with the English translation and commentary

Critical Arabic text is based on 3 manuscripts: Oxford one, translated by Halley; Istanbul one, published in [Ap12]; and Teheran one.

[Ap 8] - the detailed English exposition of all 7 books on the basis of the editions [Ap 2] and [Ap3] published by Thomas Little Heath (18611940).

[Ap9] - commented French translation of all 7 books published by Paul Ver Eecke.

[Ap10] - German translation of Books 1- 4 published by Arthur Czwalina.

[Ap11] - the Greek text of Heiberg reproduced and published with the Modern Greek translation of all 7 books by Euangelos Stamatis (1898-1990).

[Ap12] - facsimile edition of the Istanbul manuscript of the medieval Arabic translation of all 7 books by Hilal al-Himsi and Thabit ibn Qurra copied by the famous mathematician and physicist al-Hasan Ibn al-Haytham (965-ca.1050) prepared by Nazim Terzioglu (1912- 1976).

[Ap13] - commented Russian translation of 20 propositions by I. Yagodinsky (1928).

[Ap14] - commented Russian translation of all 7 books published by B. A. Rosenfeld - in press.

Many mathematicians undertook attempts of restoration of Book 8. Let us mention the attempt by Ibn al-Haytham [IH] published with the English translation by Jan Pieter Hogendijk (b.1955) and the attempt by Halley added to his translation [Ap2].

Let us mention the excellent exposition of Apollonius' Conics: [Ze] - The Theory of Conic Sections in Antiquity by Hieronymus Georg Zeuthen (1839-1920).

[Hea, pp.126-196] - in the book A History of Greek Mathematics by T.L. Heath.

[VdW, pp.241-261] - in the book The Science Awakening by Bartel Leendert Van der Waerden (1903-1996).

[VZ, pp.97-108] - in the book History of Mathematics by Michail E. Vashchenko-Zakharchenko (1825-1912).

[IM, pp.129-139] - in the book History of Mathematics from most ancient times to beginning of 19th century, vol.1 by Adolf P. Yushkevich (1906-1993). [Too] - the article Apollonius of Perga by G. J. Toomer. See also Introduction to his edition [Ap7],

[FU] Apollonius of Perga's Conica. Text, Context, Subtext by M.N.Fried and Sabetai Unguru.

[Rho] - Apollonius of Perga, Doctoral Thesis by Diana L. Rodes (2005)

[Ro3] - Apollonius of Perga (in Russian by B.A.Rosenfeld 2003). See also his article [Ro4].

G. Other mathematical works of Apollonius

Besides Conics Apollonius was the author of following mathematical works:

1) Cutting off of a ratio (Λογου αποτομα) in two books.
2) Cutting off of an area (Χωριου αποτομα) in two books.
3) Determinate section (Διωρισμενα τομα) ιν two books.
4) Inclinations (Νευσεις) in two books.
5) Tangencies (Επαφαι) in two books
6) Plane loci (Τοποι επιπεδοι) in two books.
7) Comparison of dodecahedron and isocahedron (Συγκρισις δωδεκαεδρου και εικοσαεδρου).
8) On non-ordered irrationals (Περι των ατακτων αλογων).
9) Rapid obtaining of a result (Ωκυτοκιον).
10) Screw lines (Κοξλιας).
11) Treatise on great numbers.

12) General treatise (Καθολου πραγματεια).

From these works only treatise (1) is extant in medieval Arabic translation. There are the Latin translation [Ap15] by E. Halley and the English translation [Ap16] by E.M.Macierowski of this treatise.

The short expositions of treatises (1) - (6) are given by Pappus of Alexandria (3rd c. A.D.) in Book 7 of Mathematical Collection [Pa, pp. 510 -546; Ap11, vol.1, pp.100 - 120].

The fragments of medieval Arabic translations of these treatises and English translations of these fragments are published by J.P.Hogendijk [Ho].

In works (1) and (2) the following problems are solved: given two straight lines ΑΒ and ΧΔ with fixed points Α and Χ, to find two points Β and Δ, such that, in the case of treatise (1), the ratio ΑΒ/ΧΔ would be equal to the given ratio, and, in the case of treatise (2), the product ΑΒ.ΧΔ would be equal to the given area.

In treatise (3) the problems of the following type are solved: given four points Α, Β, Χ, Δ on a straight line, to find a point Π such that ratio ΑΠ.ΧΠ/ΒΠ.ΔΠ would have the given or an extremal value. The last problem is equivalent to the problem of determining an extremum of a function that is a ratio of two quadratic polynomials.

In work (4) the problems equivalent to quadratic and cubic equations are solved by geometrical means called "inclinations".

In treatise (5) the problem of construction of a circle tangent to given objects of three kinds, which can be circles, straight lines, and points, is solved.

In treatise (6) theorems on plane loci, which is on circles and straight lines, are proven. In this treatise, homotheties, inversions with respect to circles, and other transformations mapping plane loci to plane loci are considered.

There is only the commentary on work (7) by Hypsicles (2nd -1st c. B.C.) added to Euclid's Elements as Book 14 [Ap11, vol.1, pp.60-66]. In this work, Aristaeus' treatise Comparison of five solids is mentioned, where the theorem, that if a cube and a regular octahedron are inscribed in the same sphere, then as their volumes are one to the other, so their surfaces are one to the other, is proven. Apollonius proves analogous theorem on regular dodecahedron and icosahedron inscribed in the same sphere.

The commentary by Pappus on the work (8) is extant only in the medieval Arabic translation [Ap11, vol.1, pp. 134-144]. This commentary shows that in this treatise, besides quadratic irrationals considered in Book 10 of Euclid's Elements, cubic and higher irrationals are also considered.

Work (9) is mentioned by Eutocius (6th c. A.D.) on Archimedes Measuring a circle [Ap11, vol.1, p. 48]. This information shows that in the treatise, the approximate value of the ratio of the circumference of a circle to its diameter was found in a more rapid way than in Archimedes' work.

The work (10) is mentioned by Proclus Diadochus (5th c. A.D.) in his commentary on Book 1 of Euclid's Elements [Ap11, vol.1, p. 144]. According to this information, in the work (10) screw lines in the surface of a right circular cylinder are considered.

The commentary by Pappus on the work (11) is extant in Book 2 of his Mathematical Collection [Ap.11, vol.1, pp. 70-72]. The beginning of this book containing the title of the work (11) is lost. The commentary shows that in this work a system of names of great numbers was proposed, which later was improved by Archimedes in Psammites.

The work (12) is mentioned by Marinus (5th c. A.D.) in his commentary to Euclids' Data together with Apollonius' Inclinations [Ap11, vol.1, pp.68-70] Therefore it is clear that this work is geometrical. Probably, in it, like in Inclinations, problems equivalent to algebraic equations were solved by geometrical methods. The title of the work (12) shows that these methods were more general than inclination. Probably, in this work Apollonius described the methods used by him for obtaining proportions from which he derived in Prop. I.11 - I.13 of Conics equations of parabola, hyperbola and ellipse and proportions equivalent to algebraic equations of evolutes of conics given by him in Prop. V.51 and V.52.

Some mathematicians of Western Europe undertook attempts to restore lost works of Apollonius. F.Viete (1540-1603) in [Vi] and M.Ghetaldi (1566-1622) in [Ghe1] restored Tangencies. Ghetaldi in [Ghe2] - Inclinations. F.van Schooten (1615-1660) [Sch] and P.Fermat (1601-1665)[Fe] -Plane loci.

H. Letters and their numerical values

The Greek alphabet of the classic epoch consisted of 24 letters, which had following numerical values:
Α,α -- alpha=1, Β,β -- bēta=2, Γ,γ -- gamma=3, Δ,δ -- delta =4, Ε,ε -- epsilon=5, Ζ,ζ -- zēta=7, Η,η -- ēta= 8, Θ,θ -- theta=9, Ι,ι -- iota=10, Κ,κ -- kappa=20, Λ,λ -- lambda=30, Μ,μ -- mu=40, Ν,ν -- nu=50, Ξ,ξ -- xi =60, Ο,ο -- omicron=70, Π,π -- pi=80, Ρ,ρ -- rho=100, Σ,σ -- sigma=200, Τ,τ -- tau=300, Υ,υ -- upsilon=400, Φ,φ -- phi=500, Χ,χ -- chi= 600, Ψ,ψ -- psi=700, Ω,ω -- omēga=800. Numbers 6, 90, 900 were represented by 3 archaic letters F -- wau, Ϙ -- koppa, □ – sabi or sampi.

The last of these letters was not in use in the most ancient times, the first and second ones were used during the time when the Latin alphabet was created, on the base of the Greek one. From them, Latin letters F and Q were derived. The numbers 1000, 2000,3000, etc. were represented by Α’, Β’, Γ’, etc. Apollonius used these letters for numbering propositions in Conics.

Claudius Ptolemy, who borrowed from Babylonian astronomers not only information on their observations but also sexagesimal fractions, used these letters from Α =1 to ΝΘ =59 for recording of sexagesimal fractions. Zero in these fractions was denoted by the first letter of the word ουδεν -- “nothing”, hence our figure 0 came.

The Greek letters and their names came from Phoenician letters. These letters were invented in the city of Biblos where Egyptians imported Lebanese cedars. Phoencians replaced Egyptian hieroglyphs denoted the things imaging by them by the letters denoting the first sounds of the names of these things. Phoenician letters are images of things whose names begin from these letters, for instance, the letter “aleph” meaning “bull head”, has the form of the turned A, and hence Greek alfa came, the letters “ beth” meaning “house” has the form of the rectangle with the gap in the lower side, hence Greek beta came

Phoenician letters likewise have numerical values.

The Greek letters from Α to Π have the same numerical values as corresponding Phoenician letters. The value 90 was denoted by the Phoenician letter “cade” from which the letter □ came, whose name is “sampi” or “sabi”. The numerical values of the Phoenician letters corresponding to Greek letters Ϙ, Ρ, Σ, and Τ are 100, 200, 300, and 400.

From Phoenician letters also Hebrew and Arabic letters came .The names of Hebrew letters are the same as of Phoenician ones. Arabs added to these letters, which came from Phoenician ones, six new letters. The names of Arabic letters are simplified Phoenician names. The numerical values of Hebrew and Arabic letters, which came from the same Phoenician letters, have the same values of these letters.

The names and numerical values of arabic letters are as follows:
alif – 1, ba—2, te—400, tah—500, jiv—3, tḥa, dal–4, dhal–600, ra–200, za–7, sin–60, shin–300, ṣad–90, ḍad–800, ta–9, ẓa–900, ain–70, gain–1000, fa–80, qaf–100, kaf–20, lam–30, mim–40, nun–50, waw–6, ha–5, ia–10.

In editions [Ap5] and [Ap6], Greek letters in Apollonius’ diagrams and text are represented by Latin letters. In edition [Ap5], Greek letters are represented by the different Latin letters.

In proposition 53 of Book 2 in edition [Ap5], the archaic letters ς and Ϙ are represented by the letters X’ and Y’.

In edition [Ap7] Arabic letters in Thabit ibn Qurra’s diagrams are represented by Greek letters with the same numerical values. The letter “waw” with numerical value 6 is represented not by F, but by ς, the letter "ghayn" with numerical value 1000 is represented not by A' but by ι.

Edition [Ap12] shows that in the translation by Hilal al-Himsi of Books 1-4 Greek letters in Apollonius' diagrams are represented by Arabic letters which came from the same Phoenician letters, but in the translation by Thabit ibn Qurra of Books 5-7 Greek letters of Apollonius are represented by Arabic letters according to a more complicate rule. For instance, the first three letters Α, Β, and Γ he transcribes by the first three Arabic letters "alif", "ba", and "ta".

In our translation we transcribe Arabic letters in diagrams and text in Books 5-7 by the same Greek letters as in the translation by Toomer.

COMMENTARY ON BOOK ONE

Preface to Book I

1. Apollonius dedicated Books 1-3 of Conics to his teacher Eudemus of Pergamum (see Introduction, B), with whom he discussed the structure of this work. Last books of Conics finished after Eudemus' death Apollonius dedicated to Eudemus' student Attalus.

The preface to Book1 is essentially the general preface to the whole of Conics.

2. Apollonius' information on geometer Naucrates is the only known to us mention of this scholar. Naucrates was a friend of Apollonius, visited him at Alexandria, discussed with him theory of conics, and Apollonius gave him the first variant of Conics.

The name of Naucrates is connected with the word ναυκρατια - "power of seamen". The name Naucratis of the town founded by Greek seamen in the delta of Nilus in 5th c. B.C. is connected with the same word.

3. Apollonius' words that the first four books of Conics contain the elements of theory of conics show that these four books are revisions of Euclid's Elements of conics.

4. “Three sections” are three conics which Euclid and Archimedes called “sections of right, obtuse and acute cones”. Apollonius called them a parabola, a hyperbola, and an ellipse. Like his precursors, Apollonius used the term “hyperbola” only for a single branch of a hyperbola.

Unlike his precursors, Apollonius considers two branches of a hyperbola and calls them ἀντικειμ ένaι - “opposite”. In [Ap5], [Ap6], and [Ap7], this term is translated as “opposite sections”.

5. “A locus with respect to three straight lines” l_1, l_2, and l_3 is a locus of points whose distances d_i from the lines l_i satisfy the equation

$$d_1 d_3 = k d_2^2 \quad . \qquad (1.1)$$

“A locus with respect to four straight lines” l_1, l_2, l_3, and l_4 is a locus of points whose distances d_i from the lines l_i satisfy the equation

$$d_1 d_3 = k d_2 d_4 \quad . \qquad (1.2)$$

The loci with respect to three or four straight lines are conic sections. Apollonius believes that Euclid’s proof of this fact in his Elements of Conics is not sufficient and can be completed by the theorems in Book 3 of Conics. This proof was fulfilled by R.C.Taliaferro in Appendix A to his translation of Conics [Ap5, pp.267-275].

This fact was proved by means of analytic geometry by the creator of this discipline Rene Descartes (1596 - 1650) in his Geometry.

First Definitions

6. Apollonius defines a conic surface as a surface described by a straight line of an indefinite length passing through a fixed point called the vertex and through points of the circumference of a circle the plane of which does not pass through the vertex. This surface consists of two surfaces located “vertically” on both sides of the vertex.

The straight line joining the vertex of a conic surface with the center of the circumference determining this surface Apollonius calls the axis of this surface.

Apollonius' definition differs from the one by Euclid. For Euclid a conic surface was the surface of a right circular cone formed by a rectangular triangle revolving around one of its catheti.

The Greek word "κώνος" originally meant "pine cone".

Ancient mathematicians used the terms "straight line" only for rectilinear segments, "plane"- only for bounded parts of planes, usually rectangles, "surface"- only for bounded parts of surfaces.

Ancient mathematicians never used the term "infinite" for lines, planes, and surfaces and replaced it by words "of indefinite length" and "of indefinite size".

7. Apollonius defines the cone as a solid bounded by a conic surface and the circle whose the circumference determines this surface. Apollonius calls the vertex of the conic surface "vertex of the cone", the circle with circumference determining the conic surface "the base of the cone", and the segment of the axis of the conic surface between the vertex and the base of a cone "the axis of the cone".

Unlike Euclid who considered only right circular cones, Apollonius considered cones that can be both right and oblique.

8. In the case when a plane curve has a family of parallel chords whose midpoints are on a straight line, Apollonius calls this straight line a diameter (διαμέτρος) of this plane curve.

Apollonius' definition of a diameter of a plane curve is the generalization of Euclid's definition of a diameter for the circumference of a circle. Diameters of circumferences are perpendicular to the chords bisected by them. Diameters of plane curves in a general case are not perpendicular to such chords.

If on the plane there is a system of oblique coordinates whose axis 0x coincides with a diameter of a plane curve and axis 0y is parallel to the bisected chords, the curve maps to itself by reflection

$$x' = x , \quad y' = -y . \quad (1.3)$$

This reflection is said to be either right or oblique depending on whether the coordinate angle x0y is right or acute, respectively.

The points of intersection of a diameter of a plane curve with its diameters Apollonius calls "vertices" of this curve.

The term "diameter" for curves that are not circumferences of circles was used by Archimedes, but only in the cases when the diameter is perpendicular to the bisected chords.

Diameters of conic sections are considered by Apollonius below.

9. An oblique reflection (1.3) is a particular case of an affine transformation in a plane, which is a bijective transformation in a plane mapping straight lines to straight lines.

Right and oblique contractions to a straight line and right and oblique dilatations from a straight line (0.4) are also affine transformations.

Since parallel lines have no common points, affine transformations map parallel straight lines to parallel ones. Therefore affine transformations map parallelograms to parallelograms and vectors to vectors, and if vectors x and y are mapped to vectors x' and y', the sum x + y is mapped to the sum x' + y' and a product kx by an arbitrary real number k is mapped to the product kx'. Therefore if Α, Β, Χ are three points in a straight line, the affine transformations preserve simple ratios of oriented segments $\varsigma = ΑΧ/ΑΒ$, and, in a general case, affine transformations in rectangular and oblique coordinates have the form

$$x' = \mathrm{A}x + \mathrm{B}y + \mathrm{X}, \quad y' = \Delta x + \mathrm{E}y + \Phi. \quad (1.4)$$

Under the affine transformation (1.4) the areas of all figures in the plane are multiplied by the absolute value of the determinant AE-BD. In the case when this value is equal to 1, the transformation (1.4) is an equiaffine one.

Since the determinants $\mathrm{AE}-\mathrm{B}\Delta$ of the reflections (1.3) are equal to -1, they are equiaffine transformations too. In particular, if transformation (1.3) maps a point Β to a point Β' and points Α and Χ are fixed points of this transformation, the triangles ΑΒΧ and ΑΒ'Χ have the same base and equal heights and, therefore, equal areas.

Equiaffine and general affine transformations were used by Thabit ibn Qurra and by his grandson Ibrahim ibn Sinan (908-946), respectively.

For the affine geometry and its history see [Ro1, pp.106-114] and [Ro2, pp. 130-133, 143-146].

10. The segments of the bisected chords between the curve and the diameter are called by Apollonius τεταγμένως κατ ήχθται -- “applied in order”.

Federigo Commandino (1509-1575) in his Latin translation [Ap1] of Conics wrote the above expression as “*ordinatim applicatae*” from which the term “ordinates” had come. Therefore in editions [Ap5], [Ap6], [Ap7] this Apollonius’ expression is translated as “lines drawn ordinatewise”.

11. If two plane curves have a family of parallel chords whose midpoints are on a straight line, Apollonius calls this line a “transverse diameter of the two plane curves”. The points of intersection of the transverse diameter with the curves Apollonius calls “vertices“ of these curves.

If the midpoints of the parallel straight lines joining two plane curves are on a straight line, Apollonius calls this line an “upright diameter” of these plane curves.

The segments of parallel chords between the curves and the transverse diameter are called the ordinates of points of these curves.

Transverse and erect diameters were used by Apollonius for two “opposite hyperbolas”.

12. The diameter drawn in the direction of parallel chords is called by Apollonius a conjugate one with the diameter bisecting these chords.

13. The diameter of a plane curve as well as transverse and upright diameters of two plane curves are called by Apollonius “axes” when these diameters are perpendicular to the chords bisected by them. Two perpendicular axes of one or of two plane curves Apollonius calls “conjugate axes”.

Later Apollonius considers axes of conics.

14. Modern mathematicians use the terms “diameter” and “axis” for conic sections in the same sense as Apollonius, while the term “vertex” is used in the same sense as by Apollonius’ precursors, i.e. as a point of intersection of a conic section with its axis.

Propositions I.1-I.5 on cones

15. In Prop. I.1 Apollonius proves that a straight line joining the vertex of a conic surface and any point on the latter lies entirely on this surface.

In the porism (corollary) to this proposition, Apollonius proves that the straight line joining the vertex of a conic surface with any point which is within this surface lines entirely within this surface, and the straight line joining the vertex of the conic surface with any point which is outside this surface lies entirely outside this surface.

16. In Prop. I.2 Apollonius proves that the segment joining two points of a vertical sheet of this conic surface and its continuation and not passing through the vertex of the cone lies within the cone, and continuations of this segment lie outside the cone.

Apollonius does not prove an analogous proposition: the segment joining two points of two vertical sheets of a conic surface and not passing through the vertex of the cone lies outside the conic surface and continuations of this segment lie within the conic surface.

No doubt that Apollonius did not prove this proposition since it was not in Euclid’s Elements of conics.

Note that the line which is the sum of two abscissas of a point of the ellipse (0.7) joins two points of a surface of a cone, and the line which is the difference of two abscissas of a point of the hyperbola (0.8) joins two points of different sheets of a conic surface. Apollonius calls these segments latera transversa of an ellipse and a hyperbola.

17. In Prop. I.3 Apollonius proves that the section of a cone by a plane passing through its vertex and meeting its base is a triangle.

18. In Prop. I.4 Apollonius proves that the section of the surface of a circular cone by a plane parallel to its base is the circumference of a circle.

19. In Prop. I.5 Apollonius proves that the surface of an oblique circular cone besides sections parallel to its base has another family of circular sections. It can be explained by the fact that the section of the surface of an oblique circular cone by a plane perpendicular to its axis is an ellipse, therefore the solid bounded by this plane and the conic surface is a right elliptic cone. Since the ellipse has two perpendicular axes of symmetry, the right elliptic cone and its surface have two perpendicular planes of symmetry passing through the axes of symmetry of an ellipse and the vertex of a cone.

The reflection with respect to one of these planes maps any circular section of the cone parallel to its base to itself. The reflection with respect to the second plane maps circular sections parallel to the base of the cone to circular sections of the second family. Apollonius calls the circles bounded by circumferences of different families and the planes of these circles ὑπεναντ ία , which we following P. Ver Eecke [Ap 9, p.10] translate as "antiparallel". The expressions of Apollonius "the line is equal in square to the rectangular plane" means that the square on the line is equal to mentioned plane.

20. Apollonius' abbreviations "ὑπό ΑΒΓ", "υπο ΑΒ, ΓΔ", and "ἀπό ΑΒ", which mean a rectangular plane with sides ΑΒ and ΒΓ, a rectangular plane with sides ΑΒ and ΓΔ, and a square with a side ΑΒ, we translate by the abbreviations pl.ΑΒΓ, pl.ΑΒ,ΓΔ and sq.ΑΒ, respectively.

Prepositions υπο and απο mean "under" and "on".

The expressions of Apollonius "the line is equal in square to the rectangular plane" means that the square on the line is equal to mentioned plane.

21. Prop. I.5 forms the basis for the theory of stereographic projection, that is the projection of a sphere from its point P onto the plane tangent to the sphere at its antipodal point. If a curve on the sphere, not passing through the point P under this projection is mapped onto the circumference of a circle, then the projecting lines are rectilinear generators of a circular cone.

If this cone is right, the plane of the projected curve is parallel to the plane of projection. If the cone is oblique, the plane of the projected curve is antiparallel to the plane of projection. In both cases, the projected curve is the circumference of a circle. Thus stereographic projection maps circumferences of circles on the sphere not passing through the point P to circumferences of circles on the plane.

If the circumference of a circle on the sphere passes through the point P, its plane intersects the plane of projection in a straight line, and the stereographic projection maps these circumferences to straight lines.

Apollonius knew stereographic projection. This is clear from the description by a Roman architect of the 1st c. B.C. Vitruvius Pollio in his Ten Books on Architecture of an astronomical instrument called "spider" (ἀραχνά), invented by "the astronomer Eudoxus, or as some say, Apollonius" [Vi1, p.256; Vi2, p.320].

Vitruvius wrote that the instrument contained bronze rods and "behind these rods there is a drum on which the firmament and zodiac are drawn and figured: the drawing being figured with the twelve celestial signs" [Vi1, p.261 Vi2, p. 322].

Daniele Barbaro (1513-1570) in his commentary on this Vitruvius' work describes the projection ("analemma") in a spider as follows: "Analemma is projected from the pole of the sphere onto a plane. To project the sphere onto the plane [by means of an analemma] is to describe in the plane all circles and all [zodiacal] signs that are on the sphere. Thus all that is on the sphere is represented in the plane according to the same optical mode as in making of the table of an astrolabe" [Vi2, p. 322].

These words show that the projection in a spider is stereographic. Therefore this instrument could not have been invented by Eudoxus who lived in 4th c. B.C. when the stereographic projection based on Proposition I.5 of Apollonius' Conics was not known yet.

The drum portrays the tropics, the ecliptic (the zodiacal circle), and the images of some most bright stars. These circles and images of stars form the figure similar to a spider, this fact explains the name of the instrument. The drum can rotate by means of a hydraulic machine.

The rods form a motionless part of the instrument. This part portrays the celestial equator, the tropics, the horizon, and circles of altitude over the horizon that is the parallels of the horizon. These circles form the "spider-web" in which the "spider" movies.

The ecliptic is the circumference of the great circle on celestial sphere where the visual annual way of the Sun is realized. Every day the Sun makes its way along the ecliptic about 1° . The ecliptic is divided into 12 zodiacal signs corresponding to the months. The ecliptic intersects the celestial equator at the beginnings of the signs of Aries and Libra where the Sun is on the days of the spring and autumn equinoxes. The Sun is at the maximal distance from the celestial equator at the beginnings of the signs of Cancer and Capricorn where the Sun is on the days of the summer and winter solstices. These last points under diurnal rotation of the celestial sphere describe the circumference of circles called tropics of Cancer and Capricorn.

The celestial equator as well as tropics is invariant under the diurnal rotation of the celestial sphere.

No doubt Apollonius knew that stereographic projection is conformal, that is it preserves the magnitudes of the angles between curves, because this property can be proved by means of Euclid's Elements. Let the stereographic projection with the pole Σ maps the point Ξ and circular arcs on the sphere with the tangents ΞΥ and Ξς to point Ξ' and the circular arcs on the plane with tangents Ξ'Υ' and Ξ'ς'. Let the points Υ and ς be the points of the intersection of the tangents ΞΥ and Ξς with the plane tangent to the sphere at Σ. The segments ΞΥ and ΥΣ are equal as two tangents to the sphere drawn from one point, and, analogously, Ξς =ςΣ. Therefore the triangles ΞΥς and ΣΥς are equal, because the angles ΥΞς and ΥΣς are equal.

Since the lines ΥΣ and Υ'Σ' are parallel, as the lines ςΣ and ς'Σ', the angle Υ'Ξ'ς' is equal to the angle ΥΞς. This equality means that stereographic projection is conformal.

It is well known that the celestial equator on the terrestrial equator is perpendicular to the horizon, and at the terrestrial poles it coincides with the horizon. If an observer is at the point with latitude ϕ, he sees that the celestial equator intersects the horizon under an angle equal to $90^o - \phi$.

The celestial equator and the tropics are represented by rods by circumferences of three concentric circles. Since the ecliptic touches both tropics, the image of it also touches the images of the tropics.

If the instrument is used at night, the altitude of a bright star is measured; at daytime, the altitude of the Sun is measured. The drum is turned to such position that the image of the star with the measured altitude or the image of the point of the ecliptic corresponding to the day of measuring the altitude of the Sun will be under the image of a circle of the measured altitude. Then the image of the whole firmament will be obtained at the moment of the observation and the spherical coordinates of all its points and stars can be found too. The altitude of a celestial point over the horizon is determined according to the altitude circle passing through the image of this point, the azimuth of this point is determined by the position of the image of this point on the altitude circle. In particular, the position of the “horoscope”, that is the point of intersection of the ecliptic and the eastern part of the horizon, which is important for astrological predictions, will be found.

The angle by which the drum turns determines the exact time of the moment of the observation.

Probably, the invention by Apollonius of an instrument for measuring time connected with a hydraulic machine described by an architect was known in medieval East, and therefore an Arabic treatise on a water clock dedicated to an architect (al-najjār al-muhandis) was ascribed to Apollonius. There are three manuscripts of this work entitled Treatise on construction of [an instrument with] a flute (Risala san a [ᶜāla] al-ẓamr) kept in Paris, London, and Beirut. The German translation of this treatise according to all manuscripts was published by E. Wiedemann [Wie]. When the surface of the water in this clock is dropped to a certain level, the sound of the flute is heard. F.Sezgin [Sez, p.143] also believes that this treatise is only ascribed to Apollonius.

An instrument similar to Apollonius' one called "horoscopical instrument" was described by Claudius Ptolemy in Planispherium.

Later, analogous instrument called μικρα ἀστρολάβον - "little [instrument] seizing stars" was invented by Theon of Alexandria in 4th c. A.D. He replaced Apollonius' motionless "spider-web" by a motionless metallic continuous disk called "tympanum", and the rotating drum - by a rotating fretted disk, also called "spider". Unlike in Apollonius' instrument, in Theon's "astrolabon" the motionless tympanum is located under the rotating "spider".

This instrument was very popular in the medieval East by the name "asturlab" and in medieval Europe as "astrolabium".

Now these instruments are called "astrolabes". Medieval astrolabes were portative circular cylinders, with a radius of 10 to 20 cm and a height of 4 to 5 cm. The cylinders contained 10 to 20 tympanums for different latitudes.

The operations with the medieval astrolabes were similar to ones with Apollonius' instrument.

On the lower base of medieval astrolabes the instrument for measuring altitudes of the Sun and the stars was situated. This instrument contained an alhidad with two diopters and arrows at the ends, which could rotate around the center of the cylinder base and whose arrows pointed out altitude on the degree scale on the circumference of the base. To measure the altitude of a celestial point, the astrolabe was suspended vertically, and the ahidad was directed to this point. The arrow of the alhidad showed the altitude of this point

Both Apollonius' instrument and the medieval astrolabes can be regarded as transparent nomograms, in which the role of the transparent is played by the upper part of the instrument.

On the stereographic projection and astrolabes see [Ro1, pp.116-117; 295-206] and [Ro2, pp. 121-130].

Propositions I.6 - I.10 on diameters and ordinates of conics

22. In Prop. I.6 Apollonius considers a circular cone, right or oblique, with the vertex A and the base BΓ. The triangle ABΓ containing the axis of the cone is called an axial triangle. From the point M of the circumference of the base, the perpendicular MN to the diameter BΓ of the base is dropped.

Apollonius proves that the line ΔE that is drawn from the point Δ on the surface of the cone parallel to MN and reaches the surface of the cone again is bisected by the plane ABΓ.

23. In Prop. I.7 Apollonius considers the same cone as in Prop. I.6. This cone is cut by the plane passing through the point H of the rectilinear generator AB of the cone and along the line ΔE in its base perpendicular to the diameter BΓ of the base or to continuation of this diameter. This plane cuts off from the surface of the cone the conic section ΔHE. Apollonius proves that chords of this conic parallel to ΔE are bisected by the plane ABΓ, and the line HΘ of the intersection of the planes ABΓ and ΔHE is a diameter of this conic.

Apollonius proves that these chords are perpendicular to the diameter bisecting them if the cone is right, and if the cone is oblique, and the axial triangle is perpendicular to the plane of the base of the cone.

The plane ΔHE can be inclined to the line AB under an arbitrary angle unequal to the angle of the inclination of planes parallel or antiparallel to the plane of the base of the cone. Therefore an arbitrary section of the cone that is not a pair of intersecting straight lines can be obtained from the circumference of the base of the cone by the central projection from the vertex of the cone.

24. Since every section of a cone that is not a pair of intersecting straight lines can be obtained from a circumference of a circle by a central projection, every such conic section can be obtained from the circumference of a circle by a projective transformation.

Projective transformations in a plane can be defined as follows. If the plane E is located in the space and is projected from a point Σ which is outside this plane onto another plane E', the plane E' is projected from a point Σ' onto a plane E", the plane E" is projected from a point Σ" onto a plane E''', and after several such projections the plane $E^{(k)}$ is projected from a point $\Sigma^{(k)}$ onto the plane E, we obtain a projective transformation in the plane E. This transformation is not bijective, and for it to become bijective all planes E, E' , E", . . ., $E^{(k)}$ must be supplemented by new points ,so that the supplemented planes will be in a bijective correspondence with a bundle of straight lines through a point in the space. These supplemented planes are called “projective planes”, the new points of these planes are called “points at infinity”.

The point M in the plane E is represented by the infinite straight line ΣM in the space or by vectors directed along this line. These vectors are determined up to a non-zero real multiplier. Elie Cartan (1869-1951) called these vectors “analytic points”. Points at infinity are represented by lines and vectors parallel to the plane E.

If three linearly independent vectors e_1 , e_2 , e_3 are given in the space, vectors x representing the point M in the projective plane can be written in the form of $x^1e_1 + x^2e_2 + x^3e_3$. The numbers x^i are called “projective coordinates” of points in the projective plane. These coordinates, as well as the vectors x representing the points, are determined up to a non-zero multiplier.

If vectors e_1 and e_2 are parallel to the plane E, then affine coordinates x and y of a point M in the plane E are connected with projective coordinates x^i of this point by correlations $x = x^1/x^3$, $y = x^2/x^3$.

Points in the projective plane that are on one straight line are represented by the lines of the bundle that are in one plane. These lines form a plane pencil of straight lines. Therefore straight lines in the projective plane are determined by equations

$$u_1x^1 + u_2x^2 + u_3x^3 = 0 . \quad (1.5)$$

Numbers u_i , called “tangential coordinates” of straight lines, as well as projective coordinates of points, are determined up to a non-zero real multiplier.

Since points at infinity of parallel straight lines in the plane E are represented by the same straight line of the bundle, parallel straight lines in the projective plane have a common point at infinity, that is they meet at this point. All points at infinity in the projective plane are represented by straight lines of a plane pencil and form a straight line called “the straight line at infinity”.

Projective transformations in the projective plane can be defined as bijective transformations in this plane preserving straight lines. These transformations map points at infinity to usual points, and therefore they map parallel lines to intersecting ones.

In projective coordinates, projective transformations have the form

$$'x^i = \Sigma_j A_j^i x^j . \quad (1.6)$$

In affine coordinates, these transformations have the form

$$x' = (\mathrm{A}x + \mathrm{B}y + \mathrm{X})/(\mathrm{K}x + \Lambda y + \mathrm{M}),$$
$$y' = (\Delta x + \mathrm{E}y + \Phi)/(\mathrm{K}x + \Lambda y + \mathrm{M}). \quad (1.7)$$

Since conics can be obtained from the circumferences of circles by projective transformations, these transformations map conics to conics.

Conics in the projective plane that have no common points with the straight line at infinity are ellipses. Conics that touch this line are parabolas. Conics that intersect this line at two points are hyperbolas, the straight line at infinity divides hyperbolas into two branches.

In Apollonius' Conics many theorems of projective geometry are proved, but he never uses the term "point at infinity". This term was first mentioned in Optical Part of Astronomy by Johannes Kepler (1571-1630).

Important theorems of projective geometry were proved by Pappus of Alexandria in his commentary on Euclid's Porisms.

Ibrahim ibn Sinan considered the projective transformation $x' = a^2/x$, $y' = ay/x$ mapping the circumference of a circle $x^2 + y^2 = a^2$ to the equilateral hyperbola $x^2 - y^2 = a^2$. More complicated projective transformations were considered by Abu'l-Rayhan al-Biruni (973-1048) in the theory of an astrolabe based not on a stereographic projection but on the central projection from an arbitrary point of the axis of the celestial sphere.

On projective geometry and its history see [Ro1. pp.114-122, 125-128], [Ro2, pp.116-121, 133-142, 147-150], and [RoY, pp. 470-475].

25. In Prop. I.8 Apollonius finds the conditions for conics to be continued indefinitely, that is, as modern geometers say, to extend to infinity. These conics are parabolas or hyperbolas.

26. In Prop. I.9 Apollonius proves that sections of an oblique circular cone by planes intersecting both lateral sides of an axial triangle are not circumferences of circles if these planes are not parallel or antiparallel to the plane of the base of the cone. In this proof, Apollonius supposes that the section of a cone by a plane not parallel to the plane of the base of the cone is a circumference of a circle and proves that this plane is antiparallel to the plane of the base of the cone.

27. In Prop. I.10 Apollonius proves that conic sections are convex curves.

In this proposition the notions of interior and exterior points of conics are mentioned for the first time. The propositions on these points are analogous to the propositions of Book 3 of Euclid's Elements on interior and exterior points of circles. The interior and exterior points of a circle are points whose distances from its center are less or greater than its radius, respectively. This metric definition is impossible for conics.

Apollonius does not give the definitions of interior and exterior points of conics but essentially transfers these notions from circles to conics by projective transformations.

Propositions I.11- I.16 on equations of conics

28. In Prop. I.11-I.13 Apollonius finds the equations of conics in systems of coordinates whose axis 0x is a diameter of a conic, ordinates are parallel to the diameter conjugate to this diameter and the point O is the end of first diameter.

Apollonius called any such equation συμπτώμα, meaning “case”, “coincidence”. The ordinates of points of a conic in these equations were determined in Prop. I.7. The abscissas of these points are segments of the diameter from the vertex to the ordinate. See Note 9 for the term “ordinate”. Apollonius called abscissa ἀπό τέμνομενα πρός τα κορυφή -- “cut off from the vertex “. Our term “abscissa” came from Latin translation by Commandino in [Ap1] of these words by expression “*ex verticis abscissa*.”

Apollonius proves that the equations of conics in these coordinates which can be rectangular or oblique have the same forms (0.3), (0.9), and (0.10) as given by his precursors in rectangular coordinates.

In those equations Apollonius uses the expression “the line is equal in square to a rectangular plane” (see Note 19 on this book).

29. In Prop. I.11 a circular cone has the same vertex A, base ΒΓ, and axial triangle ΑΒΓ, as in Prop. I.7. This cone may be right or oblique, the angle at the vertex of this cone can be an arbitrary angle less than 180°. This cone is cut by a plane meeting the line AB at a point H, parallel to the line ΑΓ and intersecting the diameter ΒΓ of the base of the cone at a point Θ. This plane cuts off the conic ΔΗΕ from the surface of the cone. The line ΔΕ is perpendicular to the line ΒΓ and meets it at the point Θ. The line ΗΘ is a diameter of this conic.

If K is an arbitrary point of the conic, its ordinate y = ΚΛ is parallel to the line ΔΕ and its abscissa x = ΗΛ.

To obtain the equation of the conic, Apollonius determines the line HZ, which he called ὀρθία πλεύρα - “right side” (some rectangular plane).
We, like the majority historians of mathematics, translate this term of Apollonius by Latin translation latus rectum . The line HZ is the perpendicular to the diameter ΗΘ at point H. The length of the line HZ is given by the proportion

$$HZ/HA = B\Gamma^2/BA.A\Gamma . \quad (1.8)$$

Apollonius often calls the latus rectum by one word ορθια and by long expression "the straight line of application [of rectangular planes] to which the ordinates to the diameter are equal in square", but we in all cases call this straight line "the latus rectum". We denote latus rectum by 2p. Proportion (1.8) shows that latus rectum 2p is proportional to the segment AH. Therefore every diameter of a conic corresponds to a certain value of latus rectum.

Since the segment p in the equations of conics in rectangular coordinates is called "parameter", sometimes the segment 2p in oblique coordinates is also called "parameter".

30. The term "compounded ratio" was used by ancient mathematicians for ratios of geometrical magnitudes which modern mathematicians call "products of ratios". This is explained by the fact that the term "product" ancient mathematicians used only for integer and rational numbers.

In Book 5 of Euclid's Elements, only particular cases of compounded ratios were defined - double, triple and multiple ratios, that is ratios compounded from equal ratios. The general compounded ratio was considered by Euclid only in Prop. VI.23 of Elements where in he proved that the ratio of two equiangular parallelograms is compounded from the ratios of corresponding sides of these parallelograms.

The definition of a compounded ratio in the original text of Elements was absent. It was added only by Theon of Alexandria in 4th c. A.D., who defined compounded ratio by means of multiplication of "quantities of ratios", which was not used by Euclid. But the proof of proposition VI.23 shows that a ratio A:B is compounded from ratios X:Δ and E:Φ if there are such magnitudes K, Λ, M that A:B = K:M, X:Δ = K:Λ, and E:Φ = Λ:M.

In our commentary a ratio A:B compounded from ratios X:Δ and E:Φ is denoted as A:B = (X:Δ) x (E:Φ) .

According to Prop. VI.23 of Euclid's Elements, the right hand side of proportion (1.8) is equal to a compounded ratio and this equality can be rewritten as

$$\mathrm{HZ:HA} = (\mathrm{B\Gamma:BA}) \times (\mathrm{B\Gamma:A\Gamma}) . \quad (1.9)$$

A plane parallel to the base of the cone and drawn through the line KΛ cuts off from the surface of the cone the circumference MKN of a circle with the diameter MΛN parallel to the straight line BΓ and perpendicular to the straight line KΛ.

The similarity of the triangles HML and ABΓ implies the proportion

$$\mathrm{B\Gamma:A\Gamma} = \mathrm{M\Lambda:H\Lambda} . \quad (1.10)$$

The straight line drawn from the point Η parallel to ΒΓ and equal to ΛΝ cuts off from the triangle ΑΒΓ a similar triangle. Hence the proportion

$$ΒΓ:ΒΑ = ΛΝ:ΗΑ \qquad (1.11)$$

holds. Proportions (1.10) and (1.11) imply that equality (1.9) can be rewritten as

$$ΗΖ:ΗΑ = (ΛΝ:ΗΑ) \times (ΜΛ:ΗΛ)\ . \quad (1.12)$$

According to Prop. VI.23 of Euclid's Elements, equality (1.12) can be rewritten as

$$ΗΖ:ΗΑ = ΜΛ.ΛΝ : ΗΑ.ΗΛ\ . \quad (1.13)$$

The straight line ΜΝ is a diameter of the circumference ΜΚΝ of a circle. The straight line ΚΛ = y is perpendicular to the diameter ΜΝ and divides it at L into the segments $ΜΛ = x_1$ and $ΛΝ = x_2$. Therefore according to Prop. II.14 of Elements, $ΜΛ.ΛΝ = ΚΛ^2$ and equality (1.13) can be rewritten as

$$ΗΖ/ΗΑ = ΚΛ^2/ΗΑ.ΗΛ\ . \qquad (1.14)$$

Since ΗΖ = 2p, ΗΛ = x, and ΚΛ =y, the equality (1.14) is equivalent to the equation (0.3) in rectangular, as well as in oblique coordinates.

Since the angle ΒΑΓ at the vertex of the cone now can be different from a right angle, the old name "section of right-angled cone" for the conic (0.3) no longer makes sense. Therefore Apollonius gave it a new name. The equation (0.3) shows that for each point of this conic with a given ordinate y the abscissa x of this point is obtained by the "application" to the given straight line 2p of a rectangle whose area is equal to the square on the straight line y. Apollonius called this conic παραβολή - "application", hence our term "parabola" came.

The parameter p of a parabola is equal to half of the latus rectum corresponding to the axis of this parabola.

31. In Prop. I.12 Apollonius considers the same cone as in the Prop. I.11 and the plane that meets the straight line ΑΒ at the point Η between A and B and intersects the triangle ΑΒΓ in ΗΘ, so that the angle ΗΘΒ is greater than the angle ΑΓΒ. It also intersects the base of the cone in ΔΕ perpendicular

to the straight line ΒΓ. This plane cuts off the conic ΔΗΕ from the surface of the cone. ΗΘ is a diameter of this conic.

If ΗΘ and ΑΓ are continued, they meet at the point Ζ. Apollonius calls the straight line ΗΖ πλαγία πλεύρα - "transverse side" (of some rectangular plane). Like the majority of historians mathematics we translate this Apollonius' expression by the Latin words the latus transversum . We denote this straight line by 2a.

Through the vertex Α of the cone Apollonius draws the straight line ΑΚ to the point Κ on the straight line ΒΓ and parallel to the diameter ΗΘ .

If Μ is an arbitrary point of the conic, its ordinate y = ΜΝ is parallel to ΔΕ, and its abscissa is x = ΗΝ..

To obtain the equation of the conic Apollonius determines the straight line ΗΛ and represents it by the perpendicular to the diameter ΗΘ at the point Γ. The length of ΗΛ is given by the proportion

$$\mathrm{HZ} / \mathrm{H\Lambda} = \mathrm{AK}^2 / \mathrm{BK.K\Gamma}. \quad (1.15)$$

The straight line ΗΛ Apollonius called by the same expressions as the straight line HZ in Prop.I.1. We call this straight line the latus rectum and denoted it by 2p.

The proportion (1.15) shows that the ratio 2a/2p depends on the position of the diameter ΗΘ. Therefore each diameter 2a of a conic corresponds to a certain value of the latus rectum 2p.

According to Prop.VI.23 of Euclid's Elements, the right hand side of proportion (1.15) is equal to a compounded ratio and equality (1.15) can be rewritten as

$$\mathrm{HZ:H\Lambda} = (\mathrm{AK:BK}) \times (\mathrm{AK:K\Gamma}). \quad (1.16)$$

A plane parallel to the base of the cone and drawn through ΜΝ cuts off from the surface of the cone the circumference ΡΜΣ of a circle with the diameter ΡΝΣ parallel to ΒΓ and perpendicular to ΜΝ.

The similarity of the triangles ΗΡΝ and ΑΒΚ implies the proportion

$$\mathrm{AK:BK} = \mathrm{HN:PN} \quad (1.17)$$

The similarity of the triangles ΑΚΓ and ΖΝΣ implies the proportion

$$AK:K\Gamma = ZN:N\Sigma \qquad (1.18)$$

Proportions (1.17) and (1.18) imply that equality (1.16) can be rewritten as

$$HZ:H\Lambda = (HN:PN) \times (ZN:N\Sigma) \,. \quad (1.19)$$

According to Prop. VI.23 of Elements, equality (1.19) can be rewritten as

$$HZ:H\Lambda = HN.ZN : PN.N\Sigma \,. \qquad (1.20)$$

The straight line $PN\Sigma$ is a diameter of the circumference $PM\Sigma$ of a circle, and $MN = y$ is perpendicular to the diameter $P\Sigma$ and divides it at N into the segments $PN = x_1$ and $N\Sigma = x_2$. Therefore according to Prop. II.14 of Elements $PN.N\Sigma = MN^2$ and equality (1.20) can be rewritten as

$$HZ/H\Lambda = HN.ZN/MN^2 \,. \qquad (1.21)$$

Since $HZ = 2a$, $H\Lambda = 2p$, $HN = x$, and $MN = y$, equality (1.21) can be rewritten as $2a/2p = x(2a + x)/y^2$, which is equivalent to equation (0.10) in rectangular, as well as in oblique, coordinates.

Since the angle $BA\Gamma$ at the vertex of the cone does not have to be obtuse now, the old name “section of obtuse-angled cone” for the conic (0.10) no longer makes sense. Apollonius gave it a new name. This equation shows that for each point of this conic with a given ordinate y the abscissa x of this point is obtained by the “application with excess” to the given line $2p + (p/a)x$ of the rectangle equal to the square on the line y. Therefore Apollonius calls this conic ὑπερβολή - “excess”, from which our term “hyperbola” came.

The application with excess can be fulfilled by the addition to the rectangle $H\Lambda ON$ the rectangle $\Lambda P\Xi O$ similar to the rectangle with sides $HZ=2a$ and $H\Lambda = 2p$. The diagonals $Z\Lambda$ and $\Lambda\Xi$ of these rectangles are segments of one straight line. Since $\Lambda O = x$, the side $\Lambda\Pi$ of the rectangle $\Lambda\Pi\Xi O$ is equal to $2px/2a = (p/a)x$ and the area of this rectangle is equal to $(p/a)x^2$, and the area of the rectangle $H\Pi\Xi N$ is equal to y^2.

The parameter p of a hyperbola is equal to half of the latus rectum corresponding to the axis of this hyperbola.

32. In Prop. I.13 Apollonius considers the same cone as in the Prop. I.11 and I.12, and the plane which meets the straight lines AB and $A\Gamma$ at the points E and Δ and intersects the plane of the base of the cone in the straight line ΘH

perpendicular to the continuation of the straight line ΒΓ. The angle between the continuations of ΕΔ and ΒΓ is less than the angle ΑΓΒ. This plane cuts off from the surface of the cone the conic ΔΛΕ. The straight line ΔΕ is a diameter of this conic.

If Λ is an arbitrary point of this conic, its ordinate y = ΛΜ is parallel to ΗΘ and the abscissa is x = ΕΜ.

To obtain the equation of the conic Apollonius determines the straight lines ΕΖ and ΔΕ. The straight line ΕΖ is perpendicular to ΔΕ at the point Ε. Apollonius calls the line ΕΖ πλαγία πλεύρα , that is the latus rectum , and the line ΔΕ ὀρθία πλεὐρα latus transversum of the conic.
We will denote the latera rectum and transversum by 2p and 2a.

The length of the straight line ΕΖ Apollonius determines as follows.

From the vertex Α of the cone Apollonius draws the straight line ΑΚ parallel to the diameter ΔΕ to the point Κ on the continuation of the straight line BG and determines the length of ΕΖ from the proportion

$$ΔΕ/ΕΖ = ΑΚ^2/ΒΚ.ΚΓ \quad . \quad (1.22)$$

The proportion (1.22) shows that the ratio 2a/2p depends of the position of the diameter DE, and each diameter 2a of the conic corresponds to a certain value of 2p.

According to Prop.VI.23 of Elements, the right hand side of equality (1.22) is a compounded ratio and this equality can be rewritten as

$$ΔΕ:ΕΖ = (ΑΚ:ΒΚ) \times (ΑΚ:ΚΓ). \quad (1.23)$$

If through ΛΜ a plane parallel to the plane of the base of the cone is drawn, it will cut off from the surface of the cone the circumference ΠΛΡ of a circle with the diameter ΠΜΡ parallel to ΒΓ and perpendicular to ΛΜ which divides it onto the segments ΠΜ = x_1 and ΜΡ = x_2. Therefore $ΠΜ.ΜΡ = ΛΜ^2$

The similarity of the triangles EPM and ΑΒΚ implies the proportion

$$ΑΚ/ΒΚ = ΕΜ/ΠΜ \,. \quad (1.24)$$

The similarity of the triangles ΑΧΚ and ΔΡΜ implies the proportion

$$ΑΚ/ΚΧ = ΜΔ/ΜΡ. \quad (1.25)$$

Therefore equality (1.23) can be rewritten as

$$\Delta E:E\Phi = (EM:\Pi M) \times (M\Delta:MP) . (1.26)$$

According to Prop. VI.23 of Elements, equality (1.26) can be rewritten as

$$\Delta E:E\Phi = EM.M\Delta :\Pi M.MP . (1.27).$$

Since $\Pi M.MP = \Lambda M^2$, equality (1.27) can be rewritten as

$$\Delta E/EZ = EM.M\Delta/\Lambda M^2.(1.28)$$

Since EZ = 2p, ΔE = 2a, EM = x, ΛM =y, equality (1.28) can be rewritten as $2a/2p = x(2a - x)/y^2$, which is equivalent to equation (0.9) in rectangular, as well as in oblique coordinates.

Since the angle ΒΑΓ at the vertex of the cone does not have to be acute, now the old name "section of acute-angled cone" for the conic (0.9) no longer makes sense, Apollonius gave it a new name. Equation (0.9) shows that for each point of this conic with a given ordinate y the abscissa x of this point is obtained by the "application with defect" to given line 2p - (p/a)x of the rectangle equal to the square on the line y. Therefore Apollonius calls this conic ἐλλείψις "defect", hence our term "ellipse" came.

The "application with defect" can be fulfilled by subtraction from the rectangle EZNM the rectangle OZNΞ similar to the rectangle with sides EΔ =2a and EZ= 2p. The diagonal ΔZ of one of these rectangles contains the diagonal ZΞ of the other one.

The diagonal of this rectangle is a part of the diagonal ZΛ of the rectangle with sides ZΘ = 2a and ZΛ = 2p. The sides of the subtracted rectangle are x and (p/a)x.

The parameter p of an ellipse is equal to half of the latus rectum corresponding to the major axis of this ellipse.

Apollonius did not write how he obtained proportions (1.8), (1.15), (1.22) from which he derived the equations of conics. B.L.Van der Waerden wrote in Science Awakening : "Apollonius was a virtuoso in dealing with geometric algebra, and also a virtuoso in hiding his original line of thought. This is what makes his work hard to understand; his reasoning is elegant and crystal clear, but one has to guess at what led him to reason in this way, rather than in some other way" [VdW, p. 248]

It is probable that Apollonius described how he came to these proportions in his lost General Treatise (see Introduction, Γ).

33. Conics can also be characterized by the magnitude ε called eccentricity. The eccentricity of an ellipse is expressed through its latera rectum and transversum by the formula $\varepsilon^2 = 1 - p/a$. (1.29)

The eccentricity of a hyperbola is expressed through its latera rectum and transversum by the formula

$$\varepsilon^2 = 1 + p/a. \quad (1.30)$$

Like the magnitudes p and a, the value ε is determined for each diameter of an ellipse and a hyperbola.

Equations (0.9) and (0.10) in rectangular, as well as in oblique, coordinates can be written in the unitary form

$$y^2 = 2px + (\varepsilon^2 - 1)x^2. \quad (1.31)$$

Equation (1.31) coincides with equation (0.3) for $\varepsilon = 1$ and with equation $x^2 + y^2 = 2px$ or $(x - p)^2 + y^2 = p^2$ for $\varepsilon = 0$. Therefore it is possible to believe that for parabolas $\varepsilon = 1$ and for circumferences of circles $\varepsilon = 0$. Since for ellipses $p < a$, the eccentricity of ellipses satisfies inequalities
$0 < \varepsilon < 1$.

Since for hyperbolas $p/a > 0$, the eccentricity of hyperbolas satisfies the inequality $\varepsilon > 1$.

Modern mathematicians consider only eccentricities corresponding to major axes of ellipses and to real axes of hyperbolas.

34. The angle xOy of the coordinate system in which the equation of a conic has the form (1.31) can be determined as follows. Let the unit vectors i, j, k be directed along the diameter ΒΓ of the base of the circular cone, along the perpendicular to ΒΓ in the plane of the base of the cone, and along the perpendicular to this plane, respectively. Let the unit vectors h and l be directed along the axis of the circular cone and along the diameter of the conic. These vectors have the form $h = j \sin\alpha + k \cos\alpha$ and $l = i \cos\lambda + h \sin\lambda$. The axis Ox has the direction of the vector l, the axis 0y has the direction of vector j. Therefore the angle $\omega = xOy$ is determined by the formula

$$\cos\omega = lj = \sin\alpha . \sin\lambda. \quad (1.32)$$

Apollonius in Prop. I.7 mentioned that parallel chords are orthogonal to the diameter bisecting them in two cases - if the cone is right and if the axial triangle of the cone is orthogonal to the plane of its base. In the first case, the vector h coincides with the vector k and the angle $\alpha = 0$. In the second case, the cross product $h \times i = -k \sin\alpha + j \cos\alpha$ must be collinear with the vector j, hence $\sin\alpha = 0$.

In the case $\lambda = 0$, the vector l coincides with the vector i. In this case, the angle xOy is right, but the section of the cone is the circumference of a circle whose plane is parallel to the base of the cone.

35. The sine law of the plane trigonometry, discovered in 10th c. A.D. and described by al-Biruni,

$$\sin A/BX = \sin B/XA = \sin X/AB \qquad (1.33)$$

for the triangle ABX, allows for equalities (1.8), (1.15) and (1.22) to be expressed via angles.

Equality (1.8) can be expressed in the form

$$2p/ZA = \sin^2 A/\sin B\cdot\sin\Gamma = \sin^2(B + \Gamma)/\sin B\cdot\sin\Gamma\ . \quad (1.34)$$

If K is the angle between the plane of a hyperbola or an ellipse and the plane of the base of the cone, equalities (1.15) and (1.22) can be expressed, respectively, in the forms

$$2a/2p = \sin B$$
$$\sin\Gamma/\sin(B+K)\cdot\sin(K-\Gamma) \qquad (1.35)$$

And

$$2a/2p = \sin B\cdot\sin\Gamma/\sin(B+K)\cdot\sin(\Gamma-K)\ . \quad (1.36)$$

36. In the case where a circular cone is right, the line of intersection of the plane of the conic and of the plane ABΓ is the axis of the conic. In this case the triangle ABΓ is isosceles and the angle B is equal to the angle Γ. In this case equality (1.34) has the form

$$2p/HA = 4\cos^2 B\ . \quad (1.37)$$

Equation (0.3) of a parabola first found by Menaechmus apparently was obtained by him as follows. Menaechmus considered a right circular cone with right angle at its vertex A and intersected this cone by the plane perpendicular to the rectilinear generator AH at the point H.

The plane cuts off from the surface of the cone a parabola and from the axial triangle with the side AH = p the axis of this parabola.

If K is an arbitrary point of the parabola, the perpendicular $K\Lambda$ dropped from K to the axis is the ordinate y of the point K and the segment $H\Lambda$ is its abscissa x. Since the segments AH, $H\Lambda$, and $K\Lambda$ are mutually perpendicular, the segment AK is a diagonal of a parallelepiped built on these three segments and $AK^2 = A\Gamma^2 + \Gamma\Lambda^2 + K\Lambda^2 = p^2 + x^2 + y^2$. Since $AK = p + x$, $AK^2 = p^2 + 2px + x^2 = p^2 + x^2 + y^2$, equation (0.3) is thus obtained.

The latus rectum of this parabola is 2p.

Since the cone considered by Menaechmus is right-angled, each of the angles B and Γ is equal to 45^o. From formula (1.37) we obtain $2p/HA = 4\cos^2 45^o = 2$, that is $HA = p$.

37. In the case of right circular cone, equalities (1.35) and (1.36) have the form

$$2a/2p = \sin^2 B/(\sin^2 K - \sin^2 B) \quad (1.38)$$

for a hyperbola and

$$2a/2p = \sin^2 B/(\sin^2 B - \sin^2 K). \quad (1.39)$$

for an ellipse.

Left hand sides of equalities (1.38) and (1.39) are equal to $a/p = a^2/b^2$. Therefore formulas (1.29) for an ellipse and (1.30) for a hyperbola imply in both cases that $\varepsilon^2 = \sin^2 K/\sin^2 B$, hence

$$\varepsilon^2 = \sin K/\sin B. \quad (1.40)$$

If we denote the angle at the vertex A of the right circular cone by 2α, the angle B is equal to $90^o - \alpha$.

If the conic is cut off from the surface of a right circular cone by a plane perpendicular to its rectilinear generator, that is, if the conic is determined by precursors of Apollonius, the angle K is equal to α. Therefore for this conic formula (1.40) implies that

$$\varepsilon = \tan \alpha \,. \qquad (1.41)$$

In the case of the circumference of a circle, the role of a right circular cone is played by a right circular cylinder obtained from a cone by limiting process in which its vertex tends to infinity. In this case, $\alpha = 0^o$ and $\varepsilon = 0$. For an ellipse, $0^o < \alpha < 45^o$ and $0 < \varepsilon < 1$; for a parabola, $\alpha = 45^o$ and $\varepsilon = 1$; for a hyperbola, $\alpha > 45^o$ and $\varepsilon > 1$.

38. The possibility of obtaining different conics from the same right circular cone, proved by Apollonius, led Persian mathematician of 10th c.A.D. Abu Sahl al-Kuhi, to the invention of an instrument for drawing conics. The treatise of al-Kuhi was published with French translation by Franz Woepcke [Woe] (see also [RoY, p. 459]). The instrument called by al-Kuhi "perfect compass" was a compass whose motionless leg could be inclined to the plane of paper under an arbitrary angle β and the rotating leg forming with the motionless leg an arbitrary acute angle α could change its length so that a pencil at its end would always touch the paper. The rotating leg of this compass describes the surface of a right circular cone with angle 2α at its vertex and the plane of paper cuts off from this surface a conic described by the pencil. The angle K between the plane of a conic and the plane of the base of the cone is equal to $90^o - \beta$, the angle B is equal to $90^o - \alpha$. Therefore, from formula (1.40) we obtain that the eccentricity of the described conic is equal to

$$\varepsilon = \cos\beta / \cos\alpha \,. \quad (1.42)$$

For the circumference of a circle, $\alpha < \beta = 90^o$ and $\varepsilon = 0$; for an ellipse $\alpha < \beta < 90^o$ and $0 < \varepsilon < 1$; for a parabola, $\alpha = \beta$ and $\varepsilon = 1$; for a hyperbola, $\alpha > \beta$ and $\varepsilon > 1$.

In medieval Arabic translations of Conics, Apollonius' terms "hyperbola", "ellipse", and "parabola" are translated as "qat zaid" (surplus section), "qat naqis" (insufficient section), and "qat mukafi" (sufficient section). Apollonius' terms latus rectum and latus transversum are translated, respectively, as "dil qaim" (right side) and "dil mail"(oblique side). The last translation is explained by double meaning of the Greek word πλαγια -"transverse" and "oblique".

39. Apollonius' term έίδος, means "form, figure", preserved in Euclid's term ῥομβοείδες - rhomboid for a parallelogram which is not a rhombus, and Archimedes' terms κωνοείδες - conoid and σφαιροείδες - spheroid. It was used by

Apollonius for a rectangle with the sides 2a and 2p for an ellipse and a hyperbola.

This rectangle was considered in Prop. I.12 and I.13.

Besides the geometrical sense the word “eidos” has also a philosophical sense.

In works of Plato this word is often translated as “idea”, it means that which in interaction with “space” forms a stable phenomenon. For living beings Plato’s eidos is equivalent to soul. This notion obtained the further development in Aristotle’s “entelechy” and in Hegel’s “Absolute Idea”.

Probably, Apollonius also put into the notion of eidos a philosophical sense.

40. In Prop. I.14 Apollonius considers opposite hyperbolas and proves:

1) the diameter of one of two opposite hyperbolas lies on the continuation of the diameter of the other hyperbola,

2) the latus transversum of these hyperbolas is the same straight line, whose ends are the vertices of these hyperbolas,

3) the latera recta of these hyperbolas, called here “straight lines of application [of rectangular planes], to which ordinates drawn to the diameter are equal in square”, are equal one to other.

Apollonius, unlike his precursors, considered opposite hyperbolas as one whole formed by intersection of a plane with both vertical sheets of the conic surface.

In modern geometry, opposite hyperbolas of Apollonius are called two branches of a hyperbola, the straight line containing the axes of two opposite hyperbolas is called the real axis of a hyperbola, and the straight line perpendicular to the real axis and intersecting it in the midpoint of the segment between the vertices of the opposite hyperbolas is called the imaginary axis of the hyperbola.

41. If one of two opposite hyperbolas is determined by equation (0.10) in rectangular or oblique coordinates, then the second of these hyperbolas is determined by the same equation. If two points of two opposite hyperbolas have equal ordinates y, then the abscissas x and x’ of these points are connected by the correlation $x' = -2a - x$. Since two opposite hyperbolas have the same latus transversum 2a and equal latera recta 2p and the point with coordinates x, y satisfies equation (0.10), the point with coordinates x’, y satisfies the equation $y^2 = 2p(-2a - x) + (p/a)(-2a-x)^2 = -4pa - 2px + (p/a)(4a^2-4ax+x^2) = 2px + (p/a)x^2$, that is the point with coordinates x’, y also satisfies equation (0.10).

42. In Prop. I.15 Apollonius considers an ellipse AΔBE with conjugate diameters AB = 2a and ΔE = 2b in two systems of coordinates whose origins are the points A and Δ, and the axes of abscissas are the diameters AB and ΔE. The

latera recta corresponding to these diameters are the lines AN = 2p and ΔH = 2q.

In the first system of coordinates the ellipse is determined by equation (0.9), in the second one - by the equation $w^2 = 2qz - (q/b)z^2$. The eidos corresponding to the diameter AB is the rectangle with the diagonal BN, the eidos corresponding to the diameter ΔE is the rectangle with the diagonal EH.

Apollonius proved that the latus rectum AN = 2p corresponding to the diameter AB = 2a is equal to $2p = (2b)^2/2a$, equivalent to the equality $p = b^2/a$, and latus rectum ΔH = 2q corresponding to the diameter DE = 2b is equal to $2q = (2a)^2/2b$, equivalent to the equality $q = a^2/b$.

The second equation can be obtained from equation (0.9) by the substitution $x = a - w$, $y = b - z$, where latera recta p and q are connected by the correlation $p/a = b/q$.

43. In Prop. I.16 Apollonius proves that the straight line drawn through the midpoint of a transverse diameter of two opposite hyperbolas in the direction of ordinates dropped to this diameter is also a diameter of these hyperbolas and ordinates dropped to it are parallel to the first diameter, that is the drawn diameter is conjugate to the first one.

Second Definitions

44. Apollonius defines the center (κέντρον) of an ellipse, a hyperbola, and opposite hyperbolas as the midpoint of the latus transversum of these conics. The word κεντρον first meant a stick with sharp end, and after invention of compass this word became to mean the leg with sharp end of a compass, and the center itself of a circle.

The segment of the latus transversum of a conic between the center and the vertex of this conic Apollonius calls εκ του κέντρου -- "radius". This expression literally meaning "from the center" coincides with Euclid's term for a radius of a circle.

The fact that all centers of a conic coincide is proved in Prop. I.30.

45. Apollonius introduces the term "second diameter" of a conic for a segment of the diameter conjugate to the diameter containing the latus transversum of a conic equal to the mean proportional between the latus rectum and the latus transversum of the conic. If the second diameter is denoted by 2b, it is determined by the proportion

$$2a:2b = 2b:2p \quad (1.43)$$

equivalent to the formula

$$(2b)^2 = 2p \cdot 2a \ . \quad (1.44)$$

The right hand side of this equality is equal to the area of the eidos corresponding to the diameter 2a. Therefore the area of this eidos is equal to $4b^2$.

The particular case of the correlation (1.44) for the latus rectum corresponding to the diameter ΔE of an ellipse conjugate to its diameter AB =2a was proved by Apollonius in Prop. I.15.

46. If the origin of the system of rectangular or oblique coordinates, where an ellipse and a hyperbola are determined by equations (0.9) and (0.10), is moved to the center of the conic, an ellipse is determined by the equation

$$x^2/a^2 + y^2/b^2 = 1, \quad (1.45)$$

a hyperbola and opposite hyperbolas are determined by the equation

$$x^2/a^2 - y^2/b^2 = 1, \quad (1.46)$$

where 2a are latera transversa of an ellipse and a hyperbola, and 2b are second diameters of these conics connected with latera recta and transversa of these conics by the correlation (1.44) which implies the equality

$$p/a = b^2/a^2 \ . \quad (1.47)$$

Therefore formulas (1.29) and (1.30) for eccentricities of an ellipse and a hyperbola, respectively, can be rewritten in the form

$$\varepsilon^2 = 1 - b^2/a^2 = (a^2 - b^2)/a^2 \ , \quad (1.48)$$
$$\varepsilon^2 = 1 + b^2/a^2 = (a^2 + b^2)/a^2 \ . \quad (1.49)$$

Formulas (1.48) and (1.49) show that for ellipses $0 < \varepsilon < 1$, and for hyperbolas $\varepsilon > 1$. Since the circumference of a circle can be regarded as an ellipse where a = b, we obtain that for the circumference of a circle $\varepsilon = 0$. Since a parabola can be obtained by a limiting process from both ellipses and hyperbolas, we find that for a parabola $\varepsilon = 1$.

Propositions I.17 - I.31 on interior and exterior points of conics

47. Interior and exterior points of conics were first considered by Apollonius in Prop. I.10.

In Prop. I.17 the conic ΑΓ with the vertex Α and the diameter ΑΒ is considered. Apollonius proves that all points of the straight line drawn through the point Α in direction of chords bisected by the diameter ΑΒ besides the point Α are exterior points of the conic. Apollonius calls this straight line tangent to the conic at the point Α.

Therefore exterior points of a conic can be defined as points in the plane from which tangent lines to the conic can be drawn, and interior points of a conic can be defined as points in the plane from which tangent lines to the conic cannot be drawn.

Straight lines containing parallel chords bisected by the diameter ΑΒ intersect the conic at two points that are in both sides of this diameter. Therefore the tangent line to the conic at the point Α can be obtained by the limiting process from the lines containing chords when points of their intersections with the conic tend to the point Α. Thus a line tangent to the conic can be defined as limit position of a line intersecting the conic at two points when these points tend to the point of tangency.

48. In Prop. I.18 Apollonius proves that a segment consisting of interior points of a conic and parallel to a line tangent to this conic or to a straight line meeting the conic at two points, if continued, will intersect the conic and will contain exterior points of this conic.

49. Prop. I.19 is a particular case of Prop. I.18 where the segment consisting of interior points is a part of an ordinate and one end of this segment is a point of the diameter.

50. In Prop. I.20 Apollonius proves that as the square of the ordinate y_1 of a point of a parabola is to the square of the ordinate y_2 of its another point, so the abscissa x_1 of the first point is to abscissa x_2 of the second point. The assertion of this proposition is a consequence of the equality

$$2p = y^2/x \qquad (1.50)$$

equivalent to equation (0.3) of a parabola in rectangular or oblique coordinates.

51. In Prop. I.21 Apollonius proves that as the square of the ordinate y_1 of a point of an ellipse or a hyperbola is to the square of the ordinate y_2 of its another point, so the product of two abscissas of the first point is to the product of two abscissas of the second point. These products for the ellipse are

$x_1(2a - x_1)$ and $x_2(2a - x_2)$ and for the hyperbola are $x_1(2a + x_1)$ and $x_2(2a + x_2)$. The assertions of this proposition are consequences of the equalities

$$2p/2a = y^2/x(2a - x) \quad (1.51)$$

for an ellipse and

$$2p/2a = y^2/x(2a + x) \quad (1.52)$$

for a hyperbola, which are equivalent to equations (0.9) and (0.10) for an ellipse and a hyperbola in rectangular or oblique coordinates.

Here and further Apollonius under the word "hyperbola" means one branch of a hyperbola and two branches of a hyperbola he calls "opposite hyperbolas". Apollonius' does not mean that the circumferences of circles are particular cases of ellipses, since the circumferences of circles are "plane loci" and ellipses are "solid loci". But modern mathematicians believe the circumferences of circles as particular cases of ellipses, therefore in our commentary we include the circumference of a circle in the notion of ellipse.

52. In Prop. I.22 a parabola or a hyperbola with the vertex Α and the diameter ΑΒ is considered. Apollonius proves that if the straight line ΓΔ joining two points Γ and D of the conic does not meet the diameter ΑΒ at an interior point of the conic, then the continuation of the line ΓΔ meets the continuation of the diameters ΑΒ at an exterior point of the conic.

53. In Prop. I.23 an ellipse ΑΓΒΔ with the diameters ΑΒ and ΓΔ is considered. Apollonius proves that if the straight line ΕΖ joins the points Ε and Ζ of the ellipse which lie between the ends of both diameters, then its continuation intersects the lines ΑΒ and ΓΔ at exterior points of the ellipse.

In the proof of this proposition Apollonius supposes that the diameters ΑΒ and ΓΔ are conjugate, but actually the theorem is right for arbitrary two diameters that are not parallel to the line ΕΖ. No doubt that this theorem is a generalization of a theorem of Apollonius' precursors for two axes of an ellipse, which were called by them "diameters".

54.The equation of a conic in an affine coordinate system of whose axes 0x and 0y are two non-conjugate diameters has the form

$$Ax^2 + 2Bxy + Cy^2 + F = 0\ . \quad (1.53)$$

In the most general system of affine coordinates the equation of a conic has the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 . \quad (1.54)$$

Equations (1.1) and (1.2) of loci with respect to three and four straight lines are particular cases of equation (1.54).

Equation (1.54) can be written in the vector form

$$x\Phi x + 2Vx + F = 0 , \quad (1.55)$$

AB where Φ is a linear operator with matrix B C , $V = Di + Ej$ is a vector, $x\Phi x$ and Vx are the inner products of the vectors x, Φx and V with the vector x .

The eigenvectors of the operator Φ determine the directions of the axes of a conic, the eigenvalues of this operator for an ellipse are equal to $1/a^2$ and $1/b^2$, for a hyperbola are equal to $1/a^2$ and $-1/b^2$, and for a parabola are equal to 0 and 1.

On different equations of conics and quadrics see [Ro1, pp. 136-152].

55. In Prop. I.24, a parabola or a hyperbola with the vertex A and the diameter AB is considered. Apollonius proves that the straight line tangent to the conic at an arbitrary point E different from the point A intersects the continuation of the diameter AB at an exterior point of the conic

56. In Prop. I.25 the same ellipse as in Prop. I.23 is considered. Apollonius proves that the straight line EZ tangent to the ellipse at a point H which lies between the ends of both diameters intersects the lines AB and ΓΔ at exterior points of the ellipse.

In the proof of this proposition, Apollonius supposes that the diameters AB and ΓΔ are also conjugate, but actually the theorem is right for arbitrary two diameters which are not parallel to the line EZ .No doubt that this theorem is a generalization of a theorem of Apollonius' precursors for two axes of an ellipse.

57. In Prop. I.26 a parabola or a hyperbola is considered.

Apollonius proves that a straight line parallel to its diameter, if continued, will intersect this conic at one point. In the case of a parabola, the directions of all diameters coincide; in the case of a hyperbola the directions of diameters are different.

58. In Prop. I.27 Apollonius proves that each straight line passing through an interior point of a parabola and having the direction different from the direction of its diameter, if continued, will intersect the parabola at two points.

59. In Prop. I.28 Apollonius proves that each straight line passing

through an interior point of one of two opposite hyperbolas and parallel to the straight line tangent to other from these hyperbolas, if continued, will intersect the first hyperbola at two points.

60. In Prop. I.29 Apollonius proves that if a straight line passing through a center of two opposite hyperbolas intersects one of them, then if continued, this line will intersect the second of these hyperbolas.

61. In Prop. I.30 Apollonius proves that all centers of an ellipse and of opposite hyperbolas coincide.

This proposition implies that an ellipse and opposite hyperbolas are invariant under reflection with respect to the centers of these conics. This reflection can be reduced to the form

$$x' = -x, \quad y' = -y . \quad (1.56)$$

The reflection (1.56) is a particular case of a homothety

$$x' = kx , \quad y' = ky. \quad (1.57)$$

The reflection (1.56) can be regarded as a turn by 180°.

62. Apollonius uses following terms of Euclid concerning ratios and proportions. The inversion of a ratio A/B is the transition from this ratio to the ratio B/A, the composition of a ratio A/B is the transition from this ratio to the ratio (A + B)/B, the separation of a ratio A/B is the transition from this ratio to the ratio (A - B)/B, the conversion of a ratio A/B is the transition from this ratio to the ratio A/(A - B) (Definitions V.13 - V.16 of Euclid's Elements) [Euc, pp. 99 -100]. The application of these operations to both ratios of a proportion A/B = C/D leads to new proportions.

The alternation of a proportion A/B = C/D is the transition from this proportion to the proportion A/C = B/D (Definition V.12 of Elements) [Euc. p.99].

63. In Prop. I.31, a hyperbola BΔ with the vertex B and the latus transversum AB = 2a is considered. On the segment AB such a point Γ is taken that AΓ < a. From the point Γ the straight line ΓΔ intersecting the hyperbola is drawn. Apollonius proves that ΓΔ, if continued, contains interior points of the hyperbola.

Propositions I.32 - I.40 on tangent straight lines

and on inversions with respect to conics

64. In Prop. I.32 Apollonius, like in Prop. I.17, draws the straight line tangent to a conic at its vertex parallel to ordinates, and proves that another straight line does not fall between the conic and the tangent line.

The angle between a conic section and the straight line tangent to it is a so-called “horn-formed angle”, the assertion of Apollonius means that horn-formed angles are less than any rectilinear angle. If multiplication of rectilinear and horn-formed angles by numbers is defined, this fact implies that rectilinear and horn-formed angles together form a non-Archimedean number system, that is the system where the axiom of Archimedes (for any two numbers a and b, a > b, such integer n exists that nb > a) is not fulfilled.

65. In Prop. I.33, Apollonius proves that if Γ is a point of a parabola with the diameter AB and the vertex E, and ΓΔ is the ordinate dropped from Γ to the diameter, and if AE = EΔ, the straight line AΓ is tangent to the parabola at Γ.

In modern projective geometry, the straight line ΓΔ is called the polar of the point A, and A is called the pole of ΓΔ.

66. In Prop. I.34 Apollonius proves that if Γ is a point of a hyperbola or an ellipse with the latus transversum AB, if ΓΔ is an ordinate, and if E is such a point on the straight line AB that the ratios AΔ/ΔB and AE/EB are equal, the straight line EΓ is tangent to the conic at Γ.

In modern projective geometry, the straight line ΓΔ is called the polar of the point E, and E is called the pole ΓΔ.

It is said about the four points A, B, Δ, E on the straight line AB that these points form a “harmonic quadruple”, or that points A and B “harmonically divide” the points Δ and E.

Modern mathematicians represent segments of a straight line by real numbers and consider oriented segments represented by positive and negative numbers. Therefore modern mathematicians define harmonic quadruples by the equality

$$A\Delta/\Delta B : AE/EB = -1. \quad (1.58)$$

Apollonius, like all ancient mathematicians, did not consider negative magnitudes and therefore believed that the ratio AΔ/ΔB is equal to the ratio AE/EB.

Apollonius called segments of harmonic quadruples “ὀμολόγα “, literally meaning “with the same ratio”, from “ὀμος“ - ”the same” and “λόγος” - “ratio”.

67. The word ὁμολόγα with other meaning of the word λογος was used by Euclidi in Definition V.11 [Euc, p. 99] in the sense “corresponding”. The words “homologous” and “homology” meaning “corresponding” and “correspondence” came into use in all European languages.

In projective geometry, homology is a special case of projective correspondence with a straight line consisting of fixed points and a plane pencil of straight lines consisting of invariant lines. The line of fixed points is called the “axis of the homology”, the center of the pencil of invariant lines is called the “center of the homology”.

The famous theorem of Girard Desargues (1591-1661) about triangles ABX and A’B’X’, such that lines AA’, BB’, and XX’ meet at one point and the points of intersection of the lines AB and A’B’, BX and B’X’ , XA and X’A’ are on one straight line, is called the “theorem on homologous triangles”.

The most important are affine homologies, which is homologies that are affine transformations. The axis or the center of these homologies must be at infinity. There are two kinds of affine homologies with axes not passing through centers:

1) A contraction to a straight line or a dilation from a straight line, which in rectangular or oblique coordinates can be reduced to the form (0.5). The center of this homology is at infinity.

2) A homothety which can be reduced to the form (1.57).
The axis of this homology is at infinity.

There are also two kinds of affine homologies with axis passing through center:

3) A parallel translation

$$x' = x + a, \quad y' = y + b. \qquad (1.59)$$

Both the axis and the center of this homology are at infinity;

4) A shift that can be reduced to the form

$$x' = x + ky, \quad y' = y. \qquad (1.60)$$

The center of this homology is at infinity.

The term “homology” (homologie) for a special kind of projective transformation was introduced by Jean Victor Poncelet (1788-1867).

By analogy with the term of Poncelet Michel Chasles (1793-1880) proposed to call an arbitrary projective correspondence homographie.

The Chasles' term was taken by Italian geometers for linear operators in the form omografia. In particular, Cesare Burali-Forti (1861-1931) defined "principal homography" connected with each point of a surface in the space: if a surface is determined by vector equation $x = x(u,v)$, then at each point of this surface the tangent plane with unit normal vector n are determined. The differentials x and dn are directed in the tangent plane, and dn is a linear vector-function of dx , $dn = Kdx$, (1.61) where K is Burali-Forti's "principal homography" in whose name we see a trace of Apollonius' term.

The founder of algebraic topology Henri Poincare (1856-1912) called the most important notion of this mathematical discipline "homology".

By analogy with Poincare's theory, in 20th c. Henri Cartan (b. 1904) and others have created different parts of "homological algebra".

The term "homologous" in the sense "corresponding" was used by Dmitri Mendeleev in chemistry and by Nikolay Vavilov in biology.

68. The left hand side of equality (1.58) is called the "cross-ratio" of four points A, B, Δ, E of a projective straight line. This cross-ratio can also be defined as follows. If points A, B, Δ, E on a straight line are represented by vectors a, b, d, e, with common beginning point S, then the vectors d and e are linear combinations of the vectors a and b having the form

$$d = \alpha a + \beta b, \quad e = \gamma a + \delta b . \qquad (1.62)$$

Then the cross-ratio of these four points can be defined as

$$\lambda = (\delta/\gamma) : (\beta/\alpha). \qquad (1.63)$$

Formula (1.63) shows that the cross-ratio λ does not change if the vectors a, b, d, e are multiplied by arbitrary non-zero real numbers.

Under projective transformations, the vectors a, b, d, e undergo linear transformations. This fact implies that the cross-ratio (1.63) does not change under projective transformations.

For the proof that expression (1.63) coincides with the left hand side of equality (1.58) it is sufficient to take $a = \Sigma A$, $b = \Sigma B$, $d = \Sigma\Delta$, $e = \Sigma E$ in equality (1.62).

Since projective transformations map straight lines to straight lines, a projective correspondence between two projective lines can be defined as a bijective transformation preserving cross-ratios of quadruples of points in these lines. These transformations, as well as projective transformations in a projective line, can be written in the form

$$x' = (Ax + B)/(Cx + D). \qquad (1.64)$$

The cross-ratio of points A, B, Δ, E admits the following geometrical interpretation. If the points A and B divide the points Δ and E and on segments AB and ΔE two circumferences of circles are constructed as on the diameters, the cross-ratio of these four points is connected with the angle ϕ between these circles by the formula

$$A\Delta/\Delta B : AE/EB = -\cot^2(\phi/2) \ . \quad (1.65)$$

In the case of a harmonic quadruple, $\phi = 90^o$.

If the points A and B do not divide the points Δ and E then cross-ratio is positive, and angle ϕ is imaginary number equal to $i\psi$. Since

$$\cot(i\psi) = i\coth \psi , \qquad (1.66)$$

in this case, formula (1.65) can be written in the form

$$A\Delta/\Delta B : AE/EB = \coth^2(\psi/2) \ . \quad (1.67)$$

69. In Prop. I.33 and I.34, the point Δ is obtained from the point A in the first case and from the point E in the second case as the intersection of the polar of the point Δ with the diameter passing through Δ. These transformations are called inversions with respect to a parabola, a hyperbola, or an ellipse.

These inversions are expressed by rational functions of coordinates and are involutive transformations, that is coincide with transformations inverse to them. Therefore they are called “birational transformations”.
These transformations are also called “Cremona transformations”, since Luigi Cremona (1830-1903) created the general theory of these transformations.

On the history of Cremona transformations see [Ro2, pp.114-116, 142, 347-348].

70. Prop. I.35 is inverse to Prop. I.33. Here Apollonius proves that if a point A is obtained from a point Δ by the inversion with respect to a parabola with the vertex E, the segment AE is equal to the segment $E\Delta$..

71. Prop. I.36 is inverse to Prop. I.34. Here Apollonius proves that if a point E is obtained from a point Δ by the inversion with respect to a hyperbola

or an ellipse with latus transversum AB, the points E and Δ harmonically divide the points A and B.

72. In Prop. I.37, the same conics, as in Prop. I.34 and I.36, are considered. The centers of these conics are the points H.

Apollonius proves that if the point E is obtained from the point Δ by the inversion with respect to this conic, the product of the segments $x = H\Delta$ and $x' = HE$ is equal to the square of the radius $HA = a$, that is $xx' = a^2$.

In Prop. I.37 Apollonius proves also the proportion

$$| x | . | x' - x | / y^2 = 2a/2p . \quad (1.68)$$

In the case of the ellipse $| x | < a$ and the proportion (1.68) can be rewritten in the form $(a^2-x^2)/y^2 = a^2/b^2$, or alternately $(a^2-x^2)/a^2 = y^2/b^2$ equivalent to the equation (1.45) of the ellipse .

In the case of the hyperbola $| x | > a$ and the proportion (1.68) can be rewritten in the form $(x^2-a^2)/y^2 = a^2/b^2$, or alternately $(x^2-a^2)/a^2 = y^2/b^2$ equivalent to the equation (1.46) of the hyperbola .

73. The most important of the transformations considered in Prop. I.34 and I.37 is the inversion with respect to a circumference of a circle, also called "the inversion with respect to a circle". In this case, $a = b = p$, and equality $x' = a^2/x$ follows from the similarity of the triangles ΓΔH and GEH and from the equality $\Gamma H = HA$.

An inversion with respect to an ellipse can be obtained from an inversion with respect to a circle by an affine transformation. An inversion with respect to a hyperbola and parabola can be obtained from an inversion with respect to a circle by a projective transformations.

74. An inversion with respect to a circle, like stereographic projection, maps circumferences of circles and straight lines into circumferences of circles or straight lines and preserves angles between lines.

The first of these properties was known to Apollonius since Pappus of Alexandria in his Mathematical Collection wrote that Apollonius in the treatise Plane Loci described following transformations. "If two straight lines are drawn either from one given point or from two, and either in a straight line or parallel or containing a given angle, and either holding a ratio to one another or containing a given area, and the end of one touches a plane locus given in position, the end of the other will touch a plane locus given in position, sometimes of the same kind, sometimes of the other, and sometimes similarly situated with respect to the straight line, sometimes oppositely; this follows in accordance with the various assumptions"

[Pa, p. 106].

In the case when the segments considered in this fragment are two segments of one straight line and if these segments have a given ratio, the map of the ends of the segments is a homothety (1.57). If these segments contain “a given area”, that is have a given product equal to a^2, this map is an inversion with respect to a circle of radius a.

A homothety maps circles to circles and straight lines to straight lines. The inversion with respect to a circle with the center O maps circumferences not passing through O to circumferences and circumferences passing through O to straight lines.

To make the inversions with respect to the circles into bijective transformations, the plane must be supplemented by the single point at infinity. The plane supplemented by this point is called the “conformal plane”. This plane is in bijective and bicontinuous correspondence with a sphere.

Nearness of properties of an inversion with respect to a circle and of stereographic projection is explained by the fact that if an inversion with respect to a sphere is defined by analogy with an inversion with respect to a circle, the inversion with respect to the sphere $x^2 + y^2 + z^2 = a^2$ maps the sphere $x^2 + y^2 + z^2 = az$ to the plane $z = a$, and this map coincides with the stereographic projection.

Apparently Apollonius knew that inversions with respect to circles map tangent lines to tangent ones. Probably Apollonius also knew that these inversions preserve angles between lines.

75. No doubt that an inversion with respect to a circle was used by Apollonius in his treatise Tangencies. In this treatise ten problems of drawing a circle tangent to three things that can be a circle, a straight line, and a point were solved. Let us denote these problems by three letters: p (for points), l (for lines), and c (for circles). In these notations ten problems of Apollonius' Tangencies are: 1) ppp, 2) ppl, 3) pll, 4) lll, 5) ppc, 6) pcc, 7) plc, 8) llc, 9) lcc, 10) ccc .

The problems ppp and lll are problems of drawing circumscribed and inscribed circles for given triangles, other problems are more difficult.

The problem ccc of drawing a circle tangent to three given circles with the centers A, B, C and the radii r_1 , r_2 , r_3 where $r_1 \geq r_2 \geq r_3$ can be reduced to the problem pcc of drawing the circle tangent to two circles with the centers A and B and the radii $r_1 - r_3$ and $r_2 - r_3$ and passing through the point C. If the obtained circle has the center D and the radius r, the solution of the problem ccc is given by a circle with the center D and the radius $r+r_3$.

The problem pcc can be solved by the inversion with respect to a circle with the center C: this inversion maps all circles passing through C to straight lines and circles with the centers A and B and the radii $r_1 - r_3$ and $r_2 - r_3$ to the circles c_1 and c_2, and the point C to the point at infinity. Further, the straight line l tangent to the circles c_1 and c_2 must be drawn, and the same inversion must be made. This inversion will map the circles c_1 and c_2 to the circles with the centers A and B and the radii $r_1 - r_3$ and $r_2 - r_3$, and the line l to the circle giving the solution of the problem pcc.

Analogously to this problem of Apollonius, Pierre Fermat solved the problem of determining the sphere tangent to four given spheres.

76. Other transformations described by Pappus in the quoted fragment are products of homotheties and motions and products of inversions with respect to circles and motions. The former kinds of these products are called "similitudes", the latter of them are called "circular transformations". The latter transformations can also be defined as products of inversions with respect to circles, or as bijective transformations in the conformal plane mapping circumferences to circumferences, where straight lines are thought of as circumferences passing through the point at infinity.

77. Points of the conformal plane can be represented by complex numbers or by points of the Riemann sphere. In this case, circular transformations are represented by fractional linear transformations (1.64) or by compositions of these transformations with the reflection $x' = x$, with respect to the real axis.

In particular, the inversion with respect to the circumference

$$Axx + Bx + Bx + C = 0, \quad A = A, \quad C = C \qquad (1.69)$$

has the form

$$x' = (-Bx - C)/(Ax + B) , \qquad (1.70)$$

and inversion with respect to the circumference $xx = a^2$ has the form $x' = a^2/x$.

78. The cross-ratio of four points in the conformal plane represented by the complex numbers x_1, x_2, x_3, x_4 has the form

$$W = (x_1 - x_3)/(x_3 - x_2) : (x_1 - x_4)/(x_4 - x_2) . \qquad (1.71)$$

This cross-ratio, like the cross-ratio of four points in a projective straight line, is preserved under the transformations (1.64).

Therefore the cross-ratio (1.71) is real if the complex numbers x_1, x_2, x_3, x_4 represent four points on the circumference of one circle. Real cross-ratio of four complex numbers admits the geometrical interpretations with circles analogous to the interpretations (1.65) and (1.67).

79. If we put values $x_1 = x$, $x_2 = x'$, $x_3 = a$, $x_4 = b$, and $W = -1$ into equality (1.71), we obtain a circular transformation with fixed points a and b mapping every point x to such a point x' that four points x, x', a, and b form a harmonic quadruple. This transformation, called the inversion with respect to a pair of points, has the form

$$x' = ((a + b)x - 2ab)/(2x - a - b) . \quad (1.72)$$

For $a = 1$, $b = -1$, the transformation (1.72) has the form $x' = 1/x$, for $a = 0$, $b =$ transformation (1.72) has the form $x' = -x$.

On circular transformations and conformal geometry and on their history see [Ro1, pp. 204-213 and Ro2, pp. 145-147, 150-151).

80. The circular transformations (1.70) and (1.72) are involutive. Therefore these transformations determine conformal "symmetry figures". These figures are circumferences of circles and pairs of points.

The notion of symmetry figures (etres de symetrie) in the space with a group of transformations was introduced by Elie Cartan in his theory of symmetric spaces.

Analogous symmetry figures in the Euclidean plane are points and straight lines determining the reflections (1.56) and (1.3) in rectangular coordinates.

Analogous symmetry figures in the affine plane are points and "normalized straight lines", that is straight lines for which the directions of the affine reflection with respect to them are indicated. The affine reflections with respect to these figures are the transformations (1.56) and (1.3) in affine coordinates.

Analogous symmetry figures in the projective plane are pairs point + straight line which does not pass through the point. The projective reflection with respect to this figure is a homology whose center is the point of the pair and the axis is the straight line of the pair. If this figure consists of a point A and a line a, then the projective reflection with respect to this figure maps any point X to such a point X' in the line AX which together with the point X harmonically divides the point A and the point of intersection of the lines AX and a. In projective coordinates this homology has the form

$$'x^1 = x^1, \ 'x^2 = -x^2 , \ 'x^3 = x^3 \qquad (1.73)$$

In modern geometry the projective plane is often regarded not as a set of points, but as a set of pairs point + straight line. In this case, besides projective transformations (1.6) called “collineations”, projective transformations

$$u_i = \Sigma_j A_{ij} x^j \qquad (1.74)$$

mapping points to straight lines and straight lines to points are considered. The projective transformations (1.74) are called “correlations”.

The correlations (1.74) are involutive if matrices (A_{ij}) are symmetric, that is $A_{ij} = A_{ji}$. Therefore the involutive correlations (1.74) coincide with polar transformations with respect to conics determined in projective coordinates by equations

$$\Sigma_i \Sigma_j A_{ij} x^i x^j =0. \quad (1.75)$$

The conic (1.75) in affine coordinates $x = x^1/x^3$, $y = x^2/x^3$ is determined by equation (1.54), where $A = A_{11}$, $B = A_{12}$, $C = A_{22}$, $D = A_{13}$, $E = A_{23}$, and $F = A_{33}$.

The polar transformation (1.74) implies that the polar of a point with projective coordinates x_o^i with respect to the conic (1.75) is determined by the equation

$$\Sigma_i \Sigma_j A_{ij} x_o^i x^j = 0 . \quad (1.76)$$

This equation implies that the polar of a point with affine coordinates x_o and y_o with respect to the conic (1.54) is determined by the equation

$$Ax_o x + B(x_o y + y_o x) + Cy_o y +D(x + x_o) + E(y + y_o) + F = 0 . \quad (1.77)$$

Therefore conics are projective symmetry figures in projective plane regarded as a set of pairs point + straight line.

If an exterior point of a conic tends to the point of this conic, both tangent lines to this conic drawn from the exterior point tend to the tangent line at the point of a conic. Therefore the polar of an exterior point tends to the tangent line at the point of this conic, and the tangent line at a point of the conic (1.75) with projective coordinates x_o^i is determined by equation (1.76), and the tangent line to the conic (1.54) at a point with the affine coordinates x_o and y_o is determined by equation (1.77).

On symmetry figures in affine, projective and conformal geometries see [Ro1, pp.156-160, 210-213].

81. Prop. I.38 is the analogue of Prop. I.37 for the second diameter of a hyperbola and an ellipse with the center Θ. If the straight line tangent to the conic at the point E meets the line ΓΘΔ of the second diameter ΓΔ = 2b at the point H and the straight line dropped from the point E parallel to the first diameter meets the line ΓΘΔ at the point Z, the equality

$$H\Theta.\Theta Z = b^2 \qquad (1.78)$$

holds.

82. In the porism (corollary)1 to Prop. I.38 the hyperbola with the diameter AΘB and the second diameter ΓΘΔ is considered. From E of the hyperbola the tangent line meeting ΓΔ at H and EZ parallel to AB and meeting ΓΔ at Z are drawn.

Apollonius proved that the ratio ΓH/HΔ is equal to the ratio EΔ/ΓE.

This equality shows that H and Z harmonically divide two imaginary conjugate points of the second diameter at which this diameter intersects the hyperbola. The ordinates of the ends Γ and Δ of the second diameter are equal to b and -b, the ordinates of the imaginary points of intersection of this diameter with the hyperbola are equal to bi and -bi.

In modern projective geometry, a transformation in the projective straight line mapping each point X to such a point X' that the points X and X' harmonically divide two imaginary conjugate points is called an "elliptic involution" and "inversion with respect to a pair of imaginary conjugate points". An analogous transformation mapping any point X to such a point X' that X and X' harmonically divide two real points is called a "hyperbolic involution" and "inversion with respect to a pair real points".

In the porism 1 to Prop. I.38 Apollonius in fact considers an elliptic involution and shows that this involution is the composition of a hyperbolic involution with the reflection with respect to the midpoint of the segment between the fixed points of this involution.

Note that besides circumferences of circles in the conformal plane there are circumferences of circles of imaginary radius which are also symmetry figures, and the inversion with respect to a circumference of radius ri is the composition of the inversion with respect to the real circumference with the same center and radius r and the reflection with respect to this center. Its fact is an analogue of proved by Apollonius in this porism for inversions with respect to pairs of points.

83. In the end of the proof of this porism Apollonius writes οπερ εδει δει⌊αι - “what was to prove” like Euclid wrote in the ends of proofs of all theorems in Elements. Apollonius used this expression very seldom.

The abbreviation Q.E.D. of the Latin translation “quod erat demonstrandum” of this expression is often used in the ends of proofs by modern mathematicians, therefore the white square having the same sense is called also “qed”.

Arab translators of Conics in the end of each proof wrote “wa dhalika ma ardna an nabyanu” - “it is that what we wanted to prove”. Halley translated these words by the expression “quot erat demonstrandum”.

84. In Prop. I.39, the same conics as in Prop. I.37 are considered. In this proposition the point E is obtained from the point D by inversion with respect to the conic. Apollonius proves that compounded ratio

$$\Gamma E : HE = (2p : 2a) \times (E\Delta : E\Gamma) \qquad (1.79)$$

holds. If we denote $HE = x$, $E\Gamma = y$, $E\Delta = a^2/x - x$, the equality (1.79) can be rewritten in the form

$$y^2/x \mid a^2/x - x \mid = p/a . \qquad (1.80)$$

In the case of an ellipse, $x < a$ and the equality (1.80) has the form

$$y^2/(a^2 - x^2) = p/a = b^2/a^2 \qquad (1.81)$$

equivalent to equation (1.45) in rectangular or oblique coordinates.

In the case of a hyperbola, $x > a$ and the equality (1.80) has the form

$$y^2/(x^2 - a^2) = p/a = b^2/a^2 . \qquad (1.82)$$

equivalent to equation (1.46) in rectangular or oblique coordinates.

85. Prop. I.40 is the analogue of Prop. I.39 for the second diameter of an ellipse or a hyperbola. Here the ellipse or the hyperbola AB with the center Z, latera recta 2p, latera transversa $B Z \Gamma = 2a$, and the second diameters $\Delta Z E = 2b$ is considered. From the point A of the conic the ordinate AH to the second diameter is dropped, and the straight line tangent to the conic to the point Θ of the second diameter is drawn. The point Θ is obtained from the point H by the

inversion with respect to the conic. Apollonius proves that the compounded ratio

$$AH : \Theta H = (2a : 2p) \times (ZH : AH) \quad (1.83)$$

holds. If we denote $AH = x$, $ZH = y$, then, in the case of an ellipse $\Theta H = | b^2/y - y |$, and in the case of a hyperbola $\Theta H = b^2/y + y$, and the compounded ratio (1.83) can be rewritten in the form

$$(x^2/y)/(b^2/y + y) = a/p , \quad (1.84)$$

where the sign + is for a hyperbola and the sign - is for an ellipse.

In the case of an ellipse $y < b$ and equality (1.84) has the form

$$x^2/(b^2 - y^2) = a/p = a^2/b^2 \quad (1.85)$$

equivalent to equation (1.45) in rectangular or oblique coordinates.

In the case of a hyperbola, y can be arbitrary real number and equality (1.84) has the form

$$x^2/(b^2 + y^2) = a/p = a^2/b^2 \quad (1.86)$$

equivalent to equation (1.46) in rectangular or oblique coordinates.

Propositions I.41- I.45 on areas

86. In Prop. I.41 a hyperbola or an ellipse with the latus transversum $AB = 2a$, the latus rectum 2p, and the center E are considered. The abscissa $E\Delta = x$ and the ordinate $\Delta\Gamma = y$ of the point G of the conic are drawn. To the radius AE and the ordinate $\Delta\Gamma$ the segments $EH = s$ and $\Gamma\Theta = t$ under equal angles are drawn, the lengths of these segments are connected by the correlation

$$y:t = (a:s) \times (p:a) = p:s . \quad (1.87)$$

Apollonius builds three equiangular parallelograms: AH with the sides AE and ZH, $\Delta\Theta$ with the sides $\Delta\Gamma$ and $\Gamma\Theta$, and the parallelogram with the side $E\Delta$, similar to AH, and proves that the area of the third parallelogram, in the case of the hyperbola, is equal to the sum of the areas of the parallelograms AH and $\Delta\Theta$ and, in the case of the ellipse, is equal to the difference of these areas.

If we denote the sine of the equal angles AEH and ΔΓΘ by k, then the area of the parallelogram AH is equal to kas, the area of the parallelogram ΔΘ is equal to kyt =ksy^2/p, the area of the third parallelogram is equal to kasx^2/a^2 = ksx^2/a. Therefore Apollonius' assertion for the ellipse can be written in the form

$$ksx^2/a = kas - kyt, \qquad (1.88)$$

and his assertion for the hyperbola can be written in the form

$$ksx^2/a = kas + kyt . \qquad (1.89)$$

Correlation (1.87) implies that equalities (1.88) and (1.89) are equivalent to equations (1.45) and (1.46) of the ellipse and the hyperbola.

87. In Prop. I.42, a parabola BΓ with the diameter AB and the vertex B is considered. The abscissa BZ = x and the ordinate ZΓ = y of the point Γ of the parabola are determined. If the point A is obtained by the inversion with respect to the parabola from the point Z, the equality AB = BZ holds.

The abscissa BH = x_1 and the ordinate HΔ = y_1 of the point Δ of the parabola are drawn. The point Δ is joined with the point E of the diameter by the line ΔE parallel to ΓΔ, the line BΘ parallel and equal to the line ΓZ is drawn, the line HΔ is continued to the line ΓΘ. Apollonius proves that the area of the triangle ΔEH is equal to the area of the parallelogram ΘH.

If we denote the sine of the angle BZΓ by k, then the area of the parallelogram ZΘ is equal to kx_1y. Since AB = BZ, the area of the triangle AΓZ is equal to 2kxy/2 = kxy. The similarity of the triangles ΔEH and AΓZ implies that the ratio of the first of these areas to the second one is equal to $y_1^2 : y^2$. Since Γ and Δ are the points of the parabola (0.3), the proportion $y_1^2 : y^2 = x_1 : x$ holds. Therefore the area of the triangle ΔEH is equal to $kxyx_1/x = kx_1y$, that is this area is equal to the area of the parallelogram ΘH

88. In Prop. I.43 a hyperbola or an ellipse with the latus transversum AB and the center Γ is considered. The abscissa ΓZ = x and the ordinate ZE = y of the point E of the conic are drawn. The point Δ is obtained from the point Z by the inversion with respect to the conic. The abscissa ΓK= x_1 and the ordinate KH =y_1 of the point H of the conic are drawn. The line ΓE is drawn and continued to the point Λ on the straight line tangent to the conic at the point B. The ordinate HK is continued to the point M on the straight line ΓΛ. The straight line HΘ parallel to the line ΔE is drawn to the point Θ on the diameter AB. Apollonius

proves that the difference between the areas of the triangle ΚΜΓ and ΒΛΓ is equal to the area of the triangle ΗΚΘ.

If we denote the sine of the angle ΖΚΜ by k, then the areas of the triangles ΓΕΗ and ΕΗΔ are equal to kxy/2 and $k \mid a^2/x - x \mid y/2$. The similarity of the triangles ΚΜΓ and ΒΛΓ to the triangle ΖΕΓ implies that their areas are equal to $kxyx_1^2/2x^2 = kx_1^2y/2x$ and $kxya^2/2x^2 = ka^2y/2x$ and their difference is equal to $k \mid x_1^2 - a^2 \mid y/2x$.

The similarity of the triangles ΗΚΘ and ΕΖΔ implies that the area of the triangle ΗΚΘ is equal to

$$(k \mid a^2/x - x \mid y/2)(y_1^2/y^2) = k \mid x^2 - a^2 \mid y_1^2/2xy. \quad (1.90)$$

Equations (1.45) and (1.46) of an ellipse and a hyperbola can be combined into one equation

$$\mid x^2 - a^2 \mid /y^2 = a^2/b^2 . \quad (1.91)$$

Therefore for two points of these conics with coordinates x, y and x_1, y_1, we have the proportion

$$\mid x^2 - a^2 \mid /y^2 = \mid x_1^2 - a^2 \mid /y_1^2 . \quad (1.92)$$

This proportion implies that for both an ellipse and a hyperbola the area of the triangle ΗΚΘ is equal to the difference between the areas of the triangles ΚΜΓ and ΒΛΓ.

89. Prop. I.44 is the analogue of Prop. I.43 for opposite hyperbolas.

90. Prop. I.45 is the analogue of the Prop. I.43 for the second diameter.

Propositions I.46 - I.51 on transformations of coordinates

91. In Prop. I.46 Apollonius proves that in a parabola any straight line parallel to its diameter is also its diameter.

In this proposition a parabola ΒΓ with a diameter ΑΒΔ is considered. Through a point Γ of the parabola the straight line ΖΓΝΜ parallel to the diameter ΑΒΔ and the straight line ΓΑ tangent to the parabola are drawn. Through a point Λ of the parabola the ΛΕ parallel to ΓΑ is drawn. The line ΛΕ meets the diameter ΑΒΔ at E , and the line ΖΓΝΜ at Ν, and the parabola at Η.

Apollonius proves that ΛΝ = ΝΗ. Since Λ is an arbitrary point of the parabola, the chord ΛΗ is an arbitrary chord parallel to the tangent line ΑΓ, and ΖΓΝΜ bisects this chord.

Therefore the line ΖΓΝΜ is also a diameter of the parabola.

92. We translate Apollonius' term τετραπλεύρον literally mining "quadrilateral", by the word "quadrangle" since Greek word τετραγώνον mining "quadrangle" ancient mathematicians used only for squares.

93. We translate Apollonius' term πενταπλεύρον literally mining "quinquelateral", by the word "quinquangle" since Greek word πενταγώνον mining "quinquangle" ancient mathematicians used only for regular pentagon.

94. An affine reflection mapping a parabola into itself is called "parabolic turn". This transformation maps the diameters of the parabola to other diameters of this parabola. The parabolic turn mapping one diameter of the parabola to another is a composition of affine reflections with respect to the first diameter and with respect to the diameter located in the middle between both diameters. The parabolic turn mapping the diameter $y = 0$ of the parabola (0.3) to the diameter $y = h$ of this parabola has the form

$$x' = x + (h/p)y + h^2/2p, \quad y' = y + h . \quad (1.93)$$

The parabolic turn (1.93) is the product of two affine homologies which are the parallel translation (1.59), where $a = h^2/2p$ and $b = h$, and the shift (1.60) where $k = h/p$.

If we regard transformation (1.93) as a particular case of transformation (1.4), then the determinant $AE - B\Delta = 1.1 - (h/p).0 = 1$. Therefore parabolic turns are equiaffine transformations.

95. In Prop. I.47 Apollonius proves that any straight line passing through the center of a hyperbola or an ellipse is a diameter of this conic.

In this proposition, a hyperbola or an ellipse AEB with the latus transversum ABΔ and the center Γ is considered. From a point E of the conic the straight line ΔE tangent to the conic and the line EΓ are drawn. Through a point N of the conic the straight line ZNOΘ parallel to ΔE is drawn.

The line ZNOΘ meets the diameter AB at Z, the straight line ΓE at O, and the conic at Θ. Apollonius proves that NO = OΘ. Since N is an arbitrary point of the conic, the chord NΘ is an arbitrary chord parallel to EΔ, and the line ΓE bisects this chord, then ΓE is also a diameter of the conic.

96. The affine transformation mapping a hyperbola or an ellipse to itself is called, respectively, "hyperbolic turn" and "elliptic turn". These transformations map the diameters of the hyperbola or the ellipse into other diameters of this conic. The hyperbolic and elliptic turns mapping one diameter of the conic into another are compositions of affine reflections with respect to the first diameter and a diameter that is between both diameters.

The elliptic turn mapping ellipse (1.45) to itself has the form

$$x' = x\cos\phi + (a/b)y\sin\phi \quad , \quad y' = -(b/a)x\sin\phi + y\cos\phi \, . \quad (1.94)$$

The hyperbolic turn mapping hyperbola (1.46) to itself has the form

$$x' = x\cosh\phi + (a/b)y\sinh\phi \, , \quad y' = (b/a)x\sinh\phi + y\cosh\phi \quad (1.95)$$

If we regard transformations (1.94) and (1.95) as particular cases of transformation (1.4), then the determinant $\mathrm{AE} - \mathrm{B}\Delta = \cos^2\phi + \sin 2\phi = \cosh^2\phi - \sinh^2\phi = 1$.

Therefore elliptic and hyperbolic turns are equiaffine transformations.

Apollonius never mentions parabolic, elliptic, and hyperbolic turns, but no doubt that he used these transformations to generalize the results obtained by his precursors in rectangular coordinates for the cases of oblique coordinates.

97. Prop. I.48 is the analogue of Prop. I.47 for opposite hyperbolas.

98. In Prop. I.49 Apollonius proves that in the coordinate system determined by an arbitrary diameter of a parabola the coordinates of its points are connected by the same equation (0.3).

99 In Prop I.50 Apollonius proves that in the coordinate system determined by an arbitrary diameter of a hyperbola or an ellipse the coordinates of their points are connected by equations equivalent to equations (0.10) and (0.9).

100. Prop. I.51 is the analogue of Prop. I.50 for opposite hyperbolas.

Propositions I.52 - I.60 on construction of conics

101. In Prop. I.52 and I.53 the construction of the parabola with a diameter AB and a vertex A given in position and with the latus rectum 2p given in magnitude is described. In Prop. I.52, the diameter is the axis. In Prop. I.53, the general case is considered.

In Prop. I.52 Apollonius builds the right circular cone, one of the rectilinear generators of which is parallel to the plane of the parabola, and proves that this plane cuts off from the surface of the cone a parabola with the axis and the latus rectum being given lines.

In Prop. I.53 Apollonius finds two straight lines that determine the parabola with a given axis, builds this parabola by Prop. I.52, and proves that this parabola satisfies the conditions of Prop. I.53.

102. The last proportion is equivalent to the proportion (1.8). Hence the equation equivalent to the equation (0.3) can be obtained

103. In Prop. I.54 and I.55, the construction of the hyperbola with a latus transversum AB = 2a and a vertex A given in position and with the latus rectum 2p given in magnitude is described. In Prop. I.54 the diameter is the axis. In Prop. I.55 the general case is considered.

In Prop. I.54 Apollonius builds the right circular cone from which the considered plane cuts off the hyperbola, whose axis, the latus transversum, and the latus rectum are given lines.

In Prop. I.55 Apollonius finds two straight lines determining the hyperbola with given axis, builds this hyperbola by Prop. I.54, and proves that it satisfies the conditions of Prop. I.55.

104. In Prop. I.56, 1.57, and I.58, the construction of an ellipse with the latus transversum AB = 2a and the vertex A given in position, and the latus rectum 2p given in magnitude, is described.

In Prop. I.56 the latus transversum coincides with the major axis of the ellipse, in Prop I.57 the latus transversum coincides with the minor axis of the ellipse, in Prop. I.58 the general case is considered.

In Prop. I.56 Apollonius builds the right circular cone from the surface of which the considered plane cuts off the ellipse whose the major axis, the latus transverse and the latus rectum are given lines.

In Prop. I.57 and I.58 Apollonius finds two straight lines determining the ellipse with given major axis, builds the ellipse by Prop. I.56 and proves that it satisfies the conditions of Prop. I.57 and I.58.

105. In Prop. I.59 Apollonius describes the construction of two opposite hyperbolas ΑΒΓ and ΔΕΗ with the latus transversum ΒΕ = 2a, the latus rectum ΒΖ= 2p ,and an angle Θ between ordinates and the transverse diameter. Each of these hyperbolas is built by Prop. I.54 or I.55.

106. In Prop. I.60 Apollonius describes the construction of two pairs of opposite hyperbolas whose axes are conjugate axes with the latera transversa 2a and 2b of these hyperbolas corresponding to their axes.

If the equation of one pair of these opposite hyperbolas has the form (1.46) the equation of the second of these pairs has the form

$$y^2/b^2 - x^2/a^2 = 1. \qquad (1.96)$$

Apollonius indicates that the latera transversa 2a and 2b of these hyperbola are connected with their latera recta 2p and 2q by the correlations

$$(2a)^2 = 2b.2q, \quad (2b)^2 = 2a.2p . \quad (1.97)$$

Each pairs of opposite hyperbolas is built by Prop. I.59.

107. In the end of the description of this construction, Apollonius writes οπερ εδει ποιασαι - “what was to make”, like Euclid wrote in the end of all description of solutions of problems in Elements. Apollonius used this expression, like the words οπερ εδει δει⌊αι (see Note 83) very seldom.

108. Apollonius calls the hyperbolas (1.46) and (1.96), the construction of which was described in Prop. I.60, “conjugate” (συζύγεις), apparently since axes of these hyperbolas are conjugate.

COMMENTARY ON BOOK TWO

Preface to Book 2

1. On Eudemus of Pergamum see Introduction B.
2. On Apollonius' son and family see Introduction A.
3. On Philonides, Apollonius' comrade, see Introduction B.

Propositions II.1 - II.16 on asymptotes of hyperbolas

4. In Prop. II.1 Apollonius defines asymptotes of a hyperbola. He considers a hyperbola with the latus transversum ΑΒ and the center Γ. Through the point Β he draws the straight line tangent to the hyperbola and takes the points Δ and Ε on this tangent line such that the segments ΒΔ and ΒΕ are "equal in square to the quarter the eidos" corresponding to the diameter ΑΒ, that is the length b of these segments to satisfies the condition $b^2 = ap$ where a and p are the halves of the latera transversum and rectum of the hyperbola.

Apollonius proves that the straight lines ΓΔ and ΓΕ do not meet the hyperbola and calls these lines asymptotes of the hyperbola, from the word ασυμπτώτος -- "not coinciding".

Here Apollonius uses the term "asymptote" for all straight lines which do not meet the conic, for hyperbolas such straight lines are all upright diameters. However usually Apollonius used this term in the same sense as modern mathematicians, that is as limiting lines between usual diameters of a hyperbola and its upright diameters.

The equation of both asymptotes of hyperbola (1.46) has the form

$$x^2/a^2 - y^2/b^2 = 0. \qquad (2.1)$$

5. In Prop. II.2 the same hyperbola as in Prop. II.1 is considered.

Apollonius proves that any diameter of the hyperbola passing within the angle ΔΓΕ meets the hyperbola and therefore cannot be an asymptote.

6. In Prop. II.3 Apollonius proves that the straight line tangent to the hyperbola at any of its points meets both asymptotes of this hyperbola, and the point of contact bisects the segment of this line between the asymptotes, and the square of the segment of this line between the point of contact and an asymptote is equal to the quarter of the eidos corresponding to the diameter passing through the point of contact.

The assertion is evident when the point of contact is on the bisectrix of the angle ΔΓΕ. The general case can be obtained from the mentioned case by a hyperbolic turn (1.95), which transforms the hyperbola (1.46) and its asymptotes (2.1) into themselves. A hyperbolic turn is an affine transformation and maps the midpoint of a segment of the tangent straight line to the midpoint of the corresponding segment.

7. Prop. II.4 is the problem of the construction of a hyperbola with asymptotes ΑΒ and ΑΓ that passes through a given point Δ within the angle ΒΑΓ.

According to Prop. II.3, the point Δ is the midpoint of the segment ΒΓ. The point Α is the center of this hyperbola, the line ΔΑ is its diameter. If the line ΔΑ is continued to such a point Ε that ΑΕ = ΑΔ, the line ΔΕ will be the latus transversum 2a of this hyperbola. The line ΒΓ is the tangent to the hyperbola at the point Δ and is equal to 2b. The latus rectum of the hyperbola is $2p = 2b^2/a$. The ordinates dropped to the diameter ΑΔ are parallel to the line ΒΓ. Thus the problem is reduced to the problem of Prop.1.55

8. In Prop. II.5 Apollonius proves that if the diameter of a parabola or a hyperbola bisects a chord of this conic, the straight line tangent to it at the end of the diameter is parallel to this chord.

9. Prop. II.6 is analogous to Prop. II.5 for an ellipse.

10. Prop. II.7 is inverse for Prop. II.5 and II.6.

11. In Prop. II.8 Apollonius proves that if the points Α and Γ are points of the hyperbola ΑΒΓ, and the line ΑΓ continued meets the asymptotes ΕΔ and ΔΗ at the points Ε and Η, then ΓΗ = ΑΕ.

The assertion of the proposition is evident when the line ΑΓ is perpendicular to the axis of the hyperbola. The general case can be obtained from this case by a hyperbolic turn (1.95).

12. In Prop. II.9, which is a consequence of Prop. II.8, Apollonius proves that if a straight line meeting both asymptotes of a hyperbola is bisected by it, this straight line has only one common point with this hyperbola.

13. In Prop. II.10 Apollonius considers a hyperbola ΑΒΓ with the asymptotes ΔΕ and ΕΗ and a straight line ΔΑΓΗ meeting the hyperbola at the points Α and Γ and its asymptotes at the points Δ and Η, and proves that the products ΔΑ.ΑΗ and ΔΓ.ΓΗ are equal to $pa = (b^2/a)a = b^2$.

If the midpoint of the chord ΓA is Θ, the line EΘ meeting the hyperbola at B is a diameter of this hyperbola, the lines AΘ and ΘΓ are ordinates dropped to this diameter. If the equation of the hyperbola in the coordinate system whose origin is E and axis O⌊ is EBΘ is (1.46), the equation of its asymptotes is (2.1), and the line ΔAΓH is determined by equation x = h. Therefore the ordinates of the points Δ and H are y = +(b/a)h and the ordinates of the points A and Γ are $y = +(b/a)(h^2 a^2)^{1/2}$ and the equality $\Delta\Gamma.\Gamma H = (b/a)(h - (h^2-a^2)^{1/2} \cdot (b/a)(h + (h^2 - a^2)^{1/2} = (b^2/a^2)(h^2 - (h^2 - a^2)) = (b^2/a^2) \cdot a^2 = b^2$ holds.

The equality $\Delta A.AH = b^2$ can be proved analogously.

14. In Prop. II.11, Apollonius considers the hyperbola ΘBΛ with the asymptotes AΓZ and AΔK and the straight line EHΘ meeting the continued asymptote AΔK at the point E, the asymptote AΓZ at the point H and the hyperbola at the point Θ and proves that product EΘ.ΘH is equal to a^2. He draws through the point A the straight line AB parallel to EHΘ meeting the hyperbola at the point B and the straight line ΓBΔ tangent to the hyperbola at its point B. The line AB is a diameter of the hyperbola. The ordinates dropped to this diameter are parallel to the line ΓBΔ. If the equation of the hyperbola in the coordinate system whose origin is A and axis Ox is AB is (1.46), and the equation of its asymptotes is (2.1), and the line EHΘ is determined by the equation y = h, then Ox the abscissas of points H and E are x = +(a/b)h and the abscissa of the point Θ is $x = (a/b)(b^2 + h^2)^{1/2}$ and the equality $EH.ZH = (a/b)(h + (b^2 + h^2)^{1/2} \cdot (a/b)(b^2 + h^2)^{1/2} - h) = (a^2/b^2)(b^2 + h^2 - h^2) = (a^2/b^2)b^2 = a^2$ holds.

15. In Prop. II.12 Apollonius finds equations of hyperbolas in coordinate systems whose origins are centers of hyperbolas, axes are inclined to the asymptotes under arbitrary angles, and coordinates x and y of points of hyperbolas are equal to the lengths of segments parallel to these axes between the points of hyperbolas and their asymptotes.

These equations have the form xy = const. (2.2)

The constantness of the product (2.2) follows from the fact that the segments x and y are sides of parallelograms whose areas are equal to the product of the lengths x and y of these segments by sines of angles between these segments, and the hyperbolic turns transforming hyperbolas into themselves preserve the areas of these parallelograms, since these turns are equiaffine transformations.

Equation (2.2) is a particular case of equation (1.54)

The hyperbolic turn mapping the hyperbola (2.2) to itself in the coordinate system of this equation has the form

$$x' = tx, \quad y' = y/t . \qquad (2.3)$$

The important particular case of equations (2.2) is the case where the coordinate axes coincide with the asymptotes. Such was the equation of an equilateral hyperbola used by Menaechmus (see Introduction D).

The hyperbola constructed in Prop. II.4 is determined by equation (2.2) where the right hand side is equal to the product of coordinates of the point Δ. The points of this hyperbola can be obtained from the point Δ by hyperbolic turn (2.3).

16. In Prop. II.13 Apollonius proves that a straight line parallel to an asymptote of a hyperbola meets it at one point only.

The direction of an asymptote of a hyperbola is called by modern mathematicians the "asymptotic direction of a hyperbola". Analogous property holds for straight lines parallel to the axis of a parabola, therefore the direction of these lines is called the "asymptotic direction of a parabola"

17. In Prop. II.14 Apollonius proves that the asymptotes of a hyperbola and this hyperbola itself, if continued indefinitely, converge and a distance between them will be less than any given magnitude.

The assertion of this proposition is very similar to the formulations of Carl Weierstrass (1815-1897) for limits of sequences and continuity of functions: the magnitude a is the limit for sequence a_n if for any ε there exists an integer N such that for $n > N$ the inequality $| a - a_n | < \varepsilon$ holds, a function f(x) is continuous for $x = x_0$ if for any ε there exists such magnitude α that if $| x - x_0 | < \alpha$ the inequality $| f(x) - f(x_0) | < \varepsilon$ holds. Probably these formulations were created under the influence of this proposition of Conics.

18. The porism to Prop. II.14 shows that Apollonius sometimes uses the word "asymptote" not only in the same sense as modern mathematicians but also for all straight lines which do not meet conic (see Note 4 on this book).

19. Prop. II.15 is the first proposition in Book 2 of Conics where opposite hyperbolas are considered. Apollonius proves that opposite hyperbolas have the same asymptotes.

The assertion of Prop. II.15 follows from the fact that opposite hyperbolas and their asymptotes are determined by the same equations (1.46) and (2.1) as one hyperbola and its asymptotes.

20. In Prop. II.16 Apollonius proves that a straight line cutting both sides of the angle between the asymptotes of a hyperbola which is adjacent to the angle containing this hyperbola meets this hyperbola and the hyperbola opposite to it at single points.

Propositions II.17 - II.23 on conjugate opposite hyperbolas

21. In Prop. II.17 Apollonius first considers "conjugate opposite hyperbolas" whose definition was given in Prop. I.60 (see Note 108 on the Book 1).

In Prop. II.17 Apollonius proves that the asymptotes of conjugate opposite hyperbolas (1.46) and (1.96) coincide. Both these equations imply that the asymptotes of these hyperbolas are determined by equation (2.1).

22. In Prop. II.18 Apollonius proves that a straight line tangent to one of conjugate opposite hyperbolas meets each of the second opposite hyperbolas at one point.

23. In Prop. II.19 Apollonius proves that the segment of the tangent straight line considered in Prop. II.18 between the points of meeting with the opposite hyperbolas is bisected at the point of contact.

This assertion is evident when the point of contact is on an axis of the hyperbolas. The general case can be obtained from the mentioned case by a hyperbolic turn mapping each of the conjugate opposite hyperbolas to itself.

24. In Prop. II.20 Apollonius proves that if a straight line is tangent to one of the conjugate opposite hyperbolas, the diameter passing through the point of contact and the diameter parallel to the tangent straight line are conjugate upright and transverse diameters of the opposite hyperbolas.

25. In Prop. II.21 Apollonius proves that if segments AB and $\Gamma\Delta$ are conjugate diameters of conjugate opposite hyperbolas, the straight lines AE and $\Gamma\mathrm{E}$ tangent to these hyperbolas at the ends of these diameters meet at the point E of an asymptote of these hyperbolas.

This proposition follows from Prop. II.1 and the equalities $\mathrm{AB} = 2a$ and $\Gamma\Delta = 2b$.

26. Prop. II.22 is the analogue of Prop. II.9 and II.10 for conjugate opposite hyperbolas.

27. In Prop. II.23 Apollonius considers conjugate opposite hyperbolas $\mathrm{A}, \mathrm{B}, \Gamma, \Delta$ with conjugate transverse diameters $\mathrm{AB} = 2a$ and $\Gamma\Delta = 2b$ and the center X, and the straight line $\mathrm{KNM}\Lambda$ parallel to $\Gamma\Delta$ and meeting three adjacent hyperbolas: the hyperbola Γ at the point K, the hyperbola A at the points M and N, and the hyperbola Δ at the point Λ.

Apollonius proves that the product $\mathrm{KM.M}\Lambda = 2b^2$.

If the axis Ox of the coordinate system is the line AB and the axis Oy is the line $\Gamma\Delta$, and the line $\mathrm{KMN}\Lambda$ is determined by the equation $x = h$, then ordinates of the points K and Λ are $+(b/a)(h^2+a^2)^{1/2}$, the ordinates of the points M and N are $+(b/a)(h^2-a^2)^{1/2}$. Therefore the equality $\mathrm{KM.M}\Lambda = (b/a)((h^2+a^2)^{1/2}$ -

$(h^2-a^2)^{1/2}) \cdot (b/a)((h^2+a^2)^{1/2} + (h^2-a^2)^{1/2}) = (b^2/a^2)((h^2+a^2) - (h^2-a^2)) =$
$(b^2/a^2) \cdot 2a^2 = 2b^2$ holds.

Propositions II.24 - II.43 on chords and diameters of conics

28. In Prop. II.24 Apollonius proves that if, in a parabola, two chords are drawn and the arcs cut off by these chords have no common points, then the continuations of these chords meet at an exterior point of this parabola.

29. Prop. II.25 is the analogue of Prop. II.24 for a hyperbola.

There also exists an analogue of Prop. II.24 and II.25 for an ellipse.

Apollonius does not consider this proposition because the analogous assertion is well known for the circumference of a circle, and the assertion for an ellipse can be obtained from this assertion by a contraction of a circle to its diameter.

30. In Prop. II.26 Apollonius proves that one of two chords of an ellipse not passing through its center cannot bisect the second chord since chords of an ellipse bisecting another chord are diameters of this ellipse.

31. In Prop. II.27 Apollonius proves that if two straight lines tangent to an ellipse touch it at two ends of a diameter, they are parallel. This assertion follows from the fact that the midpoints of all chords of an ellipse parallel to a straight line tangent to it are points of a diameter whose vertex is the point of contact.

Since a point of meeting of two straight lines tangent to a conic is the pole of the straight line joining the points of contact, and since two parallel straight lines meet at infinity, diameters of a conic can be regarded as the polars of points at infinity.

32. In Prop. II.28 Apollonius proves that a straight line bisecting two parallel chords of a conic is a diameter of this conic.

33. In Prop. II.29 Apollonius proves that a straight line joining the point of intersection of two straight lines tangent to a conic with the midpoint of the chord between the points of contact is a diameter of the conic.

This proposition implies that a straight line joining the midpoint of a chord of a conic with the pole of the line of this chord is a diameter of the conic.

34. In Prop. II.30 Apollonius proves that the diameter of a conic drawn through an exterior point of a conic bisects the segment of the polar of this point between the points of meeting of the polar with the conic.

This proposition is the inverse of Prop II.29.

35. Prop. II.31 is the analogue of Prop. II.27 for opposite hyperbolas.

36. In Prop. II.32 Apollonius considers opposite hyperbolas and two straight lines tangent to each of them or intersecting each of them at two points, and proves that continuations of these straight lines meet within the angle between the asymptotes of these hyperbolas adjacent to the angle containing one of these hyperbolas.

37. In Prop. II.33 Apollonius proves that a straight line meeting one of the opposite hyperbolas at two points does not meet the other hyperbola, and besides the angle containing the first hyperbola, this straight line will fall inside the angles adjacent to this angle.

38. In Prop. II.34 Apollonius proves that if a straight line is tangent to one of the opposite hyperbolas and a line parallel to it meets the other hyperbola at two points, the straight line joining the point of contact and the midpoint of the segment of the parallel line between its points of meeting with the hyperbola is a transverse diameter of the opposite hyperbolas.

39. Prop. II.35 is inverse to Prop. II.34.

40. Prop. II.36 is the analogue of Prop. II.28 for opposite hyperbolas.

41. In Prop. II.37 Apollonius proves that if a straight line not passing through the center of opposite hyperbolas meets both these hyperbolas, a straight line joining the midpoint of the segment of this line between the opposite hyperbolas with the center of these hyperbolas is an upright diameter of these hyperbolas, and a straight line parallel to the first line and drawn through the center is a transverse diameter of these hyperbolas conjugate to the upright diameter.

42. In Prop. II.38 the same opposite hyperbolas and straight line intersecting these hyperbolas, as in Prop. II.37, are considered.

Apollonius proves that the upright diameter of opposite hyperbolas joining the midpoint of the segment of a straight line between both hyperbolas with the pole of this line is conjugate to the transverse diameter parallel to this line.

43. Prop. II.39 is the analogue of Prop. II.29 for opposite hyperbolas.

44. In Prop. II.40 the opposite hyperbolas ΓΑΗ and ΔΒΗΘ with the center Χ are considered. From the points Γ and Δ the straight lines ΓΕ and ΔΕ tangent to the hyperbolas are drawn, through the points Η and Θ the straight lines ΗΖ and ΘΖ tangent to the hyperbolas are drawn. The straight lines ΑΧΒ, ΓΘΔ, and ΦΖΕΗ are parallel. Apollonius proves that straight lines ΑΧΒ and ΕΧΖ are conjugate transverse and upright diameters of the opposite hyperbolas. In this proposition the lines ΓΔ and ΗΘ are the polars of the points Ε and Ζ.

45. Prop. II.41 is the analogue of Prop. II.26 for opposite hyperbolas.

46. Prop. II.42 is the analogue of Prop. II.26 for conjugate opposite hyperbolas.

47. Prop. II.43 is the analogue of Prop. II.37 for conjugate opposite hyperbolas.

Propositions II.44 - II.46 on finding diameters, center, and axes of conics

48. In Prop. II.44 Apollonius finds a diameter of a conic as the straight line joining midpoints of two parallel chords.

49. On synthesis and analysis of problems see Introduction E.

50. In Prop. II.45 Apollonius finds the center of an ellipse and a hyperbola as the point of intersection of two diameters.

51. In Prop. II.46 Apollonius finds the axis of a parabola. First Apollonius draws a diameter of this parabola. If it is not the axis, he draws a chord perpendicular to it, and the axis of the parabola is the perpendicular erected at the midpoint of this chord.

52. In Prop. II.47 Apollonius finds the axes of a hyperbola or an ellipse. First Apollonius draws a diameter. If this diameter is not an axis, the center of the conics is found. From this center an arc of the circumference of a circle meeting the conic at two points is described. These points are joined by a chord. The perpendicular to this chord at its midpoint is one of the axes of the hyperbola or the ellipse. The second axis is the diameter perpendicular to the first axis.

The problem solved in this proposition is equivalent to finding eigenvectors of a linear operator Φ of equation (1.55) of a conic.

53. In Prop. II.48 Apollonius proves that there are no other axes of a parabola, a hyperbola, and an ellipse besides axes found in Prop. II.46 and II.47.

Propositions II.49 - II.53 on drawing straight lines tangent to conics

54. Prop. II.49 is the problem of drawing straight lines tangent to conics from given points. In modern mathematics this problem is solved by methods of differential geometry. Apollonius solves this problem as a problem of synthetic geometry.

If the given point is a point of a conic, Apollonius draws the axis of this conic, drops the perpendicular from the given point to the axis, and finds the point on the axis corresponding to the foot of the perpendicular in the inversion with respect to the conic. The tangent straight line joins the found point with the given point.

If the given point is an exterior point of the continuation of the axis of a conic, Apollonius finds the point of the axis corresponding to the given point in the inversion with respect to the conic, at this point the perpendicular to the axis is erected, and the point of meeting of this perpendicular with the conic is found. The tangent straight line joins the found point with the given point.

If the given point is an arbitrary exterior point of the conic, Apollonius draws through this point a diameter of the conic, finds the point of this diameter corresponding to the given point in the inversion with respect to the conic. At the point of meeting of the diameter with the conic Apollonius draws the straight line tangent to the conic, through the point corresponding to the given point in the inversion with respect to the conic, Apollonius draws the line parallel to the tangent straight line and joins the found point with the given point.

In drawing of straight lines tangent to a hyperbola Apollonius considers three cases: when the given exterior point is within the angle between the asymptotes containing the hyperbola, when the given point is a point of an asymptote, and when the given point is a point of the angle between the asymptotes adjacent to the angle containing the hyperbola. In the first of these cases, from the given point two tangent straight lines to the hyperbola can be drawn. In the third of these cases, from the given point only one tangent straight line to the hyperbola can be drawn. In this case, from the given point also a straight line tangent to the opposite hyperbola can be drawn. In the case when the given point is within the angle between the asymptotes vertical to the angle containing the hyperbola, from this point no tangent straight line to the hyperbola can be drawn, but if the point is exterior for the opposite hyperbola, from this point two straight lines tangent to the opposite hyperbola can be drawn.

The method of drawing straight lines tangent to a hyperbola described above cannot be used if the given point is a point on an asymptote. In this case, from the given point on an asymptote an arbitrary straight line intersecting another asymptote is drawn, the segment of this line between the asymptotes is bisected, from the midpoint of this segment a straight line parallel to the second asymptote is drawn. According to Prop. II.3, the tangent straight line joins the point of intersection of the line parallel to the second asymptote with the conic and the given point.

In the cases of a parabola and an ellipse, from any exterior point of these conics two tangent straight lines to these conics can be drawn, the same is the property of two opposite hyperbolas. If an exterior point is a point on an asymptote, the role of the second straight line is played by this asymptote itself. If the given point is the center of the opposite hyperbolas, the role of two tangent straight lines is played by both asymptotes.

55. In Prop. II.50 Apollonius draws a straight line tangent to a conic and forming an angle equal to a given acute angle with the axis of the conic.

This problem, as well as the problem of Prop. II.51 and II.52, is equivalent to the solution of a differential equation. In this case, if the equation of a conic in the coordinate system whose the axis Ox is the axis of the conic has the form y = f(x), the problem is equivalent to the solution of the equation y' = k. Apollonius solves these problems by methods of the synthetic geometry on the basis of Prop. II.49.

56. Ex aequalis is the transition from the proportions A/B = X/Δ and X/Δ =E/Φ to the proportion A/B = E/Φ (Definition V.12 of Elements) [Euc., p. 100].

57. In Prop. II.51 Apollonius draws a straight line tangent to a parabola or a hyperbola and forming an angle equal to a given acute angle with a diameter of the conic passing through the point of contact.

58. In Prop. II.52 an ellipse with the major axis AB, the minor axis XΔ, the center E, and the straight line HΓΛ tangent to the ellipse at a point G are considered. The lines AX, XB, and ΓE are drawn. The line HΓΛ meets XB at the point Λ and the line AB at the point H. Apollonius proves that the angle ΛΓE is not less than the angle ΛXA.

Apollonius expression "the straight lines deflected at the middle of the section" means that two rectilinear segments AX and XB form the broken line AXB joining the ends of the major axis of the ellipse with one of the ends of its minor axis.

59. Prop. II.53 is the analogue of Prop. II.51 for an ellipse.

60. In the last diagram to Prop. II.53, there is a rectilinear segment Ω□oς with the archaic Greek letters (see Introduction, H).

In the extant editions of the Greek text of Conics instead of the letter □ with the numerical value 900, the letter A' with the numerical value 1000 is written, but in the edition [Ap12] this letter is transcribed by the Arabic letter "sad" corresponding to the letter □.

Probably, in the original Greek text, instead of the little letter ς, the great letter F with the same numerical value 6 was written, and the mentioned segment in original text had the form Ω □ oF.

61. See Note 107 on Book I.

COMMENTARY ON BOOK THREE

Propositions III.1 - III.15 on areas

1. In Prop. III.1 Apollonius considers a conic ΑΒ, draws the straight lines ΑΕΓ and ΒΕΔ tangent to it and the diameters ΑΔ and ΒΓ and proves the equality of the areas of the triangles ΑΔΕ and ΕΒΓ.

This equality is evident in the case where these triangles are symmetric with respect to one of the axes of the conic. For a parabola where the diameters ΑΔ and ΒΓ are parallel, the general case of this equality can be obtained from the mentioned case by a parabolic turn. For a hyperbola or an ellipse where the diameters ΑΔ and ΒΓ meet at the center of the conics, the general case of this equality can be obtained from the mentioned case by a hyperbolic or an elliptic turn.

2. In Prop. III.2 in the same conic as in Prop. III.1, from a point Θ of the conic the straight lines ΘΚΛ and ΘΜΗ are drawn parallel to the tangent lines ΑΕΓ and ΒΕΔ. The line ΘΜΗ meets the line ΑΓ at the point Ι. Apollonius proves that the areas of the triangle ΑΙΜ and of the quadrangle ΣΛΘΙ are equal.

The proof for a parabola is based on Prop. I.42, the proofs for a hyperbola and an ellipse are based on Prop. I.43.

3. In Prop. III.3 in the same conic as in Prop. III.1, from the points Η and Θ of the conic the straight lines ΗΖΚΛ, ΝΗΙΜ, and ΝΘΞΟ, ΘΖΠΡ are drawn parallel to the tangent lines ΑΕΓ and ΒΕΔ to the conic.

Apollonius proves that the areas of the quadrangles ΛΟΘΖ and ΜΠΖΗ are equal, and the areas of the quadrangles ΡΜΝΘ and ΚΗΝΟ are also equal.

The proof is based on Prop. III.2.

4. In Prop. III.4 Apollonius considers opposite hyperbolas Α and Β, with the center Δ, draws the tangent straight lines ΑΓ and ΒΓ meeting at Γ the diameters ΑΔ and ΒΔ meeting the lines ΒΓ and ΑΓ at the points Η and Θ, joins the line ΑΒ and ΓΔ, the line ΓΔ continued meets the line ΑΒ at the point Ε. Apollonius proves the equalities of the areas of the triangles ΑΘΔ and ΒΔΗ and of the triangles ΑΓΗ and ΒΓΘ. These equalities are evident in the case where these pairs of the triangles are symmetric with respect to the line ΓΔΕ. The general cases of

these equalities can be obtained from the mentioned cases by a hyperbolic turn around the center Δ.

5. In Prop. III.5 Apollonius considers opposite hyperbolas Α and Β with the center Γ, and through Ε and Η of these hyperbolas he draws the tangent straight lines ΕΔ and ΗΔ meeting at Δ, joins the lines ΓΔ, ΓΕ, ΓΗ and ΕΗ. Through Θ of one hyperbola, Apollonius draws the straight lines ΘΖΚΛ parallel to the line ΕΗ and the line ΘΜ parallel to the line ΔΗ. The line ΘΖΚΛ meets the continuations of the lines ΓΔ at the point Ζ, the line ΗΔ at the point Κ, and the line ΗΓ at Λ. The line ΑΓ meets a continuation of the line ΓΔ at Μ. Apollonius proves that the difference between the areas of the triangles ΘΖΝ and ΚΘΔ is equal to the area of the triangle ΚΛΗ.

The proof is based on Prop. I.45.

6. In Prop. III.6 opposite hyperbolas ΑΒ and ΓΔ with the diameters ΝΑΘΕΓ and ΒΜΗΕΔ, and the tangent straight lines ΙΑΖΗ and ΒΛΖΘ are considered. Through a point Κ of the hyperbola ΑΒ the straight line ΚΛΜ parallel to the tangent line ΑΗ and the straight line ΙΚΝΞ parallel to the tangent line ΒΘ are drawn. Apollonius proves that the areas of the quadrangle ΚΙΗΜ and the triangle ΑΙΝ are equal.

The proof is based on Prop. III.2.

7. In Prop. III.7 opposite hyperbolas ΑΒ and ΓΔ with the tangent straight lines ΑΗ and ΒΘ to one of them are considered. Through a point Κ of the hyperbola ΑΒ and a point Λ of the hyperbola ΓΔ the straight lines ΜΚΠΡΧ and ΝΣΤΛΩ parallel to the tangent line ΑΖ and the straight lines ΝΙΟΚΞ and ΧΦΥΛΨ parallel to the tangent line ΒΗ are drawn.

Apollonius proves that the areas of the quadrangles ΛΥΕΤ and ΙΚΡΕ are equal and that the areas of the quadrangles ΚΦΥΙ and ΡΧΛΤ are also equal.

The proof is based on Prop III.2.

8. In Prop. III.8 opposite hyperbolas ΑΒ and ΓΔ with the diameters ΑΘΕΙΓ and ΒΗΕΟΔ and the tangent straight lines ΑΖΗΤ and ΒΖΘΧΓ are considered. From the point Γ the straight lines ΓΟ and ΓΤ parallel to the tangent lines ΑΗ and ΒΘ are drawn. Through Δ the straight lines ΔΞ and ΔΙ parallel to the same tangent lines are drawn.

Apollonius proves that the areas of the quadrangles ΔΕΘΞ and ΗΕΓΤ are equal, and the areas of quadrangles ΞΔΙΘ and ΟΓΤΗ are also equal.

These equalities are evident in the case of symmetry of equal quadrangles with respect to the transverse axis of the opposite hyperbolas.
The general case can be obtained from this case by a hyperbolic turn.

9. In Prop. III.9 opposite hyperbolas ΑΒ and ΓΔ with the diameters ΑΘΕΜΓ and ΒΗΕΟΔ and the tangent straight lines ΑΖΗ and ΒΖΘ are considered. Through Γ and Κ of the hyperbola ΓΔ, where the point Κ is situated between the points Γ and Δ, the straight lines ΓΟ and ΚΛ parallel to the tangent straight line ΑΗ, and the straight lines ΓΛ and ΚΜ parallel to the tangent line ΒΘ are drawn . Apollonius proves that the areas of the triangle ΗΕΟ and the quadrangle ΚΕ are equal, and the areas of the quadrangles ΛΟ and ΛΜ are also equal.

The proof is based on Prop. III.5.

10. In Prop. III.10 the same opposite hyperbolas ΑΒ and ΓΔ as in Prop. III.9 are considered. From Κ of the hyperbola ΑΒ and from Λ of the hyperbola ΓΔ, the straight lines ΡΚΧ and ΛΥΤ parallel to the tangent line ΑΗ, and the straight lines ΚΞΙ and ΟΛϑΧ parallel to the tangent line ΒΘ are drawn. Apollonius proves that the areas of the quadrangles ΛΤΡΨΧ and ΦΧΚΙ are equal.

The proof is based on Prop I.44 and III.1.

11. Prop. III.11 is the analogue of Prop. III.5 for opposite hyperbolas.

12. Prop. III.12 is the analogue of Prop. III.3 for opposite hyperbolas.

13. Prop. III.13 is the analogue of Prop. III.1 for conjugate opposite hyperbolas.

14. Prop. III.14 is the analogue of Prop. III.5 for conjugate opposite hyperbolas.

15. Prop. III.15 is also the analogue of Prop. III.5 for conjugate opposite hyperbolas.

Propositions III.16 - III.29 on powers of points and their generalizations

16. In Prop. III.16 a conic ΑΒ is considered. From the points Α and Β the tangent straight lines ΑΓ and ΒΓ are drawn. Through an arbitrary point Δ of the conic the straight line ΗΔΕ parallel to the tangent line ΒΓ is drawn. The line ΗΔΕ meets the conic at Η and the tangent line ΑΓ at Ε. Apollonius proves that the proportion

$$\mathrm{B\Gamma}^2/\mathrm{A\Gamma}^2 = \mathrm{HE.E\Delta}/\mathrm{AE}^2 \qquad (3.1)$$

holds. The proof is based on Prop. I.46 for a parabola and I.47 for a hyperbola and an ellipse, and on Prop. III.1 and III.2 in all cases.

In the case of the circumference of a circle $\mathrm{A\Gamma} = \mathrm{B\Gamma}$ and $\mathrm{HE.E\Delta} = \mathrm{AE}^2$.

The magnitude $HE.E\Delta = AE^2$ is called the "power of the point E with respect to a circle", therefore this proposition is called "theorem on the power of a point".

17. In Prop. III.17 a conic AB is considered. From the points A and B the tangent straight lines $A\Gamma$ and $B\Gamma$ are drawn. Through an arbitrary points Δ and E of the conic the straight lines $\Delta H\Theta Z$ and EHIK parallel to the tangent lines $B\Gamma$ and $A\Gamma$ are drawn. These lines meet the conic at Z and K . The straight line $\Delta HZ\Phi$ meets the diameter $BO\Xi\Pi$ at Θ the straight line EHIK meets the diameter $A\Lambda MN$ at I. Apollonius proves that the proportion

$$A\Gamma^2/B\Gamma^2 = KH.HE/ZH.H\Delta \quad (3.2)$$

holds. The proof is based on Prop. I.46 for a parabola and on Prop. I.47 for a hyperbola and an ellipse, and on Prop. III.1 and III.3 in all cases.
In the case of the circumference of a circle $A\Gamma= B\Gamma$ and $KH.HE =ZH.H\Delta$.
The magnitude $KH.HE = ZH.H\Delta$ is called the "power of the point H with respect to the circle", therefore this proposition is also called "theorem on power of a point".

The point H can be an interior and an exterior point of the conic.

This proposition is also called "Newton theorem" since I. Newton revealed this proposition in his Mathematical Principles of Natural Philosophy , but he mentioned that this theorem is borrowed from ancient mathematicians.

Prop. III.16 can be considered as a limiting case of Prop. III.17, and the role of the point H of Prop. III.17 in Prop. III.16 is played by the exterior point E, the role of the points Δ and Z is played by the points Δ and H, and the role of the points E and K is played by the point A.

18. Prop. III.18 is the analogue of Prop. III.16 for opposite hyperbolas.

19. Prop. III.19 is the analogue of Prop. III.17 for opposite hyperbolas.

20. In Prop. III.20 opposite hyperbolas AB and $\Gamma\Delta$ with the center E are considered. From the points A and Γ the tangent straight lines AH and ΓH are drawn, the points A and Γ are joined by the straight line $A\Gamma$. The diameters EA and EH and the straight line $B\Theta N$ parallel to the line $A\Gamma$ are drawn. From the point K of the hyperbola AB the straight line $K\Lambda\Sigma MN\Xi$ parallel to the line $A\Gamma$ is drawn, it meets the continuation of the line AH at Λ and the hyperbola $\Gamma\Delta$ at Ξ. Apollonius proves that the proportion

$$BH.H\Delta/HA^2 = K\Lambda.\Lambda\Xi/A\Lambda^2 \quad (3.3)$$

holds. The proof is based on Prop. II.38, II.39, III.1, and III.5.

21. In Prop. III.21 opposite hyperbolas ΝΑΘΚΒ and ΓΟΔ are considered. From the points Α and Γ the tangent straight lines ΑΗ and ΓΗ are drawn, the line ΑΓ is joined. From Η the straight line ΔΗ parallel to the line ΑΓ is drawn. Through Θ and Κ of the hyperbola ΑΒ the straight lines ΝΞΘΟΠΡ and ΚΣΤ parallel to the line ΑΗ, and the straight lines ΘΛΜ and ΚΟΦΙΧΩΘ parallel to the line ΑΓ are drawn. Apollonius proves that the proportion

$$\mathrm{B\Gamma.\Gamma\Delta/\Gamma A^2 = AO.O\Omega/NO.OH} \quad (3.4)$$

holds. The proof is based on Prop. II.38, II.39. III.2, and III.12.

22. In Prop. III.22 opposite hyperbolas ΑΞΛΜ and ΘΒ are considered. From Α and Β the parallel tangent straight lines ΑΓ and ΒΔ are drawn, the line ΑΒ is joined. Through an interior point Ε the straight line ΚΕΛΜ parallel to the lines ΑΓ and ΒΔ, and the line ΘΞΕ parallel to the line ΑΒ are drawn. The straight line ΚΕΛΜ meets the continuation of the line ΑΒ at Λ. Since the lines ΑΓ and ΒΔ are parallel, the line ΑΒ is a diameter, and the segment ΑΝ is the latus transversum 2a. The latus rectum of the opposite hyperbolas is equal to 2p. Apollonius proves that the equality

$$2a/2p = \mathrm{\Theta E.E\Xi/KE.EM} \quad (3.5)$$

holds.

23. Prop. III.23 is the analogue of Prop. III.17 for conjugate opposite hyperbolas.

24. In Prop. III.24 conjugate opposite hyperbolas Α, Β, Γ, and Δ with the center Ε and the conjugate diameters ΑΓ = 2a and ΒΔ = 2b are considered. Through a point Ξ situated among all four hyperbolas the straight lines ΘΞΛ and ΡΞΜ parallel to the lines ΑΓ and ΔΒ are drawn, these lines meet the hyperbolas Α, Β, Γ, and Δ at the points Η, Ρ, Λ, Μ, respectively.

Apollonius proves that the sum of the product ΖΞ..ΞΛ and the magnitude z, determined by the proportion $\mathrm{P\Xi.\Xi M}/z = b^2/a^2$, is equal to $2a^2$, that is

$$\mathrm{Z\Xi.\Xi\Lambda} + (a^2/b^2)\,\mathrm{P\Xi.\Xi M} = 2a^2. \quad (3.6)$$

In the case where the point Ξ is on the diameter ΑΓ, and the conjugate opposite hyperbolas Α, Γ and Β, Δ are determined by equations (1.46) and (1.96) in the coordinate system whose the axes Ox and Oy are the diameters ΑΓ and ΒΔ, the abscissas of the points Γ, Ξ, Λ are equal to -a, x, a, and the ordinates of the points P, Ξ, M are equal to $(b/a)(a^2 + x^2)^{1/2}$, 0, $-(b/a)(a^2 + x^2)^{1/2}$ .Therefore in this case ΓΞ = a + x, ΞΛ = a - x, PΞ = ΞM = $(b/a)(a^2 + x^2)^{1/2}$, ΓΞ.ΞΛ = $a^2 - x^2$, PΞ.ΞM $=(b^2/a^2)(a^2 + x^2)$, and the left hand side of equality (3.6) has the form $a^2 - x^2 + a^2 + x^2 = 2a^2$.

25. The words "as the square on ΔE is to the square on AE" are absent in the Greek text. The gap was filled by Halley [Ap2, p.184].

26. Prop. III.25 and III.26 are the analogues of Prop. III.24 for the cases where the point Ξ is situated within one of hyperbolas.

In Prop. III.25 the point Ξ is situated within the hyperbola B or Δ, in Prop.III.26 it is situated within the hyperbola A or Γ.

In Prop. III.25 Apollonius proved that

$$(a^2/b^2)\mathrm{P\Xi.\Xi M} - \mathrm{O\Xi.\Xi N} = 2a^2. \quad (3.7)$$

In the case where the point X is situated on the diameter BΔ, and the conjugate opposite hyperbolas A, Γ and B, Δ are determined by equations (1.46) and (1.96) in the coordinate system whose the axes Ox and Oy are the diameters BΔ and AΓ, the abscissas of the points O, Ξ, N are equal to -a, x,a and the ordinates of the points P, Ξ, M are equal to $(b/a)(a^2 + x^2)^{1/2}$, 0, $-(b/a)(a^2 + x^2)^{1/2}$. Therefore in this case OΞ = a + x, ΞNZΘ= x - a, PΞ = ΞM = $(a/b)(a^2 + x^2)^{1/2}$, RX.XM = $(b2/a2)(a2 + x2)$,and the left hand side of equality (3.7) has the form $a^2 + x^2 - x^2 + a^2 = 2a^2$.

Since Prop. III.26 differs from Prop. III.25 only by the replacing the hyperbolas B and Δ by the hyperbolas A and Γ, the assertion of Prop. III.26 can be written as

$$(b^2/a^2)\Lambda\Xi.\Xi H - \mathrm{P\Xi.\Xi}\Theta = 2b^2 . \quad (3.8)$$

The multiplication of both of the parts of equality (3.8) by a^2/b^2 transforms equality (3.8) into the equality

$$\Lambda\Xi.\Xi H - 2a^2 = (a^2/b^2)\mathrm{P\Xi.\Xi}\Theta \quad (3.9)$$

exactly corresponding to the formulation of Prop.III.26.

27. Prop. III.27 is the analogue of Prop. III.24 for an ellipse.
In this proposition an ellipse ABΓΔ with the center E, the erect diameter AEΓ = 2b, and the transverse diameter BEΔ = 2a is considered.
Through a point H the straight lines NHZΘ parallel AΓ and through Λ the line

$K Z \Lambda M$ parallel to $B\Delta$ are drawn.

Apollonius proves that the sum of NZ^2, $Z\Theta^2$, and the areas of the figures described on KZ and ZM similar and similarly situated to the eidos corresponding to $A\Gamma$ is equal to $4a^2$.

Since the latus transversum of the mentioned eidos is equal to 2b and the latus rectum of this eidos is equal to $2a^2/b$, the area of the first plane is equal to the product of KH by the magnitude z, determined by the proportion $KH/b = z/(a^2/b)$, that is $z = (a^2/b^2)KH$. Therefore the area of this plane is equal to $(a^2/b^2)KH^2$. Analogously we obtain that the area of the second plane is equal to $(a^2/b^2)HM^2$. Therefore the assertion of this proposition can be written in the form

$$NH^2 + HZ^2 + (a^2/b^2)(KH^2 + HM^2) = 4a^2 . \quad (3.10)$$

In the case where the point H is situated on the diameter $B\Delta$ and the ellipse $AB\Gamma\Delta$ is determined by equation (1.45) in the coordinate system whose axes Ox and Oy are the diameters $B\Delta$ and $A\Gamma$, the abscissas of the points N, H, Z are equal to a, x, -a and the ordinates of the points K, H, M are equal to $(b/a)(a^2 - x^2)^{1/2}$, 0, $(b/a)(a^2 - x^2)^{1/2}$. Therefore in this case $NH = a + x$, $HZ = a - x$, $KH\Theta = \Theta M = (b/a)(a^2 - x^2)^{1/2}$, $NH^2 + HZ^2 = (a + x)^2 + (a - x)^2 = 2a^2 + 2x^2$, $KH^2 + HM^2 = 2(b^2/a^2)(a^2 - x^2)$, and the left hand side of equality (3.10), $2a^2 + 2x^2 + 2a^2 - 2x^2 = 4a^2$.

In this proposition Apollonius uses the abbreviation $\alpha\pi o$ $NZ\Theta$ for the sum of the squares of the lines NZ and $Z\Theta$. Analogous abbreviation Apollonius uses in following propositions.

28. In Prop. III.28 conjugate opposite hyperbolas A, Γ, and B, Δ with the conjugate diameters $A\Gamma = 2b$ and $B\Delta = 2a$ are considered. Through an exterior point Θ of all hyperbolas the straight lines $\Lambda\Theta MN$ and $H\Theta ZK$ parallel to the diameters $A\Gamma$ and $B\Delta$ are drawn. These lines meet the hyperbolas at the points Λ, N, H, K. Apollonius proves that

$$(\Lambda\Theta^2 + \Theta N^2)/(H\Theta^2 + \Theta K^2) = A\Gamma^2/B\Delta^2 . \quad (3.11)$$

In the case where the point H is located on the diameter $B\Delta$, and the opposite hyperbolas A, X and B, Δ are determined by equations (1.46) and (1.96) in the coordinate system whose axes Ox and Oy are the diameters $B\Delta$ and AX, the abscissas of the points Γ, H, K are equal to -a, x, a and the ordinates of the points Λ, H, N are equal to $(b/a)(a^2+x^2)^{1/2}$, 0 , $-(b/a)(a^2+x^2)^{1/2}$ Therefore in this case ΓH = a+x, HK = a - x, $\Lambda H = HN = (b/a)(a^2+x^2)^{1/2}$, $\Gamma H2+HK2 = (a+x)^2 + (a-x)^2 = 2(a^2+x^2)$, $\Lambda H^2+HN^2 = 2(b/a)(a^2+x^2)$ and both parts of equality (3.11) are equal to b^2/a^2 .

29. In Prop. III.29 the same conjugate opposite hyperbolas A, Γ and B,Δ, as in Prop. III.28, are considered. Through the center E of these hyperbolas their asymptotes $E\Xi$ and EO meeting the line ΛN at Ξ and O are drawn.

The assertion of this proposition can be expressed by the equality

$$(\Xi\Theta^2+\Theta O^2 +2AE^2)/(H\Theta^2+\Theta K^2) = B\Delta^2/A\Gamma^2. \quad (3.12)$$

In the case where the point Θ is situated on the diameter $B\Delta$ in the same coordinate system as in Prop. III.28, $H\Theta$ = a+x, ΘK = a-x, $\Xi\Theta = HO$ =(b/a)x, AE = b. Therefore both sides of equality (3.12) are equal to $b2/a2$.

Propositions III.30 - III.40 on poles and polars

30. In Prop. III.30 a hyperbola $AB\Gamma$ with the center H and the asymptotes EH and $H\Theta$ is considered. From A and Γ of this hyperbola the tangent straight lines $A\Delta$ and $\Gamma\Delta$ are drawn. The line $A\Gamma$ is joined. Through the point Δ the straight line $\Delta K\Lambda$ parallel to the asymptote HE is drawn, it meets the hyperbola at K and the line $A\Gamma$ at Λ.

Apollonius proves that $\Delta K = K\Lambda$.

The point Δ is the pole of the straight line $A\Gamma$.

In Prop. I.36 Apollonius proved that for each straight line drawn from the pole Δ, this pole and the point of meeting of this line with its polar $A\Gamma$ harmonically divide the points of intersection of this line with the hyperbola. The fact that the line $\Delta K\Lambda$ is parallel to an asymptote of the hyperbola implies that the point of intersection of this line with the hyperbola coincides with the point at infinity of this line. Therefore the points Δ and Λ harmonically divide the point K and a point at infinity, hence the equality $\Delta K = K\Lambda$ follows.

31. Prop. III.31 is the analogue of Prop. III.30 for opposite hyperbolas.

32. The parallelism of straight lines tangent to an ellipse at the ends of its diameter implies that the poles of all diameters of an ellipse are points at infinity. Therefore the line at infinity can be regarded as the polar of the center of an ellipse.

The asymptotes of a hyperbola can be regarded as the straight lines tangent to the hyperbola drawn from its center, since the asymptotes touch the hyperbola at its points at infinity. Therefore the straight line at infinity can also be regarded as the polar of the center of a hyperbola, and the points at infinity can be regarded as the poles of diameters of a hyperbola and of opposite hyperbolas.

33. In Note 80 on Book 1 we saw that polar of the points M_o with coordinates x_o, y_o with respect to the conic (1.54) is the straight line (1.77). We also saw that if the point M_o is a point of the conic (1.54), the straight line (1.77) is the straight line tangent to this conic at the point M_o.

Equation (1.54) determines a conic if the determinant

$$\Delta = \begin{vmatrix} A & B & \Delta \\ B & X & E \\ \Delta & E & \Phi \end{vmatrix} \quad (3.13)$$

is not equal to 0. Besides real conics, equation (1.54) can also determine imaginary ellipses. If the determinant (3.13) is equal to 0, equation (1.54) determines a pair of real, imaginary conjugate or coinciding straight lines.

The left hand side of equation (1.77) is symmetric with respect to coordinates x, y and x_o , y_o of the points M and M_o. Therefore if the point M_o describes a straight line, its polar rotates around the pole of this straight line.

In the case where the conic (1.54) is parabola (0.3), ellipse (1.45), or hyperbola (1.46), equation (1.77) has the forms: for a parabola

$$y_o y = p(x + x_o) , \quad (3.14)$$

for an ellipse

$$x_o x/a^2 + y_o y/b^2 = 1 \quad , \quad (3.15)$$

and for a hyperbola

$$x_o x/a^2 - y_o y/b^2 = 1 \quad . \quad (3.16)$$

By means of equations (3.14), (3.15) and (3.16) it is easy to check that if the point Δ can be obtained from the point E by the inversion with respect to the conic, and the point Δ is the pole of the straight line ΑΓ, the point E is the pole of the straight line ZH passing through Δ and parallel to ΑΓ

34. In Prop. III.32, a hyperbola ΑΒΓ with the center Δ and the asymptote ΔE is considered. From A and Γ the straight lines AH and ΓH tangent to the hyperbola are drawn, the line HΔ is joined. This line contains a diameter, the point

B, and the latus transversum BΘ of the hyperbola and meets the chord ΑΓ at Z bisecting this chord. From H the straight line HK parallel to ΑΓ and to the straight line BE tangent to the hyperbola is drawn. Apollonius draws the straight line ZΛK parallel to the asymptote ΔE. The line ZΛK meets the hyperbola at Λ. Apollonius proves the equality ZΛ = ΛK.

The line ΑΓ is the polar of the point H, and the line HK is the polar of the point Z. Therefore the points Z and K harmonically divide the points of intersection of the line ΦK with the hyperbola. The equality ZΛ = ΛK follows from the fact that the first of these points of intersection is the point Λ, and the second of these points is the point at infinity of the asymptote ΔE.

The point H is an exterior point of the hyperbola ΑΒΓ, and the point Z is an interior point of this hyperbola. In this proposition Apollonius first considers the polar of an interior point of a conic.

35. Prop. III.33 is the analogue of Prop. III.32 for opposite hyperbolas.

36. In Prop. III.34, a hyperbola AB with the center Δ and the asymptotes XΔ and ΔE is considered. From a point X of the asymptote XΔ the straight line XBE tangent to the hyperbola is drawn. Through the point B the straight line ΓBH parallel to the asymptote XΔ is drawn. From the point X the straight line XAH parallel to the asymptote ΔE is drawn. Apollonius proves the equality XA = AH.

The point X is the pole of the line BH joining the point B of the hyperbola with the point at infinity of the asymptote XΔ. Since the line XAH is parallel to the asymptote ΔE, it meets the hyperbola at the point at infinity of this asymptote. The equality XA = AH follows from the fact that the points X and H harmonically divide the points of intersection of the line XH with the hyperbola, one of these points of intersection is the point A, and the second of them is the point at infinity of the asymptote ΔE.

37. In Prop. III.35, a hyperbola AB with the asymptotes XΔ and ΔE is considered. Through a point B of the hyperbola the straight line ΦBΛ parallel to the asymptote XΔ, and from a point X the straight line XAΛΓ are drawn. The line XAΛΓ meets the hyperbola at the points A and Γ.

Apollonius proves the proportion

$$\Gamma X/XA = \Gamma\Lambda/\Lambda A. \quad (3.17)$$

The point X is the pole of the straight line ΦBΛ joining the point B of contact of the hyperbola with the line XB and the point at infinity of asymptote XΔ.

Therefore, equality (3.17) follows from the fact that the points Χ and Λ harmonically divide the points Α and Γ of the hyperbola.

38. Prop. III.36 is the analogue of Prop. III.35 for opposite hyperbolas.

39. In Prop. III.37, a conic ΑΒ is considered. From the points Α and Β the tangent straight lines ΑΧ and ΧΒ to the conic are drawn, and ΑΒ is joined. Through the point Χ the straight line ΧΔΕΓ meeting the conic at the point Δ and Γ and the straight line ΑΒ at the point Ε is drawn.

Apollonius proves the proportion

$$\Gamma X/X\Delta = \Gamma E/E\Delta \,. \quad (3.18)$$

The point Χ is the pole of the straight line ΑΒ. Therefore, equality (3.18) follows from the fact that the points Χ and Ε harmonically divide the points Δ and Γ of the conic.

In Prop. I.35 Apollonius considered a special case of this proposition where the conic is a parabola and the straight line ΧΔΕΓ is a diameter of this parabola. This diameter meets the chord ΑΒ at the point Ε and the parabola at the point Δ and at the point Γ at infinity. This last point is the common point of all parallel diameters of the parabola, this point at infinity is the point of tangency of the parabola and the line at infinity. Therefore the points Χ and Ε harmonically divide the point Δ and a point at infinity. Hence the equality $X\Delta = \Delta E$ follows.

40. In Prop. III.38 the same conic ΑΒ, as in Prop. III.37, with the tangent straight lines ΑΧ and ΒΧ and the chord ΑΒ is considered. Through the point Χ the straight line ΧΟ parallel to the line ΑΒ and the diameter ΧΕ are drawn. This diameter bisects the chord ΑΒ at the point Ε. Through the point Ε the straight line ΓΕΔΟ meeting the conic at the points Δ and Γ is drawn. Apollonius proves the proportion

$$\Gamma O/O\Delta = \Gamma E/E\Delta \,. \quad (3.19)$$

The point Χ is the pole of the line ΑΒ, the point Ε is the pole of the line ΧΟ. Therefore the points Ο and Ε harmonically divide the points Δ and Γ, hence equality (3.19) holds.

In this proposition Apollonius considers the polar of an interior point Ε of a conic.

41. Prop. III.39 is the analogue of Prop. III.37 for opposite hyperbolas.

42. Prop. III.40 is the analogue of Prop. III.38 for opposite hyperbolas, but in this proposition both poles considered by Apollonius are exterior points of the hyperbolas.

Propositions III.41 - III.44 on drawing tangent straight lines to a conic by means of projective correspondence between two straight lines

43. In Prop. III.41 the parabola ΑΒΓ with tangent straight lines ΑΔΕ, ΕΗΧ, and ΔΒΗ is considered.
Apollonius proves that these lines are cut in the same ratio, that is

$$\Gamma H/HE = E\Delta/\Delta A = HB/B\Delta. \qquad (3.20)$$

The tangent straight line DBH cuts off from the straight lines ΕΑ and ΕΓ the segments z = ΕΔ and z' = ΕΘ. The point Δ divides the segment ΕΑ in the same ratio as the point Θ divides the segment ΕΗ. Therefore if we denote ΕΓ = kΕΑ, segments z and z' are connected by the formula

$$z' = kz . \qquad (3.21)$$

The correspondence (3.21) is a particular case of projective correspondence (1.64).

Prop. III.41 shows that straight lines tangent to a parabola join points of two fixed tangent lines to this parabola connected by a projective correspondence.

The problems analogous to the problem of Prop. III.41 Apollonius solved in his treatise Cutting off of a Ratio.

44. In Prop. III.42 a hyperbola, an ellipse, or opposite hyperbolas with latus transversum ΑΒ = 2a and latus rectum 2p is considered. From the points Α and Β the straight lines ΑΓ and ΒΔ parallel to ordinates are drawn. Through an arbitrary point Ε of the conic the tangent straight line ΓΕΔ to the conic is drawn. Apollonius proves the equality

$$A\Gamma . B\Delta = ap . \qquad (3.22)$$

In the cases of an ellipse and opposite hyperbolas the lines ΑΓ and ΒΔ are tangent to the conics. The product ap is equal to b^2

In the case of the ellipse (1.45) the equations of the lines ΑΓ and ΒΔ are $x = -a$ and $x = a$. If coordinates of the point Ε are x_o and y_o the equation of the line ΓΕΔ has the form (3.15) and the ordinates of the points Γ and Δ are

$$y_1 = (b^2/y_o)(1 + x_o/a)\ ,\ \ y_2 = (b^2/y_o)(1 - x_o/a)\ . \quad (3.23)$$

Therefore

$$y_1\, y_2 = (b^2/y_o)^2\, y_o^2\, /b^2 = b^2\ . \quad (3.24)$$

For a hyperbola and opposite hyperbolas equality (3.22) can be proven analogously.

Equality (3.22) implies the formula

$$y_2 = b^2/y_1\ . \quad (3.25)$$

The correspondence (3.25) is also a particular case of projective correspondence (1.64).

Prop. III.42 shows that straight lines tangent to a hyperbola, an ellipse, or opposite hyperbolas join points of two fixed tangent straight lines ΑΓ and ΒΔ to a conic connected by a projective correspondence.

45. In Prop. III.43 the hyperbola ΑΒ with the center Δ, the asymptotes ΓΔ and ΔΕ, and axis ΒΔ is considered. Through the vertex Β and an arbitrary point Α of the hyperbola tangent lines ΖΒΗ and ΓΑΘ meeting the asymptotes at the points Η, Θ Γ, and Ζ are drawn. Apollonius proves the equality

$$\mathrm{Z}\Delta.\Delta\mathrm{H} = \Gamma\Delta.\Delta\mathrm{Z}. \quad (3.26)$$

If we denote ΓΔ = x, ΔΘ = y, equality (3.26) can be rewritten in the form (2.2). This equality, like the same equality in Prop. II.12, can be proven by means of a hyperbolic turn (2.3) mapping the hyperbola and its asymptotes to themselves.

Equation (2.2) implies the correspondence $y = k/x$ which is also the particular case of the projective correspondence (1.64).

Prop. III.43 shows that straight lines tangent to a hyperbola join points of two asymptotes connected by a projective correspondence.

The problems analogous to the problems of Prop. III.42 and III.43 Apollonius solved in the treatise Cutting off an Area.

46. In Prop. III.44 a hyperbola or opposite hyperbolas ΑΒ with the center Δ and the asymptotes ΓΔ and ΔΕ is considered. From the points Α and Β the tangent straight lines ΓΑΖΗ and ΕΒΖΘ meeting at Ζ are drawn. These lines meet the asymptotes ΓΔ and ΔΕ, respectively, at the points Η, Γ and Θ, Ε. The lines ΑΒ, ΗΘ, and ΓΕ are drawn.

Apollonius proves that these three lines are parallel.

Prop. III.43 implies that three triangles ΖΑΒ, ΦΖΗΘ and ΖΓΕ are similar, therefore the three mentioned lines are parallel.

Propositions III.45 - III.52 on foci and directrices of conics

47. In the propositions on foci and directrices only systems of rectangular coordinates whose axes 0x and Oy are axes of conics are considered, and all latera transversa and recta, second diameters and eccentricities of conics correspond to axes of these conics.

In Prop. III.45 a hyperbola, an ellipse, or opposite hyperbolas with the axis ΑΒ is considered, at the vertices Α and Β the straight lines ΑΓ and ΒΔ are drawn at right angles to the axis ΑΒ. At the points Α and Β of this axis inside the conics two rectangular planes with the area b^2 are applied. The areas of these planes are equal to the quarter of the eidos corresponding to the axis. The horizontal sides of these planes are denoted by ΑΖ and ΒΗ, the vertical sides of these plane are equal to the segment ΖΒ and ΑΗ, respectively. In the case of ellipses the points Ζ and Η are situated between the points Α and Β, in the case of hyperbolas and opposite hyperbolas the points A and B are situated between the points Ζ and Η. The points Ζ and Η in both cases satisfy to the conditions

$$\mathrm{AZ} = \mathrm{HB}\ ,\ \mathrm{AH} = \mathrm{ZB}\ , \quad (3.27)$$

$$\mathrm{AZ.ZB} = \mathrm{AH.HB} = b^2\ . \quad (3.28)$$

Apollonius calls the points Ζ and Η τα εκ τας παραβολας γεναθεντα σαμεια - “the points of beginnings of applications”. In modern geometry these points are called “foci” of ellipses and hyperbolas.

In the case of ellipse (1.45) and hyperbola (1.46) the abscissas of the points Α and Β are -a and a, and if the abscissa of the point Ζ or Η is x, in the case of the ellipse (1.45)the distances ΑΖ = HB is equal to a - |x| and the distances ΑΗ = ΖΒ is equal to a +|x|, and therefore ΑΖ.ΗΒ =ΑΗ.ΖΒ =b^2=

a^2-x^2, hence $x^2=a^2-b^2$, and hyperbola (1.46)the distances AZ = HB is equal to |x| -a and the distances AH = ZB is equal to a +|x|, and therefore AZ.HB =AH.ZB $=b^2=x^2-a^2$, hence $x^2=a^2+b^2$.

Therefore in both cases the distances between the center of the conic and the point Z or H is equal to aε, where ε is the eccentricity of the ellipse or the hyperbola.

In Prop. III.45 through an arbitrary point E of the conic the tangent straight lines ΓEΔ are drawn. Apollonius proves that for all points E both angles ΓZΔ and ΓHΔ are right.

The proof of this assertion is based on the similarity of the rectangular triangles ΓAZ, BZΔ, ΔBH, and HAΓ.

In the case of ellipse (1.45) and hyperbola (1.46) the ordinates of the points of these conics with the same abscissas as the points Z and H are equal to $p = b^2/a$.

48. Apollonius does not define focus of a parabola. This focus can be defined analogously to the definition of foci for an ellipse and a hyperbola.

If a parabola with the vertex A and the axis AB is determined by equation (0.3), the equation of the straight line ΓEΔ tangent to it at an arbitrary point E of this parabola with coordinates x_o and y_o has the form (3.14). The straight line tangent to the parabola at the point A has the equation x = 0. The ordinate of the point Γ of meeting of this tangent line is $y = px_o/y_o$. Let us prove that on the axis AB of this parabola there is such point Z that for all points E the straight line ΓZ is perpendicular to the line ΓEΔ. The angle coefficient of the line ΓEΔ is equal to $k_1= p/y_o$, the ordinate of the point Γ is equal to $y=px_0/y_o$, and if the abscissa of the point Z is equal to x, the angle coefficient of the line ΓZ is equal to $k_2 = -px_0/y_0x$. The condition of orthogonality of the lines ΓEΔ and ΓZ has the form $k_1k_2 = -1$ that is $p^2x_0/y_0^2x = 1$, hence $x = p^2x_0/y_0^2$. Since E is a point of the parabola, its coordinates satisfy to equation (0.3), $x = p^2x_0/2px_0 = p/2$. The point Z with the abscissa p/2 is called the focus of the parabola (03).

If Δ is the point as infinity of the line ΓEΔ, the line ΓZ is perpendicular to the line ZΔ parallel to ΓEΔ, and the angle ΓZΔ is right.

The ordinate of the point of the parabola (03) with the abscissa x = p/2 is equal to p.

The point at infinity of the axis AB can be regarded as the second focus of the parabola.

49. In Prop. III.46 the same ellipse and opposite hyperbolas as in Prop. III.45 are considered. Apollonius proves that the angle AGZ is equal to the angle ΔΓH and the angle ΓΔZ is equal to the angle BΔH.

The proof is based on Prop. III.45.

50. The analogue of Prop. III.46 also holds for a parabola. If we denote the points at infinity of the axis of the parabola by B and of the straight line ΓE by Δ, the angle ΑΓΖ is equal to the angle between the tangent line ΓΕΔ and the axis ΑΒ.

51. In Prop. III.47 the same ellipse and opposite hyperbolas, as in the Prop. III.45 and III.46, are considered, and the lines ΓΗ and ΔΖ meeting at the point Θ and the line ΘΕ are drawn.

Apollonius proves that the line ΘΕ is perpendicular to the line ΓΕΔ.

In modern geometry the line intersecting a conic and perpendicular at its point of contact to the tangent line is called "normal" to the conic at this point. Therefore the line ΘΕ is the normal to the conic at its point Ε.

The proof is based on Prop. III.45 and III.46.

52. The analogue of Prop. III.47 also holds for a parabola. If the diameter ΓΒ and the straight line ΖΔ parallel to the tangent line ΓΕΔ meet at the point Θ, the line ΘΕ is a normal to the parabola. Since the angle coefficient of the tangent line ΓΕ is equal to p/y_0, the angle coefficient of the normal ΕΘ is equal to $-y_0/p$, hence we obtain that the distance between the foot of the perpendicular dropped from the point Ε to the axis and the point of intersection of the axis with the normal ΕΘ, called "subnormal" of the point Ε of the parabola, is equal to p.

53. In Prop. III.48 the same ellipse and opposite hyperbolas, as in Prop. III.45 , III.46, and III.47, are considered, and the straight lines ΖΕ and ΗΕ are drawn. Apollonius proves that the angle ΖΕΘ equal to the angle ΗΕΘ and the angle ΓΕΧ is equal to the angle ΗΕΔ.

In the case of the ellipse this proposition means that light rays issuing from one focus reflected from the ellipse will fall into other focus. Since light rays have a heat, when they gather at the other focus, they will bring to this focus so much heat that if at this point a combustible substance is situated, it will burn. This fact explains the term "focus" - the Latin word for hearth or place of the fire.

In the case of opposite hyperbolas this Proposition means that light rays issuing from one focus reflected from one of hyperbolas so, that the continuations of reflected rays will gather at the other focus of opposite hyperbolas.

The proof is based on Prop. III.45 - III.47.

54.The analogue of Prop. III.48 also holds for a parabola ΑΕ with the axis ΑΒ. In this case the role of second focus Η is played by the point at infinity of the axis ΑΒ, and the angle ΓΕΖ is equal to the angle between the continuation of the line ΓE and the diameter drawn from the point E parallel to AB. Therefore light rays parallel to AB reflected from the parabola will gather at its focus Z .

This fact is the reason why medieval Arabic mathematicians called parabola "burning mirror" and its focus called "point of ignition". The last term was used in the Book of Optics by Ibn al-Haytham and in Optics by the Polish physicist of 13th c. Witelo written under the influence of Ibn al-Haytham. The term "focus" was introduced by J.Kepler in his Optical Part of Astronomy regarded by him as "supplement to Witelo". In this book Kepler considered the focus at infinity of a parabola called by him "blind focus".

Parabolic burning mirror was invented by Archimedes who used it during the defense of Syracuse in 214-212 B.C. where this city was besieged by Romans. Archimedes placed soldiers with brilliant copper shields so that their shields formed a part of the surface of a paraboloid of revolution whose axis was directed to the sum, and the focus was situated on Roman ship.

and directed the axis of this paraboloid to the sunrise. The focus of this paraboloid was located on a Roman ship. Since Archimedes was killed by Romans after the capture of Syracuse, he could not describe this action, and Apollonius could not know about it.

In 1968 E.Stamatis organized the burning of a wooden vessel in the bay of Thessalonika by the method of Archimedes. The experiment made by the engineer Joannis Sakas showed the effectiveness of Archimedes' method.

The surface of a paraboloid of revolution is formed by rotation of a parabola around its axis, the focus and the vertex of a parabola are the focus and the vertex of the paraboloid. Therefore the focus of the paraboloid is situated on its axis, and to burn a ship it is necessary for the vertex of the paraboloid, the ship and Sun to be on one straight line. It is possible only during sunrise or sunset. Siracuse is on the Eastern coast of Sicily and Thesalonika is on the Eastern coast of Greek Macedonia, and in both towns Sun rises over the sea. Therefore the burning of Roman ship by Archimedes and of wooden ship by Sakas were possible.

G.J.Toomer in the paper [Too] wrote that in the monastery at Bobbio a Greek manuscript with quotation of an Apollonius' treatise On Burning Mirrors (Περι πυριος) was found, and on the basis of this paper H.Flaumenhaft in his note [Ap5. pp.XIX] also wrote that Apollonius was an author of a physical treatise. But later Toomer has discovered that the treatise ascribed in the Bobbio manuscript to Apollonius in fact was written by Diocles in 1st c. B.C. The treatise by Diocles extant only in medieval Arabic translation was published by Toomer with his English translation [Di] and in his introduction to edition [Ap7]

Toomer wrote that this treatise was written not by Apollonius, but by Diocles. In this treatise, parabolic burning mirrors were described; moreover a parabola is called by the Archimedes' term "section of right-angled cone" and it is written that the problem of burning mirrors was studied by Archimedes' friend Dositheus to whom Archimedes dedicated some his works.

55. In Prop. III.49 the same ellipse and opposite hyperbolas, as in Prop. III.45 - III.48, are considered. From the point Η onto tangent straight line ΓΕΔ the perpendicular ΗΘ is dropped and the straight lines ΑΘ and ΒΘ are drawn.

Apollonius proves that the angle ΑΘΒ is right.

The proof is based on Prop. III.45.

56. The theorem analogous to Prop. III.49 also holds for a parabola. If from the point Ζ a perpendicular is dropped onto the tangent straight line ΓΕΔ, the foot of this perpendicular coincides with the point Γ, and the angle ΑΓΔ is right.

57. In Prop. III.50 the same ellipse and opposite hyperbolas as in Prop. III.45 - III.49 are considered. The continuations of the axis ΑΒ and of the straight line ΓΔ meet at the point Κ, the straight line ΕΖ is drawn, from the center Θ the straight line ΘΛ parallel to the line ΕΖ is drawn, this line meets the line ΓΕΔ at Λ.

Apollonius proves that the segment ΘΛ is equal to the half of the latus transversum 2a of the section.

The proof is based on Prop. III.45.

58. In Prop. III.50 the point K is the pole of the straight line $x = x_o$, therefore the segment ΘΚ is equal to a^2/x_o, and segment ΚΖ is equal to $| a^2/x_o - a\varepsilon$.

Since the triangles ΚΖΕ and ΚΘΛ are similar, the proportion

$$ZE/\Theta\Lambda = KZ/K\Theta \quad (3.29)$$

holds. Therefore for the focus Z of the ellipse we obtain that

$$ZE = KZ.\Theta\Lambda/K\Theta = (a^2/x_o - a\varepsilon)\, a/(a^2/x_o) = a - x_o\varepsilon . \quad (3.30)$$

Analogously, for the focus Η of the ellipse we obtain the equality

$$HE = a + x_o\varepsilon \quad . \quad (3.31)$$

Analogously, we find that for the opposite hyperbolas in the case where $x_0 > 0$ obtain the equalities

$$ZE = x_0\varepsilon + a , \quad (3.32)$$
$$HE = x_0\varepsilon - a , \quad (3.33)$$

in the case where $x_0 < 0$ the segment ZE is equal to the product of the right hand part of equality (3.33) by -1 and the segment HE is equal to the product of the right hand part of equality (3.32) by -1.

The segments ZE and HE in modern geometry are called "focal radii-vectors of the point E".

59. Equalities (3.30) and (3.31) for ellipse (1.45) imply that

$$ZE = a - x_0\varepsilon = \varepsilon(a/\varepsilon - x_0), \quad (3.34)$$
$$HE = a + x_0\varepsilon = \varepsilon(a/\varepsilon + x_0). \quad (3.35)$$

Equalities (3.32) and (3.33) for opposite hyperbolas (1.46) imply that

$$ZE = \varepsilon x_0 + a = \varepsilon(x_0 + a/\varepsilon), \quad (3.36)$$
$$HE = \varepsilon x_0 - a = \varepsilon(x_0 - a/\varepsilon), \quad (3.37)$$

and in the case where $x_0 < 0$ the segment ZE is equal to the product of the right hand part of equality (3.37) by -1 ,and the segment HE is equal to the product of the right hand part of equality (3.36) by -1.

In modern geometry the straight lines $x_0 = a/\varepsilon$ and $x_0 = -a/\varepsilon$ are called "directrices of the ellipse and the opposite hyperbolas".

Magnitudes $a/\varepsilon - x_0$ and $a/\varepsilon + x_0$, or the ellipse, magnitude $x_0 + a/\varepsilon$, $x_0 - a/\varepsilon$ for the opposite hyperbolas with $x > 0$ and magnitudes $a/\varepsilon - x0$ and $-x0 - a/\varepsilon$ for the opposite hyperbolas with $x0 < 0$ are equal to the distamces from the point E of the conic to directrices.

Therefore equalities (3.34) , (3.35) , (3,36), and (3.37) show that ellipses opposite hyperbolas are the loci of points whose distances from the foci and directrices are proportional, and if the feet of perpendiculars dropped from E onto directrices are the points Y and W, these proportionalities can be written in the form

$$ZE = \varepsilon E\Xi, \; HE = \varepsilon E\Psi . \quad (3.38)$$

The coefficients of these proportionalities are equal to the eccentricities of ellipses and hyperbolas. Since for ellipses $\varepsilon < 1$ and for hyperbolas $\varepsilon > 1$, the distances of points of ellipses from the foci are less than their distances from the directrices, and distances of points of hyperbolas from the foci are greater than their distances from the directrices.

The first equality (3.38) also holds for parabola (0.3), since for it the role of the directrix is played by the line $x = -p/2$ and

$$ZE^2 = (x_o - p/2)^2 + y_o^2 = (x_o - p/2)^2 + 2px_o = (x_o + p/2)^2 = E\Psi^2 . \quad (3.39)$$

Therefore parabola (0.3) is the locus of point equidistant from the focus with the abscissa $p/2$ and from the directrix determined by equation $x = -p/2$

The directrices of ellipses, hyperbolas, and parabolas are polars of foci of these conics, since if we put in equation (3.14) of polar with respect to parabola (0.3) values $x_o = p/2$, $y_o = 0$, we will obtain the equation $x = - p/2$, and if we put in equations (3.15) and (3.16) of polars with respect to ellipse (1.45) and hyperbola (1.46) values $x_o = +a\varepsilon$, $y_o = 0$, we will obtain equations $x = +a/\varepsilon$.

Although the existence of directrices of conics follows from Prop.III.50 of Conics, Apollonius never mentioned directrices.

Foci and directrices of ellipses, hyperbolas and parabolas were mentioned in Mathematical Collection by Pappus in the survey of Euclid's work Loci on Surfaces (Τοποι προς επιφανειαις).

Some historians of mathematics believe that foci and directrices were already considered in this Euclid's work, which also was never mentioned in Conics. No doubt that in fact foci and directrices were mentioned by Pappus who knew Prop. III.50 of Conics.

Note that Germinal Pierre Dandelin (1794-1847) proved that the foci and the directrices of conics can be obtained as follows. If a conic is cut off from the surface of a right circular cone, Dandelin inscribed in this surface two spheres tangent to it along the circumferences of circles and tangent to the cutting plane at a point. These points of contact are the foci and the lines of intersection of the cutting plane with the planes of circles are the directrices.

60. In Prop. III.51 hyperbola and opposite hyperbolas (1.46) with transverse axis 2a and the foci Δ and E are considered.

Apollonius proves that for an arbitrary point Z of the opposite hyperbolas the equality

$$| EZ - \Delta Z | = 2a \qquad (3.40)$$

holds. Equality (3.40) follows from equalities (3.32) and (3.33).

61. In Prop. III.52 ellipse (1.45) with the major axis 2a, and the foci Γ and Δ is considered.

Apollonius proves that for an arbitrary point E of this ellipse the equality

$$\Gamma E + E\Delta = 2a \qquad (3.41)$$

holds. Equality (3.41) follows from equalities (3.30) and (3.31).

Equality (3.41) is the base for the “gardener’s method” for construction of elliptic flower-beds.

Propositions III.53 - III.56 on construction of conics by means of projective correspondence between two plane pencils of straight lines

62. In Prop. III.53 an ellipse or opposite hyperbolas ABΓ with the latus transversum AΓ = 2a and the latus rectum 2p are considered. From the vertices A and Γ the tangent straight lines AΔ and ΓE and the straight lines ABE and ΓBΔ through an arbitrary point B of the conic are drawn.

Apollonius proves the equality

$$A\Delta . EH = (2a)(2p) \; . \qquad (3.42)$$

The proof of this proposition is based on Prop. I.12 and I.13.
The symmetry of equation (1.5) with respect to coordinates x_i and coefficients u_i shows that in projective plane the duality principle holds. This principle means that for every theorem of projective geometry in this plane there is the dual theorem differing from the first one by replacing words “point” by “straight line” and vice versa and expressions “a point on a straight line” by “a straight line through a point” and vice versa.

In the duality principle points on a straight line correspond to straight lines in a plane pencil and points of a conic correspond to straight lines tangent to a conic. The coefficients u_i of equations of straight lines tangent to conic (1.75) satisfy to equation

$$\Sigma_i \Sigma_j \, (B^{ij}) \, u_i u_j = 0, \qquad (3.43)$$

where the matrix (B^{ij}) is inverse to the matrix (A_{ij}). This fact follows from equality (1.74).

For quadruples of straight lines of a plane pencil, like for quadruples of points of a straight line, cross-ratios can be determined. If the straight lines p, q, r, s meet an arbitrary straight line at the points Π, Θ, P, Σ, the cross-ratio of the lines p, q, r, s is determined as the number equal to the cross-ratio of the points Π, Θ, Ρ, Σ.

Therefore between two plane pencils of straight lines a projective correspondence can be established.

The affine coordinates of the points Π, Θ, Ρ, Σ on their straight line can be regarded as affine coordinates of the straight lines p, q, r, s of a pencil.

The projective correspondence between two plane pencils of straight lines can be determined by transformation (1.64) where x and x' are affine coordinates of straight lines of two pencils.

In Prop. III.53 there are two plane pencils of straight lines with centers Α and Χ. The affine coordinates of the lines of the first pencil are determined by points of the line ΧΕ, the affine coordinates of the lines of the second pencil are determined by points of the line ΑΔ.

The correlation (3.43) determines a projective correspondence between pencils of lines with centers Α and Χ. Therefore in this proposition the conic is obtained as a locus of points of meeting of corresponding straight lines of two pencils connected by a projective correspondence.

This construction is a particular case of the construction of a conic section according to the theorem of Jacob Steiner (1796-1863) on the generation of conics by means of two projective plane pencils of straight lines.

The constructions of straight lines tangent to conics in Prop. III.41 - III.43 are particular cases of the constructions by means of the theorem dual to Steiner theorem.

63. In Prop. III.54 a conic ΑΒΓ is considered. From the points Α and Γ the straight lines ΑΔ and ΓΔ tangent to it are drawn. The line ΑΓ is joined and bisected at the point Ε. The line ΔΒΖ is drawn, through the point Α the line ΑΗ parallel to the line ΓΔ is drawn, from the point Γ the line ΓΗ parallel to the line ΑΔ is drawn Through an arbitrary point Ζ of the conic the lines ΑΖ and ΓΖ are drawn and continued to the points Θ and Η. Apollonius proves that

$$\mathrm{AH.\Gamma\Theta} : \mathrm{A\Gamma}^2 = (\mathrm{EB}^2 : \mathrm{B\Delta}^2) \times (\mathrm{A\Delta.\Delta\Gamma} : \mathrm{AE}^2). \qquad (3.44)$$

The proof is based on Prop. II.29 and III.16.

The point Δ is the pole of the line ΑΓ. Since the chord ΑΓ is bisected at the point Ε, the line ΔΒΕ is a diameter of the conic.

In Prop. III.54 there are two plane pencils of straight lines with the centers Α and Γ. The affine coordinates of the lines of the first pencil coincide with the affine coordinates of points of the line ΓΘΗ, the affine coordinates of lines of the second pencil coincide with the affine coordinates of points of the line AH. Since lines AΓ, EB, BΔ, AΔ, ΔΓ, and AE are independent of the position of point Z, equality (3.44) determines a projective correspondence between pencils of lines with the centers A and Γ. Therefore in this proposition the conic is also obtained as a locus of points of meeting of corresponding straight lines of two plane pencils connected by a projective correspondence, that is as in a particular case of Steiner theorem

64. Prop. III.55 is the analogue of Prop. III.54 for opposite hyperbolas ABΓ and ΔEH with tangent straight lines at points A and Δ of both hyperbolas.

65. Prop. III.56 is the analogue of Prop. III.54 for opposite hyperbolas AB and ΓΔ with tangent straight lines at points A and B of first hyperbola.

COMMENTARY ON BOOK FOUR

Preface to Book 4

1. Book 4 and the following books of Conics were finished by Apollonius after the death of Eudemus of Pergamum. Apollonius sent these books to Attalus. Some historians of mathematics identify Attalus with one of three kings of Pergamum having this name. This opinion is impossible since Apollonius never names Attalus "king". E.Stamatis who identified Attalus with a king of Pergamum in his translation Conics inserted before the name of Attalus the word "[king] (βασιλεα)" [Ap11, vol. 3, p. 101].

Probably, Attalus to whom Apollonius sent books of Conics was a student of Eudemus and a comrade of Apollonius.

2. Conon of Samos was a well-known Alexandrian mathematician, on him see Introduction, C.

Nicoteles of Cyrena and Thrasydaeus are known only from this preface.

Propositions IV.1 - IV.23 on poles and polars

3. In Prop. IV.1 a conic $AB\Gamma$ with the tangent straight line ΔB at the point B is considered. From the point Δ the straight line $\Delta E\Gamma$ meeting the conic at the points E and Γ is drawn. On this line the point Z is found such that $\Gamma Z/ZE = \Gamma\Delta/\Delta E$. Apollonius proves that the straight line BZ meets the conic at such point A that the straight line ΔA is tangent to the conic.

This proposition is inverse to Prop. III.37, and is proved by the reduction to absurd. The line AB is the polar of the point Δ.

On the term "with the same ratio" (ομολογους) which Apollonius used in the definition of harmonic quadruples of points see Note. 66 on Book 1.

4. Prop. IV.2 is a particular case of Prop. IV.1 where the conic is a hyperbola, the point B of contact is between the points Γ and E and the point Δ is

within the angle between the asymptotes containing the hyperbola.

5. Prop. IV.3 is also a particular case of Prop. IV.1 where the conic is a hyperbola, the point Β of contact is not between the points Γ and Ε, and the point Δ is within the angle between the asymptotes containing the hyperbola.

6. Prop. IV.4 is also a particular case of Prop. IV.1 where the conic is a hyperbola, the point Β of contact is between the points Γ and Ε, and the point Δ is within the angle between the asymptotes adjacent to the angle containing the hyperbola.

7. Prop. IV.5 is also a particular case of Prop. IV.1 where the conic is a hyperbola, the point Δ is on an asymptote, and the line ΒΖ is parallel to this asymptote.

In this case, the point Α is at infinity.

8. Prop. IV.6 is a limit case for Prop. IV.1 where the conic is a hyperbola, the line ΔΕΖ is parallel to an asymptote of the hyperbola, the point Δ is within the angle between the asymptotes containing the hyperbola, and ΔΕ = ΕΖ.

In this case G is a point at infinity.

9. Prop. IV.7 is a limit case for Prop. IV.1 where the conic is a hyperbola, the line ΔΕΖ is parallel to an asymptote of the hyperbola, the point Δ is within the angle between the asymptotes adjacent to the angle containing the hyperbola, and ΔΕ = ΕΖ.

In this case Γ is a point at infinity.

10. Prop. IV.8 is a limit case for Prop. IV.1 where the conic is a hyperbola, the line ΔΕΖ is parallel to an asymptote of the hyperbola, the point Δ is on this asymptote, and ΔΕ =ΕΖ.

In this case Γ is a point at infinity.

11. In Prop. IV.9 a conic ΑΒ is considered. From the point Δ two straight lines ΔΕΦ and ΔΖΗ meeting the conic at the points Ε, Φ, Ζ, and Η are drawn. On the line ΔΕΦ the point Κ is taken so that ΔΕ/ΕΚ = ΔΦ/ΦΚ. On the line ΔΖΗ the point Λ is taken so that ΔΖ/ΖΛ = ΔΗ/ΗΛ.

Apollonius proves that the straight line ΚΛ meets the conic at such points Α and Β that the straight lines ΔΑ and ΔΒ are tangent to the conic.

This proposition is also inverse to Prop. III.37, and is proven by the reduction to absurd. The line ΑΒ is the polar of point Δ.

Apollonius calls harmonic quadruples Δ, Ε, Κ, Φ and Δ, Ζ, Λ, Η ομολογους .

12. Prop. IV.10 is a particular case of Prop. IV.9 where the conic is a hyperbola, the points of meeting of the conic with one of these straight lines are between the points of meeting of the conic with the other straight line, and the point Δ is within the angle between the asymptotes containing the hyperbola.

13. Prop. IV.11 is also a particular case of Prop. IV.9 where the conic is a hyperbola, the points of meeting of the conic with one of these straight lines is not between the points of meeting of the conic with other straight line, and the point Δ is within the angle between the asymptotes containing the hyperbola.

14. Prop. IV.12 is a particular case of Prop. IV.9 where the conic is a hyperbola, the points of meeting of the conic with one of the straight lines are between the points of meeting of the conic with the other of these straight lines, and the point Δ is within the angle between the asymptotes adjacent to the angle containing the hyperbola.

In this case the polar of the point Δ meets the given hyperbola and the hyperbola opposite to it.

15. Prop. IV.13 is a limit case of Prop. IV.9 where the conic is a hyperbola, the point Δ is on one of its asymptotes and the points of meeting of the hyperbola with one of the straight lines are between the points of meeting of the conic with the other of these straight lines.

In this case the polar of the point Δ is parallel to the asymptote passing through the point Δ.

16. Prop. IV.14 is also a limit case of Prop. IV.9 where the conic is a hyperbola, the point Δ is on one of its asymptotes, and the straight line ΔH is parallel to the other asymptote and meets the hyperbola at a single point H, and
$H\Lambda = \Delta H$

In this case Z is a point at infinity.

17. In Book 4 of Conics the important proposition in which the point Δ is within the angle between the asymptotes of the hyperbola AB, and from the point Δ the straight lines ΔA and ΔB parallel to the asymptotes are drawn and on these lines the segments AK and BΛ equal to the segments ΔA and ΔB are drawn, respectively, is absent. In this proposition the fact that the line KΛ meets the hyperbola at the points of contact of straight lines joining these points with the point Δ is proven. This proposition is a limit case of Prop. IV.9 where the second points of meeting of the lines ΔA and ΔB with the conic are at infinity. The analogue of this proposition for the case where the point Δ is within the angle adjacent to the angle containing the hyperbola, and the lines ΔA and ΔB parallel to the asymptotes meet both opposite hyperbolas, is Prop. IV.23.

18. Prop. IV.15 is the analogue of Prop. IV.1 for opposite hyperbolas, where the point Δ is within the angle between the asymptotes containing one of opposite hyperbolas.

Here Apollonius also calls the segments of harmonic quadruples ομολογους

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19. Prop. IV.16 is the analogue of Prop. IV.1 for opposite hyperbolas where the point Δ is within the angle between the asymptotes adjacent to the angle containing one of opposite hyperbolas.

20. Prop. IV.17 is the analogue of Prop. IV.1 for opposite hyperbolas where the point Δ is on one asymptote.

21. Prop. IV.18 is the analogue of Prop. IV.9 for opposite hyperbolas where the point Δ is within the angle between the asymptotes containing one of opposite hyperbolas.

22. Prop. IV.19 is the analogue of Prop. IV.9 for opposite hyperbolas where the point Δ is within the angle between the asymptotes adjacent to the angle containing one of opposite hyperbolas.

23. Prop. IV.20 is the analogue of Prop. IV.9 for opposite hyperbolas where the point Δ is on one asymptote.

24. Prop. IV.21 is the analogue of Prop. IV.14 for opposite hyperbolas.

25. Prop. IV.22 is the limit case of Prop. IV.19 where the line ΔB is parallel to an asymptote of opposite hyperbolas, and ΔB = BK.

In this case the line ΔB is tangent to opposite hyperbolas at a point at infinity.

26. Prop. IV.23 is the limit case of Prop. IV.19 where the lines ΔA and ΔB are parallel to both asymptotes of opposite hyperbolas, ΔA = AH and ΔB = BK.

In this case the lines ΔA and ΔB are straight lines tangent to opposite hyperbolas at their points at infinity.

Propositions IV.24 - IV.57 on intersections and tangencies of conics

27. In Prop. IV.24 Apollonius proves that two conics cannot have a common arc.

28. In Prop. IV.25 Apollonius proves that two conics cannot have more than four points of intersection.

29. If to regard the circumferences of circles as particular case of conics, then their equations can be written in the form

$$A(x^2 + y^2) + 2Dx + 2Ey + F = 0. \quad (4.1)$$

Equation (4.1) in projective coordinates has the form

$$A(x_1^2 + x_2^2) + 2Dx_1x_3 + 2Ex_2x_3 + Fx_3^2 = 0. \quad (4.2)$$

If the straight line at infinity is determined in projective coordinates by equation $x_3 = 0$, the intersection of this line with any circumference of circle (4.2) is determined by equation $x_1^2 + x_2^2 = 0$. This equation shows that all circumferences of circles meet the line at infinity at the same imaginary points. These points J.V.Poncelet called "cyclic points" of the projective plane. Therefore any two intersecting circumferences of circles have four common points - two real points and two imaginary cyclic points at infinity. Two circumferences of circles which have no common real points have four common imaginary points - two cyclic points and two imaginary conjugate points which, in general, are not cyclic ones.

In the case where these imaginary conjugate points are also cyclic points, two circles are concentric and the circumferences of these circles can be regarded as tangent at the cyclic points.

30. Since the distance d of a point M_o with coordinates x_o and y_o from the straight line

$$Ax + By + C = 0 \qquad (4.3)$$

is equal to

$$d = |(Ax_o + By_o + C)| / (A^2 + B^2)^{1/2}, \qquad (4.4)$$

equations (1.1) and (1.2) of loci with respect to 3 and 4 straight lines (see Note 4 on Book 1) have form (1.54), that is these loci are conics.

This solution of the problem on these loci was found by R. Descartes in his book Geometry.

If A, B, C, and D are points of a conic, we denote the straight lines AB, BC, CD, and DA, respectively, by l_1, l_2, l_3, and l_4. These points satisfy following conditions: for A $d_4 = d_1 = 0$, for B $d_1 = d_2 = 0$, for C $d_2 = d_3 = 0$, for D $d_3 = d_4 = 0$. The coordinates of all these points satisfy equation (1.2) for all values of coefficient k. Therefore points A, B, C, D are common points of all conics determined by condition (1.2). Hence we obtain that five points determine a conic passing though these points: four points determine loci with respect to four straight lines for any value of coefficient k in equation (1.2), and fifth point determines the value of k where (1.2) is the equation of the conic passing through 5 given points.

The assertion that a conic is determined by 5 points also follows from the fact that general equation (1.54) of a conic contains 6 coefficients determined up to non-zero multiplier. Therefore if these coefficients are divided by one of them, then equation (1.54) will become a linear equation with 5 unknown magnitudes, and if we put in this equation the coordinates of 5 given points, we will obtain the system of 5 linear equations with 5 unknown magnitudes. The solution of this system will determine all coefficients of the equation of the conic passing though 5 given points.

No doubt that Apollonius knew how to find a conic passing through five given points, since he knew the locus with respect to four lines. Apollonius does not disclose the solution of this problem apparently owing to a very great number of particular cases of this problem.

The conics passing through four given points are parameterized by one parameter k in equation (1.2). This 1-parameter family of conics is called a "pencil of conics".

The most important case of a pencil of conics is a pencil of circumferences of circles. Four points determining this pencil are two cyclic points and two real or imaginary points of intersection of all circumferences of a pencil. The centers of all circumferences of a pencil are on one straight line that is the axis of symmetry of non-cyclic common points of circumferences of the pencil.

A pencil of circumferences is called elliptic, hyperbolic, and parabolic when the non-cyclic common points of circumferences of the pencil are, respectively, real, imaginary conjugate, or coincide. The circumferences of an elliptic or a hyperbolic pencil cut off from the line of their centers, respectively, an elliptic or hyperbolic involution (see Note 82 on Book1). Parabolic pencil consists of tangent circumferences. Elliptic pencils do not contain circumferences of zero radius, hyperbolic pencils contain two such circumferences, parabolic pencil contains one such circumference.

Apollonius considered hyperbolic pencils of circumferences in his treatise Plane loci, where he defined circumferences of these pencils as the loci of points with constant ratios of distances from two fixed points.

31. In Prop. IV.26 Apollonius proves that if two conics are tangent at one point, they cannot meet at more than two other points.

This proposition shows that a point of contact is equivalent to two points of intersection.

32. In Prop. IV.27- IV.29 Apollonius proves that if two conics are tangent at two points, they cannot have other common point.

In Prop. IV.27 conics AMB and AHB tangent at points A and B are considered and the supposition that these conics meet at a point Z which is not between the points A and B is refuted.

The straight lines AΛ and BΛ tangent to both conics meet at the point Λ. The point Λ is the pole of the straight line AB. If these conics meet at a third point Z, the straight line ΛZ meets the conics AMB at the point M, and AHB at the point H, and the straight line AB at a point N. Prop. III.37 implies that for the conic AMB the proportion $ΛZ/ZN = ΛM/MN$ holds, and for the conic AHB the proportion $ΛZ/ZN = ΛH/HN$ holds. But the ratios $ΛM/MN$ and $ΛH/HN$ are not equal, therefore the conics cannot have a third common point.

Apollonius' words "as in the first diagram", "as in the second diagram", and "as in the third diagram" in Prop. IV.28 and IV.29 show that Apollonius originally regarded these three propositions as three parts of a single proposition.

33. In Prop. IV.28 the case where tangent lines AI and BΛ are parallel is considered. This case is possible only where conics AMB and AHB are ellipses or one of them is the circumference of a circle. In this case Λ is a point at infinity. Apollonius proves that the line AB is a diameter of both conics.

34. In Prop. IV.29 the supposition that the point Γ, in which the conics AMB and AHB meet, is between points A and B is refuted.

Here, besides the general method of solving this problem exposed in Prop. IV.27, Apollonius states another method: he bisects the line AB at the point Z and draws the straight line ZΛ. The point Λ is the pole of the line AB and corresponds to the point Z in the inversion with respect to both conics. Therefore the line ZΛ is a diameter of both conics.

If these conics meet at the point Γ, Apollonius draws from Γ the straight line ΓKHM parallel to AB and meeting ZΛ at the point K. Both segments ΓM and ΓH must be bisected at the point K. Since it is impossible, the conics AMB and AHB cannot have a third common point Γ.

35. In Prop. IV.30 Apollonius proves that two parabolas can be tangent only at one point.

In this proposition the existence of two parabolas AMB and AHB tangent at the points A and B is supposed. The straight lines AΛ and BΛ tangent to both parabolas meet at the point Λ. In both cases the point Λ is the pole of the straight line AB, and if the line AB is bisected by the point Z, the point Λ is obtained from point Z by the inversions with respect to both parabolas. The line ΛZ meets the parabola AMB at the point M and the parabola AHB at the point H. Prop. I.33 implies the equalities $ΛM = MZ$ and $ΛH = HZ$, that is both points M and H bisect the segment ΛZ. Since the segment ΛZ can be bisected only at one point, the supposition on the existence of two parabolas tangent at two points is impossible.

If two parabolas touch at a finite point, they also touch at a point at infinity, and therefore have a common diameter joining both points of tangency.

36. In Prop. IV.31 Apollonius proves that a parabola that is in the exterior domain of a hyperbola cannot be tangent to it at two points.

37. In Prop. IV.32 Apollonius proves that an arc of a parabola that is in the interior domain of an ellipse cannot be tangent to it at two points.

38. In Prop. IV.33 Apollonius proves that two hyperbolas with the same center cannot be tangent at two points.

39. In Prop. IV.34 Apollonius proves that if two ellipses with same center are tangent at two points, the straight line joining the points of contact is a diameter of both ellipses.

40. At the end of Prop. IV.34 Apollonius wrote "what was to prove" (see Note 83 to Book 1).

41. In Prop. IV.35 Apollonius proves that two conics whose convexities are in the same direction cannot meet at more than two points.

42. In Prop. IV.36 Apollonius proves that if a conic meets one of opposite hyperbolas at two points and the arcs between the points of meeting have concavities in the same direction, then the conic does not meet the other of opposite hyperbolas.

43. In Prop. IV.37 Apollonius proves that if a conic meets one of opposite hyperbolas, it meets the other hyperbola at no more than two points.

44. Prop. IV.38 is the analogue of Prop. IV.25 for a conic and opposite hyperbolas.

45. In Prop. IV.39 Apollonius proves that if a conic is tangent to one of opposite hyperbolas in its concave part, it does not meet the other of the opposite hyperbolas.

46. In Prop. IV.40 Apollonius proves that if a conic is tangent to each of opposite hyperbolas at one point, it does not meet these opposite hyperbolas at other point.

47. In Prop. IV.41 Apollonius proves that if a hyperbola meets one of opposite hyperbolas at two points, and the convexities of the arcs of these hyperbolas between the points of their meeting are in the opposite directions, then the given hyperbola does not meet the other of the opposite hyperbolas.

48. In Prop. IV.42 Apollonius proves that if a hyperbola meets both opposite hyperbolas, then the hyperbola opposite to the given hyperbola does not meet one of the given opposite hyperbolas at two points.

49. In Prop. IV.43 Apollonius proves that if a hyperbola meets each of opposite hyperbolas at two points, and in both cases the convexities of the arcs of hyperbolas between the points of their meeting are in the opposite directions,

then the hyperbola opposite to the given hyperbola has no common point with either of the given opposite hyperbolas.

In this case two pairs of opposite hyperbolas have four common points.

Apollonius does not formulate the analogue of Prop. IV.43 for the case where the convexities of the arcs of the hyperbolas in the meeting of the given hyperbola with one of opposite hyperbolas are in the opposite directions, and in the meeting of the given hyperbola with second of the opposite hyperbolas are in the same direction.

50. In Prop. IV.44 Apollonius proves that if a hyperbola meets one of opposite hyperbolas at four points, then the hyperbola opposite to the given hyperbola has no common point with second of the opposite hyperbolas.

In this case two pairs of opposite hyperbolas also have four common points.

51. In Prop. IV.45 Apollonius proves that if a hyperbola meets one of opposite hyperbolas at two points, and concavities of the arcs of the hyperbolas between the points of their meeting are in the same direction, and it touches the second of the opposite hyperbola at one point, then the hyperbola opposite to the given hyperbola has no common point with the opposite hyperbolas.

52. In Prop. IV.46 Apollonius proves that if a hyperbola intersects one of opposite hyperbolas at two points and touches this hyperbola at one point, then the hyperbola opposite to the given hyperbola has no common point with the second of the opposite hyperbolas.

In the Greek text for the formulation of this proposition the hyperbola ΔK is erroneously called ΔEK.

53. Prop. IV.47 is the analogue of Prop. IV.45 for the case where the concavities of the arcs of the hyperbolas between the points of their meeting have opposite directions.

In the formulation of this proposition the directions of concavities of the hyperbolas are not indicated, but these directions are clear from the diagram.

54. In Prop. IV.48 Apollonius proves that if a hyperbola touches one of opposite hyperbolas and intersects it at two points, then the hyperbola opposite to the given hyperbola has no common point with the second of the opposite hyperbolas.

55. In Prop. IV.49 Apollonius proves that if a hyperbola touches one of opposite hyperbolas and intersects it at one other point, then the hyperbola opposite to the given hyperbola intersects the second of the opposite hyperbolas at no more than one point.

In the Greek text of the formulation of this proposition the hyperbola EΘ is erroneously called EZΘ.

In Prop. IV.49, IV.50 and IV.53 Apollonius' expression δυο συζυγων is usually translated as "two conjugate pairs of hyperbolas". However the pairs of hyperbolas considered in these propositions are not conjugate in the sense defined by Apollonius in Prop. I.60 (see Note108 to Book 1). Indeed, in these propositions this expression has a more broad sense meaning two pairs of hyperbolas anyhow connected one to other.

56. In Prop. IV.50 Apollonius proves that if a hyperbola touches one of opposite hyperbolas at one point, then the hyperbola opposite to the given hyperbola intersects the second of the opposite hyperbolas at no more than two points.

57. In Prop. IV.51 Apollonius proves that if a hyperbola touches both opposite hyperbolas at one point, then the hyperbola opposite to the given hyperbola has no common point with the second of opposite hyperbolas.

58. In Prop. IV.52 two pairs of opposite hyperbolas are considered. Apollonius proves that if the hyperbolas of the first pair touch both hyperbolas of the second pair at one point, and the concavities of the tangent hyperbolas in both cases are in the same direction, then both pairs of the opposite hyperbolas have only two common points of contact.

59. In Prop. IV.53 Apollonius proves that if a hyperbola touches one of opposite hyperbolas at two points, then the hyperbola opposite to it has no common point with the second pair of opposite hyperbolas.

60. In Prop. IV.54 Apollonius proves that if a hyperbola touches one of opposite hyperbolas, and the convexities of these two hyperbolas are in the opposite directions, then the hyperbola opposite to the given hyperbola has no common point with the second of the opposite hyperbolas.

61. In Prop. IV.55 Apollonius proves that two pairs of opposite hyperbolas intersect at no more than four points.

Apollonius' proof of this proposition is based on particular cases considered by him above.

The general proof of this proposition can be obtained by application to Prop. IV.25 a projective transformation mapping both conics to pairs of opposite hyperbolas.

62. The gap was fulfilled by E. Halley

63. In Prop. IV.56 Apollonius proves that if two pairs of opposite hyperbolas touch one another at one point, then they intersect at no more than two points.

Apollonius' proof of this propositions is based on particular cases considered by him above.

The general proof of this proposition can be obtained by application to

Prop. IV.26 a projective transformation mapping both conics to pairs of opposite hyperbola.

In the Greek text of the formulation of this proposition, the opposite hyperbolas ΑΒ and Γ are erroneously called ΑΒ and ΒΓ, since two opposite hyperbolas have no common point.

64. In Prop. IV.57 Apollonius proves that if two pairs of opposite hyperbolas touch one another at two points, then they do not intersect.

Apollonius' proof of this proposition is based on particular cases considered by him above.

In the diagram of the Greek text concerning the fourth of these cases, the point Ζ where hyperbolas ΔΖ and ΕΖ touch one another is erroneously denoted by the letter Γ.

The general proof of this proposition can be obtained by application to Prop. IV.27 and IV.29 a projective transformation mapping both conics to pairs of opposite hyperbolas.

65. Let us consider the intersections and contacts of conics on the example of the classification of conics in the hyperbolic plane which by the interpretation of Felix Klein (1849-1925) can be regarded as the interior domain of a conic in the projective plane (see [Ro1, pp.257-259]). The conic bounding the image of the hyperbolic plane in the projective plane is called the "absolute of the hyperbolic plane".

The classification contains 12 kinds of conics:

1) an ellipse intersecting the absolute at 4 imaginary points;

2-3) hyperbolas with the interior and exterior centers intersecting the absolute at 4 real points;

4) a semi-hyperbola intersecting the absolute at 2 real and 2 imaginary points;

5) the circumference of a circle touching the absolute at 2 double imaginary points;

6) the equidistant of a straight line touching the absolute at 2 double real points;

7) an elliptic parabola touching the absolute at 1 double point and intersecting it at 2 imaginary points;

8-10) semi-hyperbolic parabolas with one and two branches and a hyperbolic parabola touching the absolute at 1 double point and intersecting it at 2 real points;

11) a horocycle touching the absolute at 1 quadruple point;

12) an osculating parabola touching the absolute at 1 triple point and intersecting it at 1 real point.

The circumference of a circle in the hyperbolic plane, like in the Euclidean plane is a locus of points equidistant from a point. Concentric circumferences touch the absolute at the same points.

The equidistant of a straight line in the hyperbolic plane is a locus of points in this plane equidistant from a straight line called the base. Equdistants with the same base touch the absolute at the same points.

A horocycle is an orthogonal trajectory of a pencil of parallel straight lines in the hyperbolic plane, it touches the absolute at the point of intersection of straight lines of this pencil. Horocycles can be obtained by limiting process both from circumferences of circles and from equidistants of straight lines.

In the propositions on contact of conics Apollonius considers only double points of contact and does not consider quadruple and triple points of contact, that is as the points of contact of the absolute of the hyperbolic plane with horocycles discovered by Nicolai Lobachevsky (1792 -1856) and as osculating parabolas discovered by Heinrich Liebmann (1874 - 1939).

COMMENTARY ON BOOK FIVE

Preface to Book 5

1. In this preface Apollonius writes that Book 5 contains "propositions on the maximal and minimal straight lines" and that his precursors and contemporaries considered only straight lines tangent to conics. The straight lines considered by Apollonius in this book are normals to conics, that is straight lines perpendicular to tangents at points of contact.

The fact that maximal and minimal straight lines drawn to a conic are normals to them can be explained as follows: the problem of drawing maximal and minimal straight lines to a conic is a problem of conditional extrema.

The general theory of such extremum was created by Joseph Louis Lagrange (1736-1813). According to this theory, the problem of finding the extrema of a function $f(x,y)$ whose arguments x and y are connected by the condition $F(x, y) = 0$ can be reduced to the finding of the extrema of the function

$$U(x, y) = f(x, y) + \lambda F(x, y) . \quad (5.1)$$

To solve this problem it is necessary to equate the partial derivatives $U_{x'}$ and $U_{y'}$ to 0 and to exclude the multiplier λ from the obtained equalities.

In the problem of finding extremal straight lines from a point M_o with coordinates x_o and y_o to a plane curve determined by the equation F(x,y) = 0, the function f(x, y) has the form

$$f(x,y) = (x - x_o)^2 + (y - y_o)^2 \quad (5.2)$$

where x and y are the coordinates of the point M of the curve, and the function (5.1) has the form

$$U(x,y) = (x - x_o)^2 + (y - y_o)^2 + \lambda F(x, y). \quad (5.3)$$

The necessary condition of the extremum of the function (5.3) has the form

$$U'_x = 2(x - x_0) + \lambda F'_x = 0, \qquad U'_y = 2 (y - y_o) + \lambda F'_x = 0. \quad (5.4)$$

Since F'_x and F'_y are coordinates of a normal vector to the curve F(x, y) = 0, equalities (5.4) show that the line M_0M is normal to the curve.

In the case of a conic, equation F(x, y) = 0 has the form (1.54).

Apollonius never uses the names of normals and always calls the straight lines M_oM "minimal" or "maximal". However in Prop. V.27 - V.30 Apollonius proves that minimal and maximal straight lines drawn to the conic are its normals.

2. In the edition [Ap7] by G.J.Toomer, the words "an infinite number" are the exact translation of the Arabic words "la nihaya li-'adada", but this expression was impossible for an ancient Greek mathematicians who never used the word "number" for actual infinity. This expression was written by Thabit ibn Qurra who in his answer to the question of his pupil Abu Musa al-Nasrani called actual infinity "complete number". No doubt that in the text of Apollonius, instead of these words, "indefinite number" were written.

E.Halley, who knew very well both the Greek and Arabic mathematics, in his Latin translation of Book 5 Conics mentioned Arabic word as "indefinite number".

Propositions V.1 - V.3 on areas

3. In Prop. V.1 a hyperbola or an ellipse ΒΑΓ with the diameter ΒΓ and the center Δ is considered. At the point Β the latus rectum ΒΕ is erected. The segment ΒΕ is bisected at the point H. The line ΔΗ is joined. From an arbitrary point Α of the conic the ordinate ΑΓ is dropped to the diameter ΒΓ. From the point Ζ of this diameter, the straight line ΖΘ parallel to ΒΕ to the line ΖΗ is drawn, ΒΚ meets line ΕΓ at Κ. Then line ΒΕ is bisected at Η, line ΗΔ meets ΖΚ at Θ. Apollonius proves that AZ^2 is equal to the double the area of the quadrangle ΒΖΘΗ.

The proof is based on Prop. I.12 and I.13.

4. In Prop. V.1 - V.3 E.Halley instead of "a diameter" writes "principal diameter", that is axis.

The opinion of Halley is explained by the fact that in all propositions of Books 5 - 7, besides Prop. V.1 - V.3, only rectangular coordinate systems whose axes 0x are axes of conic are used.

Therefore in the notes on these books, notations 2a, 2p, 2b, and ε mean latera transversa and recta, second diameters, and eccentricities corresponding to axes of conics, and these magnitudes corresponding to other diameters of conic are denoted 2a' , 2p' , etc.

5. Prop. V.2 is the particular case of Prop. V.1 for the ellipse where the point Ζ coincides with the center Δ.

Apollonius proves that $A\Delta^2$ is equal to double the area of the triangle ΒΖΔ.

6. Prop. V.3 is the particular case of Prop. V.1 for the ellipse where the point Ζ falls between the points Δ and ΓΖ

Apollonius proves that $A\Delta^2$ is equal to double the difference between the areas of the triangles ΒΔΗ and ΔΖΘ.

Propositions V.4 - V.26 on drawing minimal and maximal lines to conics from points of their axes

7. In Prop. V.4 the parabola ΑΒΓ with the axis ΓΖ and the vertex Γ is considered, and segment ΓΖ is equal to p, the half of the latus rectum.

Apollonius proves that the straight line ΓΖ is the minimal of the straight lines drawn from the point Ζ to the parabola, and if Α is an arbitrary point of the parabola with the abscissa x = ΓΕ and the ordinate y = ΑΕ, the equality

$$Z A^2 - Z\Gamma^2 = \Gamma E^2 \quad (5.5)$$

holds. Equality (5.5) follows from equation (0.3) of the parabola since this equation implies that $ZA^2 = y^2 + (x - p)^2 = 2px + x^2 - 2px + p^2$ $= x^2 + p^2 = \Gamma E^2 + \Gamma Z^2$.

Equality (5.5) implies that $Z\Gamma$ is the minimal of the straight lines drawn from the point Z to the parabola.

The straight line ΓZ is normal to the parabola since the straight line tangent to it at its vertex Γ is perpendicular to the axis of the parabola.

8. In Prop. V.5 the hyperbola $AB\Gamma$ with the axis ΓE, the vertex Γ, and the latus transversum 2a is considered. The segment ΓZ of the axis is equal to p, the half of the latus rectum of the hyperbola.

Apollonius proves that the straight line ΓZ is the minimal of the straight lines drawn from the point Z to the hyperbola, and if A is an arbitrary point of the hyperbola with the abscissa $x = \Gamma E$ and the ordinate $y = AE$, the equality

$$ZA^2 - Z\Gamma^2 = \Gamma E^2(p/a + 1) \qquad (5.6)$$

holds. Equality (5.6) follows from equation (0.10) of the hyperbola, since this equation implies that $ZA^2 = y^2 + (x - p)^2 = 2px + (p/a)x^2 + x^2 - 2px + p^2 . = (p/a)x^2 + x^2 + p^2$. Equality (5.6) implies that $Z\Gamma$ is the minimal of straight lines drawn from the point Z to the hyperbola.

Equality (5.6) can by rewritten

$$ZA^2 - Z\Gamma^2 = x^2(a^2 + b^2)/a^2 = \varepsilon^2 x^2 \quad . \qquad (5.7)$$

The line ΓZ is normal to the hyperbola since the tangent straight line to this hyperbola at the vertex Z is perpendicular to the axis of the hyperbola.

9. In Prop. V.6 the ellipse $AB\Gamma$ with the major axis $A\Gamma = 2a$ and the vertices A and Γ is considered. The segment $\Gamma\Delta$ of the major axis is equal to p, the half of the latus rectum.

Apollonius proves that the straight line $\Gamma\Delta$ is the minimal of the straight lines drawn from the point Δ to the ellipse and the straight line $A\Delta$ is the maximal of these straight lines, and if E is an arbitrary point of the ellipse with the abscissa $x = \Gamma\Lambda$ and the ordinate $y = E\Lambda$, the equalities

$$\Delta E^2 - \Gamma\Delta^2 = \Gamma\Lambda^2(1 - p/a), \quad (5.8)$$
$$A\Delta^2 - \Delta E^2 = (A\Gamma^2 - \Gamma\Lambda^2)\ (1 - p/a) \quad (5.9)$$

hold. Equalities (5.8) and (5.9) follow from equation (0.9) of the ellipse since this equation implies that $\Delta E^2 = y^2 + (x - p)^2 = 2px - (p/a)x^2 + x^2 - 2px + p^2 = x^2 - (p/a)x^2 + p^2$. For formula (5.9) let us mention the equality $A\Delta^2 = (2a - p)^2 = 4a^2 - 4ap + p^2 = 2a^2(1-p/a) + p^2$.

$$\Delta E^2 - \Gamma\Delta^2 = x^2(a^2 - b^2)/a^2 = \varepsilon^2 x^2 . \quad (5.10)$$

The analogous modification can be made in formula (5.9).

The straight lines $\Gamma\Delta$ and $A\Delta$ are normals to the ellipse since the tangent straight lines to this ellipse at the vertices A and Γ are perpendicular to the major axis of the ellipse.

10. Equalities (5.5), (5.7), and (5.10) show that the differences $ZA^2 - Z\Gamma^2$ in the cases of the parabola and the hyperbola and the difference $\Delta E^2 - \Delta\Gamma^2$ in case of the ellipse are equal to $\varepsilon^2 x^2$ where ε is the eccentricity of the conic and x is the abscissa of the points A or E of this conic.

11. In Prop. V.7 a conic $AB\Gamma\Delta$ with the axis ΔH is considered. On the axis the point E such that $\Delta E = p$ and the point Z between the points Δ and E are taken. Apollonius proves that the straight line ΔZ is the minimal of the straight lines drawn from the point Z to the conic.

Prop. V.7 follows from Prop. V.4, V.5, and V.6.

12. In Prop. V.8 at the parabola $AB\Gamma$ with the axis $\Gamma\Delta$ is considered. On the axis points E and Z are taken so that $\Gamma E > p$ and $ZE = p$, and the point Z is between the points Γ and E. The point H on the parabola with the ordinate ZH is taken. Apollonius proves that the straight line EH is the minimal of straight lines drawn from the point E to the parabola, and if K is another point on the parabola with the ordinate $K\Xi$, the equality

$$EK^2 - EH^2 = \Xi\Gamma^2 \quad (5.11)$$

holds. The proof is based on Prop. I.11 and V.1.

The straight line EH and the straight line symmetric to EH with respect to the axis of the parabola are normals to it, the axis itself is also a normal to this parabola.

The first and the second of these three straight lines are the minimals of straight lines drawn from the point E to the parabola, the straight line ΓE is the maximal of straight lines drawn from E to the arc of the parabola between the point H and the point symmetric to it.

The segment $EZ = p$ is called by modern mathematicians “subnormal” of the point H. Thus subnormals of all points of a parabola are equal. This property of parabola (0.3) can be proved as follows. The straight line tangent to the parabola at the point H with coordinates x_o and y_o is determined by equation (3.14). Therefore the normal to the parabola at H is determined by the equation

$$y - y_o = -(p/y_o)(x - x_o) . \quad (5.12)$$

The abscissa x of the point E of intersection of this normal EH with the axis can be found from the condition $y = 0$; hence the equality $x - x_o = p$ follows. The difference $x - x_o$ is the subnormal.

13. In Prop. V.9 the hyperbola ΑΒΓ with the axis ΩΔ, the latus transversum $ΩΓ = 2a$, and the center H is considered. On the axis the points E and Z are taken such that $ΓE = x - a > p = b^2/a$. The point Z is determined by the condition $HZ / ZE = a/p$. The point Z is situated between the points Γ and E.

Let Θ is the point of the hyperbola with the ordinate ΘZ. Apollonius proves that EΘ is the minimal of straight lines drawn from the point E to the hyperbola, and if K is another point on the hyperbola with the ordinate ΞK, the equality

$$EK^2 - EΘ^2 = ΞZ^2(p/a + 1) = ΞZ^2(a^2 + b^2)/a^2 . \quad (5.13)$$

holds. The proof is based on Prop. I.12 and V.1.

The straight line EΘ and the straight line symmetric to EΘ with respect to the axis of the hyperbola are normals to it, the axis itself is also a normal to this hyperbola.

The first and the second of these three straight lines are the minimal of straight lines drawn from the point E to the hyperbola, the segment ΓE is the maximal of straight lines drawn from E to the arc of the hyperbola between the point Θ and the point symmetric to it.

The straight line EZ is called by modern mathematicians "subnormal" of the point Θ of the hyperbola. If the coordinates of the point Θ are x_o and y_o , the proportion $HZ / ZE = a/p$ can be rewritten as $x_o/(x - x_o) = a/p = a^2/b^2$. This proportion follows from the fact that the straight line tangent to hyperbola (1.46) at its point Θ is determined by equation (3.16). Therefore the normal to the hyperbola of its point Θ is determined by the equation

$$y - y_o = -(a^2/b^2)(y_o/x_o)(x - x_o) , \quad (5.14)$$

and the abscissa x of the point E of intersection of the normal EΘ with the axis can be found from the condition $y = 0$. Hence the equality

$x - x_o = (a^2/b^2)\ x_o = (a/p)x_o$ holds.

14. In Prop. V.10 the ellipse ΑΒΓ with the major axis ΑΓ = 2a and the center Δ is considered. On the major axis points Ε and Ζ are taken such that $p < ΖΕ < a$. The point Ζ is situated between the points Γ and Ε. The point Ζ is determined by the condition $ΔΖ/ΖΕ = a/p$.
The point Η on the ellipse with the ordinate ΖΗ is taken.

Apollonius proves that ΕΗ is the minimal of straight lines drawn from Ε to the ellipse, and if Θ is another point on the ellipse with the ordinate ΡΘ, the equality

$$ΕΘ^2 - ΕΗ^2 = ΡΖ^2 (1 - p/a) = ΡΖ^2(a-b)^2/a^2 \quad (5.15)$$

holds. The proof is based on Prop. I.13 and V.2

The straight line ΕΗ and the line symmetric to ΕΗ with respect to the major axis of the ellipse are normals to it, and the major axis itself is also a normal to this ellipse. The first and the second of these three straight lines are the minimal of straight lines drawn from the point Ε to the ellipse, the straight line ΓΕ is the maximal of straight lines drawn from the point Ε to the arc of the ellipse between the point H and the point symmetric to it.

The line ΕΖ is called by modern mathematicians "subnormal" of the point Η of the ellipse. If the coordinates of the point Η are x_o and y_o , the proportion $ΔΖ/ΖΕ = a/p$ can be rewritten as $| x_o | / |x - x_o| = a/p = a^2/b^2$
This proportion follows from the fact that the straight line tangent to ellipse (1.45) at its point Η is determined by equation (3.15).

Therefore the normal to the ellipse at Η is determined by the equation

$$y - y_o = (a^2/b^2)(y_o/x_o)(x - x_o) . \quad (5.16)$$

The abscissa x of the point Ε of intersection of the normal ΕΗ with the major axis can be found from the condition $y = 0$. Hence the equality $| x - x_o | = (b^2/a^2)\ | x_o | = (p/a)\ | x_o |$ follows.

15. Note that if we denote abscissas of the points Ξ in the Prop V.8 and V.9, and the point Ρ in Prop. V.10 by x_1, the right hand sides of equalities (5.11), (5.13), and (5.15) can be rewritten in the form $(x_1 - x_o)^2\ \varepsilon^2$, where ε is the eccentricity of the conic .

Prop. V.9 and V.10 show that the subnormal of points of an ellipse and a hyperbola are equal to the products of the absolute values of the abscissas of these points by b^2/a^2.

16. In Prop. V.11 the ellipse ABXΔ with the major axis AΓ, the minor axis BΔ, and the center E is considered.

Apollonius proves that the maximal of straight lines drawn from the point E to the ellipse are EA and EΓ, and the minimal of them are EB and EΔ, and if Z in is as arbitrary point of the ellipse between the points A and B with the ordinate ZI, the equality

$$ZE^2 - BE^2 = EI^2(1 - p/a) \quad (5.17)$$

holds. The assertions of this proposition are limit cases for the assertions of Prop. V.10 where the point E tends to the center of the ellipse.

17. In Prop. V.12 the conic AB with the axis BΓ is considered, and ΓA is the minimal of the straight lines drawn from the point Γ to the conic.

Apollonius proves that if Δ is an arbitrary point of ΓA, the line ΔA is the minimal of lines drawn from the point Δ to the conic.

The proof is undertaken by reduction to absurd.

18. In Prop. V.13 the parabola AB with the axis BΓ is considered, and ΓA is the minimal of straight lines drawn from the point Γ to the parabola.

Apollonius proves that the angle BΓA is acute, and if the ordinate of the point A is AΔ, the straight line ΓΔ is equal to p. The line ΓΔ is subnormal of the point A. The equality ΓΔ = p was mentioned in Prop. V.8.

19. In Prop. V.14 the hyperbola AB with the axis BΓ and the center Δ is considered, and ΓA is the minimal of straight lines drawn from the point Γ to the hyperbola.

Apollonius proves that the angle BΓA is acute, and if the ordinate of the point A is AE, the proportion ΔE/EΓ = a/p holds.

The straight line ΓE is the subnormal of the point A. The proportion ΔE/ΓE = a/p was mentioned in Prop. V.9.

20. In Prop. V.15 the ellipse ABΓ with the major axis AΓ and the center I is considered, and IB is the minimal of straight lines drawn from the point I to the ellipse.

Apollonius proves that the straight line IB is perpendicular to the line AΓ, and if the point H is between the points Γ and I, and HΓ is the minimal of straight lines drawn from H to the ellipse, and if the ordinate of the point Γ is KΓ, then the angle ΓHI is obtuse, and the proportion IK/ KH = a/p holds.

The straight line BI is half the minor axis of the ellipse.

The straight line HK is subnormal of the point Γ. Proportion IK/ KH = a/p was mentioned in Prop. V.10.

21. In Prop. V.16 the ellipse $AB\Gamma$ with the center Π, the major axis 2a and the minor axis $A\Gamma$ = 2b whose length is greater than q = a^2/b, the half of the latus rectum corresponding to the minor axis, is considered. On the axis $A\Gamma$ the point Δ such that $\Gamma\Delta$ = q is taken.

Apollonius proves that $\Gamma\Delta$ is the maximal of straight lines drawn from the point Δ to the ellipse, and if E is an arbitrary point of the ellipse and the line EK is the perpendicular dropped from E to the axis $A\Gamma$, the equality

$$\Gamma\Delta^2 - \Delta E^2 = \Gamma K^2(q/b - 1) = \Gamma K^2(a^2 - b^2)/b^2 \qquad (5.18)$$

holds. Prop. V.16 is the analogue of Prop. V.6 for the minor axis of an ellipse.

The proof of Prop. V.16 is based on Prop. V.1 and V.3.

The difference $q/b - 1 = (a^2 - b^2)/b^2$ is equal to $\varepsilon^2 a^2/b^2$.

The condition $q < 2b$ is equivalent to the condition $a^2 < 2b^2$

22. In Prop. V.17 the ellipse $AB\Gamma$ with the center O, the major axis 2a, and the minor axis $A\Gamma$ = 2b equal to q = a^2/b is considered.

Apollonius proves that $A\Gamma$ is the maximal of straight lines drawn from the point A to the ellipse, and if B is an arbitrary point of the ellipse and BZ is the perpendicular dropped from this B to $A\Gamma$, the equality

$$A\Gamma^2 - AB^2 = \Gamma Z^2 (q/b - 1) = \Gamma Z^2(a^2-b^2)/b^2 \qquad (5.19)$$

holds. Prop. V.17 is also the analogue of Prop. V.6 for the minor axis of an ellipse. The proof of Prop. V.17 is based on Prop. V.3.

In particular, the distance from the point A to one of the ends of the major axis is equal to the hypotenuse of the rectangular triangle with the catheti a and b, that is $(a^2+b^2)^{1/2} = (3b^2)^{1/2} = 3^{1/2}b$.

23. In Prop. V.18 the ellipse ABG with the center N, the major axis 2a, and the minor axis $A\Gamma = 2b < q = a^2/b$ is considered. On the continuation of the axis $A\Gamma$, the point Δ such that $\Gamma\Delta$ = q is taken.

Apollonius proves that $\Gamma\Delta$ is the maximal of straight lines drawn from the point Δ to the ellipse, and ΔA is the minimal of them, and if is an arbitrary point of the ellipse and the perpendicular dropped from B to $A\Gamma$ is BK, the equality

$$\Gamma\Delta^2 - \Delta B^2 = \Gamma K^2(q/b - 1) = \Gamma K^2(a^2-b^2)/b^2 \qquad (5.20)$$

holds. Prop. V.18 is also the analogue of Prop. V.6 for the minor axis of the ellipse.

The proof of Prop. V.18 is based on Prop. V.3 .

24. In Prop. V.19 the ellipse ΑΒΓ with the major axis 2a and the minor axis ΑΓ = 2b is considered. On the axis ΑΓ the point Δ such that ΓΔ > q = a^2/b is taken. Apollonius proves that ΓΔ is the maximal of straight lines drawn from the point Δ to the ellipse.

Prop. V.19 is the analogue of Prop. V.7 for the minor axis of an ellipse.

25. In Prop. V.20 the ellipse ΑΒΓ with the center Ε, the major axis 2a, and the minor axis ΑΓ = 2b is considered. On the axis ΑΓ the point Δ such that b < ΓΔ < q = a^2/b, and on ΕΓ the point Μ such that ΕΜ/ΜΔ = b/q = b^2/a^2 are taken. From the point Μ the perpendicular ΜΖ to ΑΓ is erected, Ζ is a point of the ellipse.

Apollonius proves that ΖΔ is the maximal of straight lines drawn from the point Δ to the ellipse, and that, if Θ is another point of the ellipse and the perpendicular ΘΝ dropped from Θ to ΑΓ the equality

$$\Delta Z^2 - \Delta\Theta^2 = NM^2(q/b - 1) = NM^2(a^2 - b^2)/b^2 \quad (5.21)$$

holds. Prop. V.20 is the analogue of Prop. V.10 for the minor axis of an ellipse.

26. In Prop. V.21 the ellipse ΑΒΓ with the major axis 2a and the minor axis ΑΓ = 2b is considered. On the minor axis the point Δ is taken such that ΔΒ is the maximal of lines drawn from the point Δ to the ellipse. On the continuation of ΔΒ the point Ε is taken, such that ΒΕ is greater than ΔΒ.

Apollonius proves that ΕΒ is the maximal of straight lines drawn from the point Ε to the ellipse.

Prop. V.21 is the analogue of Prop. V.12 for the minor axis of an ellipse.

27. In Prop. V.22 the ellipse ΑΒΓ with the center Δ, the major axis 2a, and the minor axis ΑΓ = 2b is considered.

Apollonius proves that if the line ΔΒ is perpendicular to the axis ΑΓ, the segment ΔΒ is the maximal of straight lines drawn from Δ to the ellipse, and if Ζ is another point of the axis ΑΓ between the points Δ and Α, and ΓΗ is the maximal of straight lines drawn from the point Ζ to the ellipse, then the angle ΓΖΗ is acute, and if HK is the perpendicular dropped from Η to ΑΓ, the proportion ΔΚ/ ΓΚ = b/q = b^2/a^2 holds.

This proposition shows that the subnormal of the point H of the ellipse (1.45) on its minor axis is equal to the product of the absolute value of the ordinate of the point Η by a^2/b^2.

The proof is based on Prop. V.16 - V.20.

Unlike the subnormals of points of the ellipse, which are on its major axis, the subnormals of these points on the minor axis pass through the center of the ellipse.

28. In Prop. V.23 the ellipse ΑΒΓΔ with the major axis ΓΑ = 2a and the minor axis ΔΒ = 2b is considered.

Apollonius proves that if Κ is a point of the axis ΔΒ , and ΚΕ is the maximal of straight lines drawn from Κ to the ellipse, and ΚΕ meets the axis ΓΑ at the point Ζ, the line ΖΕ is the minimal of lines drawn from Ζ to the ellipse.

The proof is based on Prop. V.22.

29. In Prop. V.24 the parabola ΑΒ with the axis ΒΓ is considered.

Apollonius proves that to an arbitrary point Α of the parabola only one minimal straight line can be drawn from the axis.

The proof is based on Prop. V.13.

30. Prop. V.25 is the analogue of Prop. V.24 for a hyperbola and an ellipse.

The proof is based on Prop. V.14 and V.15.

31. In Prop. V.26 the ellipse ΑΒΓ with the minor axis ΑΓ is considered.

Apollonius proves that to an arbitrary point Β of the ellipse only one maximal straight line can be drawn from its axis ΑΓ .

The proof is based on Prop. V.22.

Propositions V.27 - V.34 on coincidence of minimal and maximal straight lines drawn to conics with their normals

32. In Prop. V.27 the parabola ΑΒ with the axis ΒΓ is considered. From the point Δ of the axis the minimal straight line ΔΑ to the parabola is drawn. From the point Α the tangent ΑΓ is drawn. The perpendicular ΑΗ is dropped to the axis. Apollonius proves that the minimal straight line ΔΑ drawn to the parabola is perpendicular to the straight line ΑΓ tangent to the parabola at the end Α of the minimal straight line.

The segment ΗΔ is the subnormal of the point Α and is equal to p. The segment ΗΓ is called “subtangent” of the point Α. Since the equation of the tangent to parabola (0.3) at its point Α with coordinates x_0, y_0 is determined by equation (3.14), this equation implies that the abscissa x of the point Γ is equal to $-x_0$. Therefore the subtangent of the point Α is equal to $2x_0$. Since ΑΗ $=y_0$ and $y_0^2 = 2px_0$, the equality

$$\mathrm{H\Gamma.H\Delta} = \mathrm{AH}^2 \qquad (5.22)$$

holds. This equality implies that A, Γ, Δ are points of the circumference of a circle with the diameter $\Gamma\Delta$. Hence the angle $\Gamma A\Delta$ rests on a diameter of the circle and therefore this angle is right.

This proposition shows that the minimal straight lines to a parabola are its normals.

33. Prop. V.28 is the analogue of Prop. V.27 for a hyperbola and an ellipse.

In this proposition, a hyperbola or an ellipse AB with the axis $B\Gamma$ and the center Δ is considered. From the point E of the axis the minimal straight line EA to the conic is drawn, from the point A the tangent $A\Gamma$ to the conic is drawn, this straight line meets the axis at Γ. From A the perpendicular AH to the axis is dropped. Apollonius proves that the minimal straight line EA is perpendicular to the tangent $A\Gamma$.

The segment HE is the subnormal of the point A, the segment $H\Gamma$ is the subtangent of A. According to Prop. V.14 and V.15, the proportion $\Delta H/HE = a/p = a^2/b^2$ holds.

If the conics are determined by equations (1.45) and (1.46) and the coordinates of the point A are equal to x_o, y_o, $\Delta H = |x_o|$ and the mentioned proportion implies that the subnormal HE is equal to $|x_o| b^2/a^2$.

Since the tangents to conics (1.45) and (1.46) at A are determined by equations (3.15) and (3.16), the abscissas x of Γ in both cases are equal to a^2/x_o. Therefore the subtangent ΓH of the point A of the hyperbola is equal to $x_o - a^2/x_o = (x_o^2 - a^2)/x_o$, and the subtangent ΓH of the point A of the ellipse is equal to $|a^2/x_o - x_o| = (a^2 - x_o^2)/|x_o|$.

Since $AH = y_o$ and the coordinates of the point A satisfy to equations (1.45) and (1.46), the product $\Gamma H.HE$ in the case of the hyperbola is equal to $(x_o^2 - a^2)b^2/a^2 = (x_o^2/a^2 - 1)b^2 = (y_o^2/b^2)b^2 = y_o^2$, and in the case of the ellipse is equal to $(a^2 - x_o^2)b^2/a^2 = (1 - x_o^2/a^2)b^2 = (y_o^2/b^2)b^2 = y_o^2$. Thus in both cases $\Gamma H.HE = y_o^2 = AH^2$. This equality analogous to equality (5.22) implies that A, E, Γ are points of the circumference of a circle with the diameter $E\Gamma$. Hence the angle $EA\Gamma$ rests on a diameter of the circle and therefore this angle is right.

Prop. V.28 shows that minimal straight lines drawn to a hyperbola or an ellipse are their normals.

34. Prop. V.29 contains a very elegant proof of Prop. V.27 and V.28 based on the fact that the considered straight lines are minimal.

35. Prop. V.30 is the analogue of Prop. V.27 and V.28 for the maximal lines drawn to the ellipse from the points of its minor axis. This proposition shows that maximal straight lines drawn to an ellipse are its normals.

36. In Prop. V.31 Apollonius proves that if a straight line perpendicular to a minimal straight line drawn to a conic passes through the end of the minimal straight line, is tangent to the conic.

This proposition is inverse to Prop. V.27 - V.29.

37. In Prop. V.32 Apollonius proves that if a straight line tangent to a conic passes through the end of a minimal line drawn to this conic, it is perpendicular to this minimal line.

This proposition is also inverse to Prop. V.27 - V.29.

38. In Prop. V.33 Apollonius proves that a perpendicular at the end of a maximal straight line drawn to an ellipse is tangent to this ellipse.

This proposition is inverse to Prop. V.30.

39. In Prop. V.34 Apollonius proves that the straight line tangent to an ellipse at the end of a maximal straight line drawn to this ellipse is perpendicular to this maximal straight line. This proposition is also inverse to Prop. V.30.

Propositions V.35 - V.48 on intersections of normals drawn to conics

40. In Prop. V.35 the parabola ΑΒΓ with the axis ΓΔ is considered. ΑΔ and ΒΕ are two minimal straight lines drawn from the points Δ and Ε of the axis to the parabola, and the point Ε is between the points Γ and Δ.

Apollonius proves that the angle ΑΔΓ is greater than the angle ΒΕΓ.

This assertion follows from the fact that in the triangles bounded by ordinates of points of the parabola, and normals and subnormals of these points the sides which are subnormals are equal; and the ordinates y_1 and y_2 of the points Β and Α are connected with the magnitudes φ_1 and φ_2 of the angles ΒΕΓ and ΑΔΓ by the correlations $y_i/p = \tan \varphi_i$; and the inequality $y_1 < y_2$ implies the inequalities $\tan \varphi 1 < \tan \varphi 2$ and $\varphi 1 < \varphi 2$.

41. Prop. V.36 is the analogue of Prop. V.35 for a hyperbola and an ellipse.

If on the ellipse (1.45) or on the hyperbola (1.46) two points with the abscissas $x_1$and x_2, the ordinates $y_1$and y_2 ,and the subnormals $p_1$and p_2 are taken, the magnitudes x_i and p_i are connected by the equalities $p_i = (b^2/a^2)x_i$ and the angles φ_i between the normals of these points and the axis are determines by the formulas $y_i /p_i = \tan \varphi_i$.

In the case of the ellipse the inequality $x_1 < x_2$ implies the inequalities $y_1 > y_2$, $p_1 < p_2$, and $\varphi_1 > \varphi_2$.

In the case of the hyperbola the inequality $x_1 < x_2$ implies the inequalities

$y_1 < y_2$, $p_1 < p_2$, and $\varphi_1 > \varphi_2$. But in this case $y_i^2 =$
$(b^2/a^2)(x_i^{\,2} - a^2)$ and $\tan^2 \varphi_i = y_i^{\,2}/p^{\,2} = (a^2 + b^2)(1 - a^2/x_i^{\,2})$ and the inequality $x_1 < x_2$ implies that $\varphi_1 < \varphi_2$.

42. In Prop. V.37 the hyperbola AB with the axis ΓΔ and the asymptotes ZG and ΓH is considered. From the point A of the hyperbola the minimal straight line AΔ and the perpendicular ZBH to the axis that lets the asymptotes are drawn.

Apollonius proves that the angle AΔΓ is smaller than the angle ΓZH.

The proof is based on Prop. V.14.

43. In Prop. V.38 a conic ABΓ with the axis ΓΔE is considered.
To this conic the minimal straight lines ΔA and EB are drawn.

Apollonius proves that the continuations of ΔA and EB meet on the other side of the axis.

The proof is based on Prop. V.35 and V.36.

44. In Prop. V.39 the ellipse ABΓΔ with the minor axis AΔ is considered. To this ellipse the maximal straight lines EB and ZΓ from the points E and Z of this axis are drawn.

Apollonius proves that the continuations of EB and ZΓ meet on the other side of this axis.

The proof is based on Prop. V.22.

45. In Prop. V.40 the ellipse AΓΔ with the major axis ABΓ and the minor axis OBΔ is considered. To this ellipse the minimal straight lines EΘ and ZH from the points Θ and H of the semi-axis BΓ are drawn.

Apollonius proves that the continuations of EΘ and ZH meet within the angle ΓBO.

The proof is based on Prop. V.23.

46. In Prop. V.41 the parabola or the ellipse ABΓ with the axis BΔ is considered. From the point Δ the minimal straight line ΔA to the conic is drawn.

Apollonius proves that the continuation of ΔA meets the conic on the other side of the axis.

The proof is based on Prop. I.27.

47. In Prop. V.42 the hyperbola ABΓ with the axis ΔE, the center Δ, the latus transversum 2a, and the latus rectum 2p in the case where $a \leq p$ is considered. The segment AE is the minimal of straight lines drawn from the point E to the hyperbola.

Apollonius proves that the continuation of AE does not meet the hyperbola.

The proof is based on Prop. V.37.

48. In Prop. V.43 the hyperbola ABΓ with the axis ΔE, the center D, the la-

tus transversum 2a, the latus rectum 2p in the case where a > p is considered. The segment ΑΕ is the minimal of straight lines drawn from the point Ε to the hyperbola. Apollonius proves that if the angle ΑΕΔ is smaller than the angle between the axis and an asymptote of the hyperbola, the continuation of ΑΕ meets the hyperbola on the other side of its axis, and if the angle ΑΕΔ is not smaller than the mentioned angle, the continuation of ΑΕ does not meet the hyperbola.

49. In Prop. V.44 the parabola ΑΒΓΔ with the axis ΔΗ is considered. Though the points Ζ and Ε of the axis two minimal straight lines ΒΖ and ΓΕ are drawn, these lines meet at the point O.

Apollonius proves that any other straight line drawn from the point Ο is not minimal of the straight lines drawn from this point.

The proof is based on Prop. V.37.

50. Prop. V.45 is the analogue of Prop. V.44 for a hyperbola and an ellipse.

51. Prop. V.46 is a particular case of Prop V.45 for an ellipse where one of two minimal straight lines coincides with the minor axis.

52. In Prop. V.47 Apollonius proves that if from four points of the major axis of an ellipse four minimal straight lines to this ellipse are drawn, their continuations do not meet at a single point.

The proof is based on Prop. V.46.

53. In Prop. V.48 the ellipse ΑΒΓΔ with the minor axis ΑΓ and the major axis ΒΔ is considered.

Apollonius proves that no three of the maximal straight lines drawn to the ellipse from one of its quadrants meet at a single point.

The proof is based on Prop. V.45 and V.46.

54. In the propositions on intersection of normals of conics, pairs of normals containing minimal or maximal straight lines drawn to conics are considered.

The segments of the normals between the point of their meeting and the conic can be either minimal or maximal straight lines, but the greatest of them cannot be a minimal and is the maximal straight line, and the smallest of them cannot be a maximal , but is the minimal straight line.

Propositions V.49 - V.77 on the curvatures and the evolutes of conics

55. The magnitude p equal to half the latus rectum of a conic, which is used in Prop. V.49 and V.50 and in many other propositions of Book 5, plays a very important role in the differential geometry of conics. This role is as follows.

If a plane curve is ΑΒΓ, the limit circumference of a circle passing through three points Α, Β, Γ where the points Α and Γ tend toward Β, is called the "osculating circumference" or "osculating circle" of the plane curve at its point Β. This circle is also called the "curvature circle", the center Δ and the radius r of this circle are called the "curvature center" and the "curvature radius" of the plane curve at its point Β. The line ΒΔ is the normal of this curve at the point Β.

In the Note 67 on the Book 1 we mentioned linear operator (omografia) K of a surface x = x(u, v) in the space. If x is a position vector of a point of this surface, the differentials dx of this vector and dn of the unit normal vector n of the surface at this point are connected by formula (1.61).

Formula (1.61) is also valid for plane curves x = x(t). In this case the vectors dx and dn are tangent to the curve and therefore are collinear, and in this case the role of the operator K is played by the number k. If we denote $| dx | = ds$ and $n = i \cos\alpha + j \sin\alpha$, then $| dn | = d\alpha$, and the magnitude k coincides with the derivative $d\alpha/ds$. This magnitude is called the "curvature" of a plane curve at its given point. The curvature k of a plane curve is connected with the curvature radius r by the formula $k = 1/r$.

The meaning of the half of the latus rectum of a conic corresponding to the axis of this conic for the differential geometry is explained by the fact that this magnitude is equal to the curvature radius of the conic at its vertex

If the parabola, the hyperbola, or the ellipse ΑΒΓ with the axis ΒΔ is determined by equation (1.31) in rectangular coordinates with the origin Β and the abscissas of points Α and Γ equal to h, the radius r_h of the circumference passing through the points Α, Β, Γ is determined by the equality

$$r_h{}^2 = (r_h - h)^2 + y^2 = r_h^2 - 2r_h h + h^2 + 2ph + h^2(\varepsilon^2 - 1), \qquad (5.23)$$

Hence we obtain that

$$r_h = p + h\varepsilon^2 . \qquad (5.24)$$

Since the curvature radius r is the limit of r_h where h tends to 0, in all cases

56. In Prop. V.49 the parabola ΑΒ with the axis ΒΓ is considered. On the axis ΒΓ the point Ε is taken such that $ΒΕ \le p$. At the point Ε the perpendicular

$E\Delta$ which is situated below the axis $B\Gamma$ to this axis is erected. From the point Δ the straight line $\Delta\Theta A$ intersecting the axis at Θ is drawn. Apollonius proves that $A\Theta$ is not minimal straight line.

Prop. V.49 shows that from no point of the segment ΔE a normal to the upper part of the parabola AB can be drawn.

57. Prop. V.50 is the analogue of Prop. V.49 for a hyperbola and an ellipse.

58. In Prop. V.51 the parabola $AB\Gamma$ with the axis ΓZ is considered. If the segment ΓZ of the axis is greater than p, at the point Z the perpendicular ZE to the axis below it axis, is erected. Drawing of normals to the upper part of the parabola from E with the coordinates x_0 and y_0 is considered.

On the straight line $Z\Gamma$ the point H such that $\Gamma H = x_0 - p$ is taken, at the point H the perpendicular HT to the axis is erected.

For each point E in the plane which is situated below the axis the segment K can be determined such that if $y_0 < K$ two normals from E to the upper part of the parabola can be drawn, if $y_0 = K$ only one normal from E to the upper part of the parabola can be drawn, and if $y_0 > K$ no normal from E to the upper part of the parabola can be drawn.

Analogously the segment K can be determined for points that are over the axis.

59. In order to determine the segment K we must find the point Θ of the axis such that $\Theta H = 2\Gamma\Theta$ and the point B of the parabola with coordinates $x = \Gamma\Theta$ and $y = \Theta B$. Then the segment K is determined by the proportion

$$K/B\Theta = \Theta H/HZ. \qquad (5.25)$$

Proportion (5.25) is an algebraic correlation expressing the dependence of the magnitude K on the abscissa x_0 of the point E.

Since $\Gamma\Theta = (x_0 - p)/3$, $\Theta H = 2(x_0 - p)/3$, $HZ = p$, $B\Theta^2 = 2p(x_0 - p)/3$, equality (5.25) can be rewritten in the form

$$K^2 = (8/27)(x_0-p)^3/p. \qquad (5.26)$$

Apollonius does not disclose how he came to proportion (5.25). The correctness of proportion (5.25) and of equivalent to it correlation (5.26) will be proved further, in Note 62.

60. The straight lines TH and HG are asymptotes of an auxiliary equilateral hyperbola, the hyperbola opposite to this hyperbola passes through the point E. The equation of this hyperbola has the form

$$y(x - x_o) - p(y - y_o) = 0. \quad (5.27)$$

The equations of the asymptotes of this hyperbola have the form $x = x_o - p$ and $y = 0$

The coincidence of one asymptote of this hyperbola with the axis of the parabola shows that the hyperbola and the parabola meet at a point at infinity which can be regarded as the center of the parabola.

Apollonius considers the auxiliary hyperbola only in the case where $y_o < K$ and the hyperbola intersects the parabola at two points. Therefore in Apollonius' diagram the auxiliary hyperbola is shown only in this case.

Indeed the auxiliary hyperbola can be determined in all three cases, and if $y_o = K$, this hyperbola touches the parabola, and if $y_o > K$, the points of intersection of the hyperbola and the parabola are imaginary.

In Book 4 Apollonius has proved that a parabola and a pair of opposite hyperbolas can meet at four points. Therefore in the case where the parabola and the auxiliary hyperbola meet at two real points, they have one common point at infinity, the fourth common point of the parabola with the auxiliary hyperbola and its opposite hyperbola is situated on this opposite hyperbola.

In the case where the parabola and the auxiliary hyperbola have two real common points A and M, only two normals EA and EM from the point E to the upper parts of the parabola can be drawn, one of these lines is minimal and the other is maximal. If the hyperbola touches the parabola in one point B, only one normal EB from the point E to the upper part of the parabola can be drawn. If the parabola and the hyperbola have no common point, no normal from the point E to the upper part of the parabola can be drawn.

Equation (5.27) can be obtained as follows. For parabola (0.3) function (5.3) has the form

$$U(x,y) = (x - x_o)^2 + (y - y_o)^2 + \lambda(y^2 - 2px). \quad (5.28)$$

Equations (5.4) for function (5.28) have the form

$$U'_x = 2(x - x_o - 2\lambda p) = 0 , \quad (5.29)$$
$$U'_y = 2(y - y_o + 2\lambda y) = 0 . \quad (5.30)$$

The elimination of λ from equations (5.29) and (5.30) leads to equation (5.27).

61. The point B of contact of the auxiliary hyperbola and the parabola ΑΒΓ

can be obtained by the limiting process from two points A and M of intersection of the hyperbola and the parabola where these points tend toward one another. Therefore the normal EB can be obtained by the limiting process from two normals EA and EM where they tend toward one another, and EB is the curvature radius of the parabola at its point B, and the point E is the curvature center of the parabola at the point B.

The abscissa ΓΘ and the ordinate ΘB of the point B were considered in Note 59.

62. The locus of the curvature centers of a plane curve in modern differential geometry is called the "evolute" of this curve. The evolute of a curve can be found as the envelope of the family of normals of this curve.

The envelope of a family of plane curves

$$F(x, y, t) = 0 \qquad (5.31)$$

with the parameter t can be obtained by elimination of t from equations (5.31) and $F'_t = 0$.

In the case of parabola (0.3) equation (5.31) of the family of normals can be obtained from equation (5.12) by the substitution

$$x_0 = t^2/2p, \quad y_0 = t . \qquad (5.32)$$

The envelope of the family of normals of parabola (0.3), that is the evolute of this parabola, is the semi-cubic parabola

$$y^{2/3} = (2/3)(x - p) /p^{1/3} . \qquad (5.33)$$

If in equation (5.33) we put $x = x_0$, $y = K$ and take the cubes of both parts of this equality, we will obtain equality (5.26) equivalent to Apollonius' proportion (5.25).

Curve (5.33) consists of two concave lines symmetric with respect to the axis of the parabola connected at the cuspidal point of the curve, which coincides with the curvature center of the parabola at its vertex. The coordinates of this point are $x = p$, $y = 0$.

63. Pappus in Prop. IV.30 of his Mathematical Collection wrote that in Prop. V.51 of Apollonius' Conics the auxiliary hyperbola must be replaced by the circumference of a circle, since hyperbolas are solid loci and the circumferences of circles are plane loci. This replacement was fulfilled by Christian Huygens (1629-1695). The Huygens' solution of this problem was reproduced by Toomer in his

English translation [Ap7, pp. 659-661].

Umar Khayyam (1048-1131) in his algebraic treatise proved that intersections of the circumferences of circles, parabolas with horizontal or vertical axes, and equilateral hyperbolas with horizontal and vertical axes or asymptotes can be used for resolution of cubic equations. Therefore the mutual disposition of parabola (0.3) and the auxiliary hyperbola in Prop.V.51, a parabola and the circumference of a circle considered by Huygens determine the solutions of cubic equations.

64. Prop. V.52 is the analogue of Prop. V.51 for a hyperbola and an ellipse. In this proposition an ellipse or a hyperbola ΑΒΓ with the axis ΕΓΔ and the center Δ is considered. This ellipse or hyperbola is determined by equation (1.45) or (1.46). The role, which in Prop. V.51 was played by the segment Κ, in Prop. V.52 is played by the segment Λ.

If the segment ΓΕ is greater than p, at the point Ε the perpendicular ΖΕ to the axis, which is situated below this axis is erected. The drawing of normals from the point Ζ with the coordinates x_0 and y_0 to the upper part of the hyperbola and of the ellipse is studied. Apollonius considers three cases: $y_0 > \Lambda$, $y_0 = \Lambda$, $y_0 < \Lambda$, and proves that in the first case no normal from the point **Z** to the upper part of the conic can be drawn, in the second case only one such normal ΖΒ from the point Ζ can be drawn, and in the third case only two such normals ΖΑ and ΖΡ from Ζ can be drawn, one of which is the minimal, and other is the maximal straight line.

65. For the determining the segment L, Apollonius finds on the axis ΓΕ of the conic ΑΒΓ the point H that such that ΔΗ/ ΗΕ = 2a/2p and on the segment ΓΗ the points Θ and K such that the segments ΔΘ and ΔΚ are two mean proportionals between the segments ΔΗ and ΔΓ. These segments satisfy the condition

$$\Delta\Gamma/\Delta K = \Delta K/\Delta\Theta = \Delta\Theta/\Delta H. \quad (5.34)$$

Correlation (5.34) is a particular case of condition (0.2). The segments ΔΚ and ΔΘ can be found by means of the intersection of two parabolas.

Apollonius also finds the point Β of the conic with coordinates x = ΔΚ, y = ΚΒ.

66. Apollonius determines the segment Λ by the compounded ratio

$$\Lambda/KB = (\Delta E/EH)\times(HK/K\Delta). \quad (5.35)$$

This formula is an algebraic correlation expressing the dependence of the magnitude Λ on the abscissa x_0 of the point Ζ.

In equality (5.35) $\Delta E = x_o$, and if we denote $\Delta H = h$, then in the case of ellipse (1.45) $EH = h - x_o$, the magnitude h is determined by the proportion $h/(h - x_o) = a/p = a^2/b^2$, hence the equality $h = x_o a^2/(a^2 - b^2)$ follows; and in the case of hyperbola (1.46) $EH = x_o - h$, the magnitude h is determined by the proportion $h/(x_o - h) = a/p = a^2/b^2$, hence the equality $h = x_o a^2/(a^2+b^2)$ follows.

In both cases equality (5.34) can be rewritten in the form

$a/\Delta K = \Delta K/ \Delta\Theta = \Delta\Theta /h$, hence the equality $\Delta K/a = (h/a)^{1/3}$ follows.

In the case of an ellipse $KB^2 = (b^2/a^2)(a^2 - \Delta K^2)$;

In the case of a hyperbola $KB^2 = (b^2/a^2)(\Delta K^2 - a^2)$.

The substitution of these magnitudes into the compounded ratio (5.35) gives for the ellipse

$$(b\Lambda)^2 = (ax_o)^2((a/h)^2 - 3(a/h)^{4/3} + 3 (a/h)^{2/3} - 1), \quad (5.36)$$

and for the hyperbola

$$(b\Lambda)^2 = (ax_o)^2 (1- 3(a/h)^{2/3} + 3(a/h)^{4/3} - (a/h)^2) \quad (5.37)$$

Like for proportion (5.25), Apollonius does not disclose how he came to compounded ratio (5.35). The correctness of correlation (5.35) and of equivalent to it correlations (5.36) and (5.37) will be proved further in Note 69.

67. In Prop.52, like in Prop.51, Apollonius considers the auxiliary hyperbolas only for the cases where $y_o < \Lambda$. On his diagrams this hyperbola is shown only where it intersects the considered conic at two points A and P.

The auxiliary hyperbola can also be determined when $y_o = \Lambda$ and $y_o > \Lambda$. In the first case this hyperbola touches the considered conic at one point B; in the second case this hyperbola and the considered conic have no common point.

To construct the auxiliary hyperbolas Apollonius takes on the straight line $E\Gamma$ the point N such that $\Gamma N/ NE = 2a/2p$, from the point H of the axis ΓE he draws a perpendicular $H\Omega$ to the axis, and through the point N he draws the straight line $N\Omega$ parallel to the axis. The straight lines ΩH and ΩN are the asymptotes of the auxiliary equilateral hyperbolas, whose opposite hyperbolas pass through the point Z.

The equation of the auxiliary hyperbola and of the hyperbola opposite to it for hyperbola (1.46) has the form

$$xy - xy_o b^2/(a^2 + b^2) - yx_o a^2/(a^2 + b^2) = 0 , \quad (5.38)$$

and for ellipse (1.45) has the form

$$xy + xy_0b^2/(a^2 - b^2) - yx_0a^2/(a^2 - b^2) = 0 . \quad (5.39)$$

Since the asymptotes of the auxiliary hyperbolas are perpendicular and parallel to the axis of the conic, the equations of asymptotes of hyperbola (5.38) have the form $x = x_0a^2/(a^2+b^2)$, $y = y_0b^2/(a^2+b^2)$, the equations of asymptotes of hyperbola (5.39) have the form $x = x_0a^2/(a^2 - b^2)$, $y = -y_0b^2/(a^2-b^2)$.

The equations of the asymptotes of hyperbolas (5.38) and (5.39) can be obtained from Apollonius' proportions for determining the points H and N, if the abscissa of the point H is denoted by x , the ordinate of the point N is denoted by y and the magnitude p is expressed by the formula $p = b^2/a$.

Note that in the case of hyperbola (1.46) $x > 0$, $y < 0$, and in the case of ellipse (1.45) $x < 0$, $y < 0$.

The absence of the constant term in the equations of hyperbolas (5.38) and (5.39) shows that these hyperbolas or hyperbolas opposite to them pass through the centers of hyperbola (1.46) and ellipse (1.45). In the case of hyperbola (1.46) its center is on the auxiliary hyperbola itself; in the case of ellipse (1.45) its center is on the hyperbola opposite to the auxiliary hyperbola.

In Book 4 Apollonius proved that a pair of opposite hyperbolas can have with another pair of opposite hyperbolas or with an ellipse 4 common points. Therefore in the case where the auxiliary hyperbola and hyperbola (1.46) or ellipse (1.45) meet at two real points A and P, touch one another at the point B, or have two imaginary common points, then other common points of the auxiliary hyperbola and the hyperbola opposite to it with ellipse (1.45) and hyperbola (1.46) with the hyperbola opposite to it are situated on the hyperbola opposite to the auxiliary hyperbola.

The normal ZB is the limit position of the normals ZA and ZP where they tend toward one another, and ZB is the curvature radius of the conic at its point B, and the point Z is the curvature center of the conic at this point.

The abscissa ΔK and the ordinate KB of the point B were determined in Note 65.

68. Equations (5.38) and (5.39) can be obtained as follows: for hyperbola (1.46) and ellipse (1.45) function (5.3) has, respectively, the form

$$U(x,y) = (x - x_0)^2 + (y - y_0)^2 + \lambda(x^2/a^2 - y^2/b^2 -1), \quad (5.40)$$

and

$$U(x,y) = (x - x_o)^2 + (y - y_o)^2 + \lambda(x^2/a^2 + y^2/b^2 - 1) . \quad (5.41)$$

Equations (5.4) for function (5.40) have the forms

$$U'_x = 2(x - x_o + \lambda x/a^2) = 0 , \quad (5.42)$$
$$U'_y = 2(y - y_o - \lambda y/b^2) = 0 . \quad (5.43)$$

Equations (5.4) for function (5.41) have the form (5.42) and the form

$$U'_y = 2(y - y_o + \lambda y/b^2) = 0 . \quad (5.44)$$

The elimination of λ from equations (5.42) and (5.43) leads to equation (5.38), the elimination of λ from equations (5.42) and (5.44) leads to equation (5.39).

69. In the case of hyperbola (1.46) and ellipse (1.45) equation (5.31) of the family of normals to these conics can be obtained from equations (5.14) and (5.16) by substitutions for hyperbola (1.46)

$$x_o = a \cosh t , \quad y_o = b \sinh t \quad (5.45)$$

and for ellipse (1.45)

$$x_o = a \cos t , \quad y_o = b \sin t . \quad (5.46)$$

The envelope of the family of normals of ellipse (1.45), that is the evolute of the ellipse, is the astroid

$$(xa/(a^2 - b^2))^{2/3} + (yb/(a^2 - b^2))^{2/3} = 1. \quad (5.47)$$

The astroid whose name means “similar to a star” consists of four concave lines symmetric with respect to the axes of the ellipse and connected at four cuspidal points. Cuspidal point of the astroid coincide with the curvature centers of the ellipse at its four vertices.

The coordinates of the right and left cuspidal points of the astroid (5.47) are

$$x = a - p = a - b^2/a = (a^2 - b^2)/a\ ,\ \ y = 0 \qquad (5.48)$$
$$x = p - a = b^2/a - a = (b^2 - a^2)/a\ ,\ \ y = 0. \qquad (5.49)$$

The coordinates of the upper and lower cuspidal points of the astroid (5.47) are equal to

$$x = 0,\ \ y = q - b = a^2/b - b = (a^2 - b^2)/b, \qquad (5.50)$$
$$x = 0,\ \ y = b - q = b - a^2/b = (b^2 - a^2)/b\ . \qquad (5.51)$$

The envelope of the family of normals of pair of opposite hyperbolas (1.46), that is the evolute of these hyperbolas, is the pseudoastroid

$$(xa/(a^2 + b^2))^{2/3} - (yb/(a^2 + b^2))^{2/3} = 1. \ (5.52)$$

The pseudoastroid consists of two branches, each of them is the envelope of the normals to one of two opposite hyperbolas. Each branch of the pseudoastroid consists of two concave lines symmetric with respect to the axes of opposite hyperbolas. These lines are connected at cuspidal points of the pseudoastroid. The cuspidal points of pseudoastroid coincide with the curvature centers of the opposite hyperbolas at their vertices

The coordinates of the right and left cuspidal points of the pseudoastroid (5.52) are equal to

$$x = a + p = a + b^2/a = (a^2 + b^2)/a\ ,\ \ y = 0 \qquad (5.53)$$
$$x = - a - p = - a - b^2/a = -(a^2 + b^2)/a\ ,\ \ y = 0. \qquad (5.54)$$

Formulas (5.48) and (5.53) can be rewritten in the form

$$x = a\varepsilon^2\ ,\quad y = 0\ . \qquad (5.55)$$

Formulas (5.49) and (5.54) can be rewritten in the form

$$x = -a\varepsilon^2\ ,\quad y = 0\ . \qquad (5.56)$$

Equations (5.47) and (5.52) can be rewritten in the form

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}, \qquad (5.57)$$
$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}\ . \qquad (5.58)$$

Since right hand parts of both equations (5.57) and (5.58) are equal to a^2x/h, these equations can be rewritten in the form

$$(by)^{2/3} = (ax)^{2/3}((a/h)^{2/3} - 1), \qquad (5.59)$$
$$(by)^{2/3} = (ax)^{2/3}(1 - (a/h)^{2/3}) . \qquad (5.60)$$

If we put into equations (5.59) and (5.60) values $x = x_o$ and $y = \Lambda$, and take both sides of each of these equations in cubes, we will obtain equalities (5.36) and (5.37) equivalent to Apollonius' compounded ratio (5.35).

The equivalence of Apollonius' equalities (5.25) and (5.35) to equations (5.33) of the semicubic parabola, (5.47) of the astroid, and (5.52) of a pseudoastroid was established by T.L.Heath [Ap8, pp. 174 and 177-178].

However all translators of Conics in 20th century, except G.J.Toomer [Ap7, p. LIII], did not mention the connection of equalities (5.25) and (5.35) with equations of evolutes of conics.

70. The segments K and Λ in Prop. V.51 and V.52 are equal to the ordinates of points of the evolutes of conics. Although Apollonius' correlations (5.25) and (5.35) are equivalent to the equations of these evolutes, Apollonius does not study the structure of these curves since they are neither plane nor solid loci and cannot be obtained by means of intersection or moving straight lines, like other curves considered by ancient mathematicians.

71. In Note 46 on Book 3 we saw that abscissas of the foci of ellipse (1.45) and pair of opposite hyperbolas (1.46) are equal to $+a\varepsilon$. In Note 69 on Book 5 we see that abscissas of curvature centers of these conics at their vertices, that is at the ends of latera transversa, are equal to $+a\varepsilon^2$.

Therefore in an ellipse ABΓ with the major axis 2a, the vertex B, the focus Φ, and the curvature center Δ at the point B the distance BΦ is equal to $a(1 - \varepsilon)$, and BΔ is equal to $a(1 - \varepsilon^2)$; and in a hyperbola ABΓ with the latus transversum 2a, the vertex B, the focus Φ and curvature center Δ of the hyperbola at vertex B the distances BΦ is equal to $a(\varepsilon - 1)$, and BΔ is equal to $a(\varepsilon^2 - 1)$; and in both conics

$$B\Delta = B\Phi(\varepsilon + 1) . \qquad (5.61)$$

Since for the circumference of a circle, $\varepsilon = 0$ and segment BΦ and BΔ are equal to the radius of the circle, and since for a parabola $\varepsilon = 1$, $B\Phi = p/2$, and $B\Delta = p$, formula (5.61) is also valid for the circumference of a circle and for a parabola.

72. In Book 5, like in other books of Conics, Apollonius does not disclose how he came to the results of this work.

Therefore M.E.Vashchenko-Zakharchenko in his History of Mathematics characterized Book 5 of Conics as follows: "Book 5, the most remarkable one, shows Apollonius' studies in their whole greatness; in this book, the question on geometrical significance of greatest and smallest magnitudes, that is the question on maxima and minima, appears for the first time. . . He explores particular cases, and with extraordinary skill, almost incomprehensible to us, from these particular cases derives more general rules, under which he brings all questions studied by him. With amazing skill he solves the most difficult of questions, and we unwittingly get the impression that he possesses other methods of research with whose help he found propositions and only afterwards he recast them into commonly accepted forms. It is known that, almost two thousand years later, Newton redid and reformed much of his investigations, shrouding them into forms and methods used by ancient Greek geometers." [VZ, p. 103]

Very enigmatic is the way that led Apollonius to equalities (5.25) and (5.35) equivalent to equations of evolutes of conics. It is difficult to understand how these correlations can be obtained without finding the envelopes of the families of normals of conics.

73. No doubt that J.L.Lagrange, who himself called his differential calculus "algebraic" and was under evident influence of Apollonius, created his theory of conditional extremum on the base of Book 5 of Conics whose Latin translation appeared in 1710.

74. In Prop. V.53 and V.54, the normals to an ellipse ΒΑΓ with the center Ε, the major axis ΒΓ = 2a, the minor axis 2b, one of the ends of which is the vertex Α, from a point Δ located on the minor axis or its continuation are drawn. From the point Δ, the straight line ΑΕΔ is drawn.

In Prop. V.53 Apollonius proves that if $ΑΔ/ΑΕ \geq a/p = a^2/b^2 = q/b$, only one normal ΔΑ from the point Δ to the ellipse can be drawn.

In Prop. V.54 Apollonius proves that if $ΑΔ/ΑΕ < a/p = a^2/b^2 = q/b$, three normals can be drawn from the point Δ to the arc ΒΑΧ of the ellipse, one along the line ΔΑ, one toward the right half and one toward the left half of the ellipse into which it is divided by its minor axis. The segments of these last normals between the major axis and the arc ΒΑΧ are minimal lines.

In Prop. V.77 Apollonius will prove that the segments of these two normals between the point Δ and the arc ΒΑΧ are maximal lines.

Since ΑΕ = b, the inequalities mentioned in this note are equivalent, respectively, to the inequalities $ΑΔ \geq q$ and $ΑΔ < q$. Therefore
Prop. V.54 essentially coincides with Prop. V.20, and Prop. V.53 is a supplement

to Prop. V.20.

If the point Δ coincides with the curvature center of the ellipse at its vertex A, that is, with the lower cuspidal point of astroid (5.47), then AΔ = q, and only one maximal line ΔA from the point Δ to the ellipse can be drawn.

If three normals are drawn from the point Δ to the arc BAX, and the ends of the normals, which are not directed along the minor axis, are joined, this joining line is parallel to the major axis and meets the minor axis at the point H such that ΔH/ HE = a/p = q/b. This line and the minor axis in this case play the role of the auxiliary hyperbola, these two lines can be regarded as degenerate case of the auxiliary hyperbola.

The point H and passing through it line parallel to the major axis can also be determined on the continuation of the minor axis beyond the point A. If the point H coincides with the vertex A, the line parallel to the major axis is tangent to the ellipse. If the point H is an exterior point of the ellipse, the line parallel to the major axis has no common points with the ellipse.

75. In Prop V.4 and V.8, where minimal lines to a parabola ABΓ with the axis BE from a point E were drawn, the role of the auxiliary hyperbola described in Prop. V.51 is played by a pair of perpendicular straight lines, one of which is the axis itself, the second meets the axis at the point H such that HE = p. If BE > p, these minimal straight lines join the point E with the points of meeting of the perpendicular to the axis with the parabola. If BE ≤p, the single minimal straight line drawn from the point E to the parabola is its axis. If BE = p, the perpendicular touches the parabola at the point B, if BE >p, the perpendicular does not meet the parabola.

In Prop V.5 and V.9, where minimal l straight lines to a hyperbola ABΓ with the axis ΔBE and the center Δ from a point E were drawn, the role of the auxiliary hyperbola described in Prop. V.52 is played by a pair of perpendicular straight lines, one of which is the axis itself, the second meets the axis at the point H such that ΔH/ HE = a/p. If BE > p, these minimal straight lines join the point E with the points of meeting of the perpendicular to the axis with the hyperbola. If BE ≤ p, the single minimal straight line drawn from the point E to the hyperbola is its axis. If ΔH = a, the perpendicular touches the hyperbola at the point B, if ΔH < a, the perpendicular does not meet the hyperbola.

In Prop V.6 and V.10, where minimal straight lines to an ellipse AB with the major axis BEΓ and the center Δ from a point E were drawn, the role of auxiliary hyperbola described in Prop. V.52 is played by a pair of perpendicular straight lines, one of which is the major axis itself, the second meets this axis at the point H such that ΔH/HE = a/p. If BE > p, these minimal straight lines join the point E with the points of meeting of the perpendicular to the major axis with the ellipse. If BE ≤ p, the single minimal straight line drawn from the point E is EB. If ΔH = a, the perpendicular touches the ellipse at the point B. If ΔH > a, the perpendicular does not meet the ellipse. The mentioned pairs of straight lines can be regarded as degenerate auxiliary hyperbolas.

76. In Prop V.55 - V.57 the normals to an arc of an ellipse from points of a quadrant opposite to this arc are drawn by means of the auxiliary hyperbola described in Prop. V.52.

77. In Prop V.58 the normals to a parabola from its exterior point is drawn by means of the auxiliary hyperbola described in Prop. V.51.

78. In Prop V.59 the normals to a hyperbola and an ellipse from their exterior points are drawn by means of the auxiliary hyperbolas described in Prop. V.52.

79. In Prop. V.60 the normal to a hyperbola from a point of its imaginary axis is drawn by means of a pair of perpendicular straight lines, one of which is the imaginary axis itself. This pair of perpendicular lines can be regarded as degenerate auxiliary hyperbola, as in Notes 74 and 75.

80. In Prop. V.61 the normals to a hyperbola from points which are on the other side of its imaginary axis are drawn by means of the auxiliary hyperbola.

81. In Prop V.62 the normals to a parabola from its interior point are drawn by means of the auxiliary hyperbola.

82. In Prop V.63 the normals to a hyperbola and an ellipse from their interior points are drawn by means of the auxiliary hyperbolas.

83. In Prop. V.64 and V.65 the parabola and the hyperbola ΑΒΓ with the vertex A and the point Z below their axes, from which no normal to the conic can be drawn, are considered. Apollonius proves that ZA is minimal of straight lines drawn from Z to the half AB of the section.

84. Prop. V.66 is the analogue of Prop. V.64 and V.65 for one half of the ellipse into which it is divided by the minor axis.

85. In Prop. V.67 the parabola and the hyperbola AB with the vertex A and the axis AE are considered, from the point Z below the axis the single normal ZB to the conic is drawn.

Apollonius proves that ZA is minimal of the lines drawn from Z to the upper half of the conic, if the angle ZAE is acute. The straight line ZA is not a normal to the conic, since the normal to a conic at its vertex is the axis of the conic.

Prop. V.67 shows that in this case, although the segment of the normal ZB between the conic and its axis is a minimal straight line, whole line ZB is neither minimal nor maximal straight line.

In the case of Prop. V.64 and V.65 the auxiliary hyperbolas (5.27) and (5.38) defined in Notes 60 and 67 on this book have no common point with the conic, and in the case of Prop. V.67 these auxiliary hyperbolas touch the conics at the points B.

Prop. V.67 can be obtained by the limiting process from Prop. V.64 or V.65 and V.72, by tending of two imaginary conjugate or real straight lines drawn from a point below the axis of the parabola or the hyperbola to two imaginary conjugate or real points of intersection of the conic with the auxiliary hyperbola toward one another.

86. Prop. V.64, V.65, and V.67 show that, although in Prop. V51 and V.52 Apollonius considered auxiliary hyperbolas only when they intersect the conics at two points, he, apparently, imagined auxiliary hyperbolas when they touch the conics or have no common points with them.

87. In Prop. V.68 the parabola AEB with the vertex B and the axis BΓ is considered. The tangent AΔ and ED to the parabola are drawn. Apollonius proves that if the point E is between the points A and B then ΔE is smaller than ΔA.

88. Prop. V.69 is the analogue of Prop. V.68 for a hyperbola.

89. In Prop. V.70 an ellipse ABΓΔ with the major axis AΓ and the minor axis BΔ is considered. If points P and Θ of the ellipse are between the points B and Γ, the straight lines PH and ΘH tangent to the ellipse are drawn.

Apollonius proves that if the point P is nearer to the axis BΔ than Θ, the straight line HP is greater than HΘ.

90. In Prop. V.71 the same ellipse as in Prop. V.70 is considered. If X and Φ are points of the ellipse, X is between the points A and B, and ς is between the points B and Γ, straight lines XE and ΘΦ perpendiculars to the axis AΓ, and XE >Θς, the straight lines XY and ΘY tangent to the ellipse are drawn.

Apollonius proves that the line XY is greater than the line ΘY.

91. In Prop. V.72 a parabola or hyperbola ABΓ with the vertex Γ and the axis ΓE is considered. From a point Δ below the axis, two normals ΔA and ΔB to the conic are drawn.

Apollonius proves that if B is between the points A and Γ, the segment ΔB is the maximal of straight lines drawn from Δ to the arc ΓBA, and ΔA is the minimal of straight lines drawn from Δ to the complement of the arc ΓB to the upper half of the conic.

92. In Prop V.73 the ellipse ABΓ with the center Δ, the major axis AΓ, and the minor axis BΔ is considered. From the point Z below the major axis, not on the continuation of the minor axis, and on the same side of the minor axis as A,

the single normal ZΘ to upper half of the ellipse is drawn.

Apollonius proves that ZΘ is the maximal of straight lines drawn from Z to the upper half of the ellipse, and ZA is the minimal of these straight lines.

In this case, one branch of auxiliary hyperbola (5.39) (see Note 67 on this book) has no a common point with the ellipse, and the other branch passes through the points Θ, Δ and Z. Each of these three points together with the asymptotes of the auxiliary hyperbola parallel to the axes of the ellipse entirely determine this hyperbola.

93. In Prop.V.74 the same ellipse as in Prop. V.73 is considered. From the point Z below the major axis, two normals ZH and ZΘ to the upper half of the ellipse are drawn, and the normal ZΘ intersects the minor axis, and the normal ZH does not intersects it. Apollonius proves that ZΘ is the maximal of straight lines drawn from Z to the upper half of the ellipse and the line is the minimal of these straight lines.

Since in Prop.V.74 the perpendicular ZN is equal to the segment Λ, in this case, one branch of the auxiliary hyperbola (5.39) touches the ellipse at the point H and other branch passes through the points Θ, Δ, Z and each of these three points and the point H together with the asymptotes of the auxiliary hyperbola parallel to the axes of the ellipse entirely determine this hyperbola.

In Prop. V.74 the straight lines ZA, ZO, ZΠ, ZT, ZΛ, ZH, ZP, ZΞ , ZΣ, ZB, ZΦ are drawn from Z to the upper half of the ellipse, and Apollonius proves that each from these lines is smaller than the following one. It implies that, although the segment of ZH between the ellipse and its major axis is a minimal straight line, the whole ZH is neither a minimal, nor a maximal straight line.

Prop. V.74 also can be obtained by the limiting process from Prop. V.73 and V.75, analogously to the transition from Prop. V.64 or V.65 and V.72 to Prop. V.67.

Prop. V.73 and V.74 also show that, Apollonius apparently imagined the auxiliary hyperbolas when they touch conics or have no real point common with them.

94. In Prop. V.75 the ellipse ΑΒΓ with the center Ξ, the major axis ΑΓ, and the minor axis ΒΞ is considered. From the point E below the major axis three normals EH, EΓ, EΔ to upper half of the ellipse are drawn, and the point Δ is between Γ and B, the straight line EH intersects the minor axis, EΓ and EΔ do not intersect this axis.

Apollonius proves that the EH is the maximal of all straight lines drawn from point E to the upper half of the ellipse, EΓ is the maximal of straight lines drawn from E to the arc ΓΔ, and the line EΔ is the minimal of the lines drawn from Ε to the arc ΓΗ.

In this case one branch of the auxiliary hyperbola (5.39) passes through the Δ and Γ, and the second branch passes through Η, Ξ, and Ε, and these five points entirely determine the auxiliary hyperbola.

95. In Prop. V.76 the same ellipse as in Prop. V.75, and the point Ε below the major axis on the continuation of the minor axis are considered, and it is assumed that no normal from Ε to the upper half of the ellipse differing from the minor axis can be drawn. Apollonius proves that ΕΒ is the maximal of straight lines drawn from Ε to the upper half of the ellipse.

The comparison of this proposition with Prop. V.53 shows that, in this case, $ΕΒ \geq q = a^2/b$.

96. In Prop. V.77 the same ellipse as in Prop. V.75 and V.76 and the point Ε on the minor axis below the major axis are considered, and from the point Ε to the upper half of the ellipse three normals - ΕΒ directed along the minor axis, and two normals symmetric with respect to the minor axis, can be drawn.

Apollonius proves that the last two normals are the maximal of straight lines drawn from Ε to upper half of the ellipse, and ΕΒ is the minimal of these straight lines.

The comparison of this proposition with Prop. V.54 shows that in this case $ΕΒ < q = a^2/b$.

COMMENTARY ON BOOK SIX

Preface to Book 6

1. "Equal" conics in modern geometry are called "congruent", that is, conics that can be mapped one to other by a motion in the plane. Motions in the plane which do not change the orientations of figures are transformations (1.4) where $A = E = \cos\varphi$, $B = -D = \sin\varphi$. These transformations are products of

turns, that is elliptic turns (1.94) for a = b by parallel translations (1.59). Motions in the plane which change the orientations of figures are transformations (1.4) where $A = - E = \cos\varphi$,
$B = D = \sin\varphi$.

2. “Similar” conics are conics which can be mapped one to another by similitudes in the plane, that is products of motions by homothety (1.57).

Definitions to Book 6

3. Apollonius defines equal conics as those that can be superimposed over one another without an excess and a defect. Analogous definition of “equal things” was given by Euclid in the Axiom 4 of his Elements.

4. Apollonius defines similar conics by means of proportionality of coordinates of points of both conic, that is he expresses similarity by means of a motion and a homothety.

5. Unlike modern mathematicians who use the term “arc” for any segments of all curved lines, ancient mathematicians used this term only for segments of circumferences of circles. Apollonius called arcs of conics “segments” or “parts”. Apollonius also used the term “segment” for plane figures bounded by arcs of conics and chords joining the ends of these arcs.
Apollonius called these chords “bases” of these segments.

Although the word “chord” has Greek origin, Euclid before Apollonius and Ptolemy after him called chords of circles “straight lines in a circle”. The Latin word chorda appeared as the translation of Arabic word watar meaning “string”, which is a translation of Indian word jiva, since Indians represented this line as the string of an arch. They called arcs of circumferences of circles “archs”, and lines joining the midpoints of an arc and a chord “arrows”.

From the word jiva, the term “sine” also came. Since the line of sine of angle α is equal to half the chord of the central angle 2α, Indians who introduced the line of sine first called it ardha-jiva - “half the string” but later called it jiva. Arab translators, who met this word in new sense, wrote it jib. Latin translators read this word as jayb - “pocket”, and translated it as sinus.

6. Apollonius calls “diameter of a segment” the locus of the midpoints of chords parallel to the base of the segment.

The Indian name of this diameter in a circle meaning arrow was translated by Arabic word sahm and by Latin word sagitta.

In Arabic translation of Conics, by the term sahm the axes of conics are called.

7. Apollonius calls “vertex of a segment” the end of its diameter on its arc.

The vertex of a segment of a parabola was considered by Archimedes in Quadrature of a section of right-angled cone, where he proved that the area of a segment of a parabola is equal to 4/3 of the area of triangle whose vertex and base coincide with the vertex and the base of the segment.

8. On the eidos corresponding to a diameter of a conic, that is on the rectangle whose sides are the latus transversum and the latus rectum corresponding to this diameter, see Note 39 on Book 1.

Propositions VI.1 - VI.10 on equality and inequality of conics

9. In Prop. VI.1 Apollonius proves that two parabolas in rectangular coordinate systems are equal if their latera recta are equal, and latera recta of two equal parabolas are equal.

10. Ancient mathematicians used the notion of equality of figures in the sense indicated in Note 3 on this book for straight and curved lines, but under equal polygons and polyhedra they understood not congruent, but equiareal and equivolume figures. Congruent polygons and polyhedra ancient mathemathicians called "equal and similar figures".

11. In Prop. VI.2 Apollonius proves that two hyperbolas or two ellipses in rectangular coordinate systems are equal if their *eidoi* corresponding to their axes are equal and similar, and the *eidoi* of equal hyperbolas and ellipses are equal and similar.

The equality and similarity of the *eidoi* of two conics, that is the congruency of these *eidoi*, imply equalities of their latera transversa 2a and latera recta $2p = 2b^2/a$ and are equivalent to the equalities of their axes 2a and 2b.

12. The corollaries of Prop. VI.1 and VI.2 are analogues of these propositions for two parabolas, hyperbolas and ellipses in oblique coordinate systems with equal coordinate angles.

In the translation of Thabit ibn Qurra, this corollary is joined with Prop. VI.3.

13. In Prop. VI.3 Apollonius proves that an ellipse cannot be equal to a hyperbola, or a parabola cannot be equal to a hyperbola.

14. In Prop. VI.4 Apollonius proves that each axis of an ellipse bisects it and its interior domain on two congruent parts.

15. In Prop. VI.5 Apollonius proves that each diameter of an ellipse which is not its axis also bisects ellipse and its interior domain on two congruent parts.

Prop. VI.4 and VI.5 are generalizations of well-known property of diameters of a circle indicated in Definition 17 in Book 1 of Euclid's Elements.

This property was proven in 6th c. B.C. by Thales.

16. In Prop. VI.6 Apollonius proves that if two arcs of two conics are equal, these conics themselves are also equal.

17. In Prop. VI.7 Apollonius proves that the axes of a parabola and a hyperbola bisect segments of these conics, whose bases are perpendicular to their axes, into two congruent parts.

18. Prop. VI.8 is the analogue of Prop. VI.7 for an ellipse.

19. In Prop. VI.9 Apollonius proves that the arcs of equal conics that are on equal distances from their vertices are equal, and arcs of these conics which are on unequal distances from their vertices are not equal.

20. In Prop. VI.10 Apollonius proves that in unequal conics there are no equal arcs. Propositions VI.11 - VI.27 on similarity and dissimilarity of conics

21. In Prop. VI.11 Apollonius proves that all parabolas are similar to one another.

If two parabolas do not have common axis and vertex, they can be led to this position by a motion. If two parabolas have common axis and vertex, they are determined in rectangular coordinate system by equations $y^2 = 2px$ and $y'^2 = 2p'x'$. In the system of rectangular coordinates with the origin in common vertex of the parabolas and the axis 0x directed along their common axis if $p'/p = k$, the first parabola can be mapped to the second one by homothety (1.57).

22. In Prop. VI.12 Apollonius proves that all hyperbolas with similar *eidoi* are similar one to another and all ellipses with similar *eidoi* are also similar one to another.

If two hyperbolas or two ellipses with similar *eidoi* do not have common axis and center, they can be led to this position by a motion.

If two hyperbolas or two ellipses with similar *eidoi* have common axis and center, they are determined in rectangular coordinates with the origin at their center and the axes 0x and 0y directed along their axes by the equations for hyperbolas $x^2/a^2 - y^2/b^2 = 1$ and $x'^2/a'^2 - y'^2/b'^2 = 1$, and for ellipses $x^2/a^2 + y^2/b^2 = 1$ and $x'^2/a'^2 + y'^2/b'^2 = 1$.

Since the *eidoi* of two conics are similar, the sides $2a$, $2p = (2b)^2/2a$ and $2a'$, $2p' = (2b')^2/2a'$ of these rectangles are proportional. Therefore the axes $2a$, $2b$ and $2a'$, $2b'$ also are proportional, and if $a'/a = b'/b = k$, the first hyperbola can be mapped to the second one by homothety (1.57), and the first ellipse can be mapped to the second one by the same homothety.

In the case of two hyperbolas, the condition of their similarity is equivalent to the condition of equality of angles between their asymptotes.

If a hyperbola is determined by equation (1.46) in rectangular coordinate system, its latus transversum is equal to $2a$, latus rectum is equal to $2p = 2b/a$, and if the angle between asymptotes of the hyperbola is φ , $\tan(\varphi/2) = b/a$. If the *eidoi* of these hyperbolas are similar, the ratios $2p/2a = b^2/a^2$ of these conics are equal one to other, and therefore for these conics $b/a = b'/a'$.

23. Prop. VI.11 and VI.12 show that all conics with equal eccentricities are similar one to another.

If $\varepsilon = 1$, conics are parabolas and the assertion follows from Prop. VI.11.

If $\varepsilon > 1$, conics are hyperbolas and since in this case $\varepsilon^2 = p/a + 1$, and the equality $\varepsilon = \varepsilon'$ implies proportion $p/a = p'/a'$, hyperbolas have similar *eidoi* and the assertion follows from Prop. VI.12.

If $0 < \varepsilon < 1$, conics are ellipses, and since in this case $\varepsilon^2 = 1 - p/a$, the equality $\varepsilon = \varepsilon'$ also implies proportion $p/a = p'/a'$, ellipses have similar *eidoi*, and the assertion follows from Prop. VI.12.

The assertion is also valid for circumferences of circles whose eccentricity is equal to 0.

24. Analogously to Prop. VI.12, the assertions can be proven that all hyperbolas with dissimilar *eidoi* can be obtained from one another by affine transformations, and all ellipses with dissimilar *eidoi* can be obtained from one another by affine transformations.

If two hyperbolas or two ellipses have no common axes and center, they can be mapped into this position by a motion.

If two hyperbolas or two ellipses have the same axes and the same center, they are determined in the rectangular coordinate system with the origin at the center of these conics and the axes 0x and 0y directed along the axes of these conics by the equations $x^2/a^2 - y^2/b^2 = 1$, $x'^2/a'^2 - y'^2/b'^2 = 1$ and $x^2/a^2 + y^2/b^2 = 1$, $x'^2/a'^2 + y'^2/b'^2 = 1$.

If the *eidoi* of two conics are dissimilar, the sides 2a, 2p and 2a', 2p' of these rectangles are not proportional.

If we denote $a'/a = h$, $b'/b = k$, the first hyperbola or ellipse can be mapped to the second one by the transformation

$$x' = hx , \quad y' = ky. \quad (6.1)$$

The product of a motion by affine transformation (6.1) is a general affine transformation.

Our assertion also follows from the fact that any two conics, which are not pairs of straight lines, can be mapped one to another by a projective transformation, and projective transformations preserving the straight line at infinity determine in the remaining part of the projective plane affine transformations.

25. Prop. VI.13 is the analogue of Prop. VI.12 for hyperbolas and ellipses whose equations are given in oblique coordinate systems with equal coordinate angles.

26. In the proof of Prop. VI.13, after the words “So the angles at points Γ and E are equal” Thabit ibn Qurra has added to the text of Apollonius the words: “because of what is proven in the Introduction preceding this treatise”. Under the Introduction to Conics, two supplements by editors of the Arabic translation of Conics brethren Banu Musa are understood here. These supplements are published with English translation by G.J.Toomer [Ap7,pp.619 -650]. In this case, Thabit ibn Qurra refers to Lemma 7 [Ap7, p. 642].

In the same proof, Thabit ibn Qurra made an analogous reference to Lemma 8 by Banu Musa [Ap7, p.644] after the proportion $ΓN.NΘ / ΓN^2 = IX.XO/ EΞ^2$.

27. In Prop. VI.14 Apollonius proves that parabolas cannot be similar to hyperbolas and ellipses.

28. In Prop. VI.15 Apollonius proves that hyperbolas cannot be similar to ellipses.

29. In Prop. VI.16 Apollonius proves that opposite hyperbolas are congruent.

The assertion follows from the fact that opposite hyperbolas are symmetric with respect to their imaginary axis.

30. In Prop. VI.17 Apollonius finds the conditions of similarity of segments of two similar parabolas.

31. In Prop. VI.18 Apollonius finds the conditions of similarity of segments of two similar hyperbolas and ellipses.

32. In the proof of Prop. VI.18 after the words “triangle ΓYI is similar to triangle MςX” Thabin ibn Qurra, like in Prop. VI.13, has added to the text of Apollonius a reference to Introduction by Banu Musa (see Note 26 on this book). In this case, Thabin ibn Qurra refers to Lemma 9 [Ap7, p. 646].

In the same proof, Thabit ibn Qurra made an analogous reference to the same Lemma 9 in Banu Musa’s Introduction after the words “triangle IXΓ is similar to triangle ςMO”.

33. In Prop. VI.19 Apollonius finds the conditions of similarity of arcs of two parabolas and two similar hyperbolas based of symmetry of these conics.

34. In Prop. VI.20 Apollonius finds the conditions of similarity of arcs of two similar ellipses based on symmetry of ellipses.

35. In Prop. VI.21 Apollonius considers another case of similarity of segments of two similar parabolas.

36. Prop. VI.22 is the analogue of Prop. VI.21 for similar hyperbolas and el-

lipses.

37. In Prop. VI.23 Apollonius proves that dissimilar conics do not contain similar arcs.

38. In the proof of Prop. VI.23, after the proportion $K\Pi.\Pi\Sigma:M\Pi^2 = A P.P\Xi:NP^2$, Thabit ibn Qurra has added a reference to Lemma 8 by Banu Musa (see Note 26 on this book).

39. In Prop. VI.24 Apollonius proves that arcs of a parabola cannot be similar to arcs of a hyperbola or an ellipse, and that arcs of a hyperbola cannot be similar to arcs of an ellipse.

40. In Prop. VI.25 Apollonius proves that arcs of a parabola, a hyperbola, and an ellipse cannot be similar to arcs of the circumference of a circle.

41. In Prop. VI.26 Apollonius proves that two hyperbolas cut off from the surface of a circular cone by two parallel planes are similar.

42. Prop. VI.27 is the analogue of Prop. VI.26 for ellipses.

The analogous proposition for parabolas is not formulated by Apollonius since in Prop. VI.11 he proved that all parabolas are similar.

Propositions VI.28 - VI.33 on placing conics
on surface of right circular cone

43. In Prop. VI.28 Apollonius shows how to place on a given right circular cone a parabola equal to a given parabola.

This parabola is cut off from the surface of the cone by a plane parallel to a rectilinear generator of the cone.

44. In the proof of Prop. VI.28, the words “and such that it is equal to section ΔE” absent after the words “Then I say that no other section, apart from this one, can be found in [this] cone such that the point of its vertex [which is the end of the axis] lies on AB” in the Thabit ibn Qurra’s translation of Book 6 of Conics were added by Halley [Ap2].

45. Prop. VI.29 is the analogue of Prop. VI.28 for a hyperbola.

If the angle between the axis of the cone and its rectilinear generators is α, and the eccentricity of the hyperbola is equal to $\varepsilon > 1$, the hyperbola is cut off from the surface of the cone by a plane whose angle β with the axis of the cone is connected with α and ε by the correlation (1.42).

The condition of solvability of this problem indicated by Apollonius is equivalent to the condition that the angle between the asymptotes of the hyperbola must not be greater than the angle 2α at the vertex of the cone.
The angle between the asymptotes of the hyperbola cut off from the surface of a right cone is maximal in the case where the plane of the hyperbola is perpen-

dicular to the plane of the base of the cone; in this case, the angle between the asymptotes of the hyperbola is equal to 2α.

46. Prop. VI.30 is the analogue of Prop. VI.29 for an ellipse.

47. In Prop. VI.31 Apollonius constructs a right circular cone containing a given parabola and similar to a given right circular cone.

Prop. VI.31 is inverse to Prop. VI.28.

48. Prop. VI.32 is the analogue of Prop. VI.31 for a hyperbola.

Prop. VI.32 is inverse to Prop. VI.29.

49. Prop. VI.33 is the analogue of Prop. VI.31 and VI.32 for an ellipse.

Prop. VI.33 is inverse to Prop. VI.30.

COMMENTARY ON BOOK SEVEN

Preface to Book 7

1. The eidos corresponding to a diameter of ellipse (1.45) or hyperbola (1.46) in the coordinate system whose axes are this diameter and the diameter conjugate to it is a rectangle bounded by the latus transversum 2a and the latus rectum $2p = 2b^2/a$. On this term see Note 39 on Book 1.

"Wonderful and beautiful theorems on diameters" are famous propositions of Apollonius on conjugate diameters of hyperbolas and ellipses connected with hyperbolic and elliptic turns.

2. In the preface to Book 7 there is information on lost Book 8, which contained problems solved by means of propositions of Book 7. This information was used by Ibn al-Haytham and Halley in their attempts to restore Book 8 of Conics.

Propositions VII.1 - VII.5 preparatory for proof of theorems on conjugate diameters and *eidoi* of conics

3. In Prop. VII.1 the parabola AB with the axis ΔAΓ and the vertex A is considered. The line AΔ is equal to the latus rectum 2p. From a point B of the parabola the ordinate BΓ to axis is dropped, the line AB is drawn. Apollonius proves that $AB^2 = A\Gamma.\Gamma\Delta$.

This equality follows from equation (0.3) in the system of rectangular coordinates, since $AB^2 = x^2+y^2 = x^2 +2px = x(x +2p) = A\Gamma.\Gamma\Delta$

The proof is based on Prop. I.11.

4. In Prop. VII.2 the hyperbola containing a point B with the continued axis AΓE, the latus transversum AΓ = 2a, and the latus rectum AΔ = 2p are considered. From the segment AΓ such a segment AΘ is cut off that $\Gamma\Theta//\Theta A = A\Gamma/A\Delta = a/2$. From the point B of the hyperbola ordinate BE to the axis is dropped. Apollonius proves that the proportion

$$AB^2/\Theta E.EA = A\Gamma/ \Gamma\Theta \quad (7.1)$$

holds. This equality is proven for two cases: when the vertex of hyperbola is the point A, and when the vertex is the point Γ.

Let the hyperbola be determined by equation (0.10) in rectangular coordinate system, and let the coordinates of the point B be equal to x_o and y_o.

Then in the case where the vertex of the hyperbola is the point Γ, the abscissa of the point A is equal to -2a and $(2a + x_o)^2 + y_o^2 = 4a^2 + 4ax_o + x_o^2 + 2px_o + (p/a)x_o^2 = 4a^2 + 2(2a +p)x_o^2+ x_o^2(p + a)/a$, $EA = A\Gamma + \Gamma E = 2a + x_o$, $\Theta E = \Theta\Gamma + \Gamma E = 2a^2/(p + a) + x_o$, $A\Gamma/GQ = 2a/(2a^2/(p +a)) = (p + a)/a$. $\Theta E.EA.A\Gamma/\Gamma\Theta = (2a^2/(p +a) + x_o)(2a + x_o)(p + a)/a = 4a^2 + 2(2a +p)x_o + x_o^2(p +a)/a = AB^2$,

hence equality (7.1) follows.

In the case where the vertex of the hyperbola is the point A, that is this point is the origin of coordinates,
$AB^2 = x_o^2 + y_o^2 = x_o^2 + 2px_o + (p/a)x_o^2 = 2px_o + x_o^2(p + a)/a$, $EA = x_o$, $\Theta E = \Theta A + AE = 2ap/(p + a) + x_o$, $A\Gamma/\Gamma\Theta = 2a/(2a^2/(p +a)) = (p + a)/a$, as in the first case, $\Theta E.EA.A\Gamma/\Gamma\Theta = (2ap/(p +a) + x_o)x_o(p + a)/a = 2px_o + x_o^2(p +a)/a = AB^2$,

hence equality (7.1) also follows.

5. In Prop. VII.2, segments AΘ in both cases are said to be ”with the same ratio”. Halley, according to the term used by Apollonius in Books 1, 3, and 4, translated this expression by the word “homologue” (see Note 66 to Book 1). Toomer, following Halley, translated this expression by the same word.

In our commentary we call segments AΘ and analogous segments “homologous segments”. G.J.Toomer calls points Q “homologous points”, here “homologous” means corresponding (see Note 67 to Book 1).

Thabit ibn Qurra , like al-Himsi in Books 1, 3 and 4, translated this expression by the words al-shabih al-nisba, meaning “with the similar ratio”. No doubt that Arabic translators confused the Greek words ομος - “the same” and ομοιος - “similar”, from which the mathematical terms “homomorphism” and “homeomorphism“ came. Halley, just like in Preface to Book 5 (see Note 2 to Book 5), corrected this error.

6. In Prop. VII.3 an ellipse ΑΒΓ with the major axis $A\Gamma = 2a$ and the latus rectum $A\Delta = 2p$ is considered. From the point A, on the continuation of the major axis such a segment AΘ is cut off that $\Gamma\Theta / \Theta A = A\Gamma/ A\Delta = 2a/2p$.
From a points B of this ellipse, the ordinate BE to the major axis is dropped.

Apollonius proves that proportion (7.1) holds. This equality is proven in two cases where the vertex A is on the left or right end of the major axis of the ellipse.

The proof of this proposition is the analogue of the proof of Prop. VII.2.

7. In Prop. VII.3, like in Prop. VII.2, segments AΘ in both cases are called “homologous segments” (see Note 5 on this book).

8. In Prop. VII.4 a hyperbola or an ellipse with the axis ΑΓ and the center Θ

is considered.

For an arbitrary point B of the hyperbola, the straight line BΔ tangent to it and meeting the axis at a point Δ is drawn. Through the center Θ of the hyperbola, a straight line ΘH parallel to the line BΔ and equal to half the diameter of the conjugate hyperbola which is conjugate to the diameter ΘB is drawn.

From the point B, the ordinate BE to the axis is dropped.

For an arbitrary point B of the ellipse, tangent straight line BΔ to it meeting the major axis at point Δ is drawn. Through the center Θ of the ellipse, a straight line ΘH parallel to BΔ and equal to half the diameter conjugate to the diameter ΘB is drawn. From the point B, the ordinate BE to the axis is dropped.

Apollonius proves that in both cases, the proportion

$$B\Delta^2/\Theta H^2 = \Delta E/E\Theta \quad (7.2)$$

holds. If the hyperbola or the ellipse are determined by equations (1.46) and (1.45) in rectangular coordinate system, and coordinates of the points B and H are denoted, respectively, x_0, y_0 and x_1, y_1, the parallelity of the lines BΔ and ΘH implies that these lines are the hypotenuses of two similar rectangular triangles. Therefore the ratio BΔH is equal to the ratio of horizontal catheti of these triangles.

The point Δ is the pole of the line BE and in the case of the ellipse can be obtained from the point E by the inversion with respect to this ellipse. According to Prop. I.37, the abscissa of the point Δ is equal to a^2/x_0 and the horizontal cathetus ΔE of the first triangle is equal to $a^2/x_0 - x_0 = (a^2 - x_0^2)/x_0$. The horizontal cathetus of the second triangle is equal to x_1. Since the diameters ΘB and ΘH are conjugate, $x_1 = ay_0/b$. Therefore, in the case of the ellipse, $x_1^2 = a^2y_0^2/b^2 = a^2(1 - x_0^2/a^2) = a^2 - x_0^2$, and x_1 is the mean proportional between lines ΔE and ΘE = x_0. Hence proportion (7.2) is obtained for the ellipse.

In the case of the hyperbola, proportion (7.2) can be obtained analogously.

9. In Prop. VII.5 the parabola AB with the axis AH, the vertex A, and the latus rectum AΓ perpendicular to the axis is considered. From a point B of the parabola the tangent straight line BΔ meeting the axis at a point Δ is drawn, the ordinate BZ to the axis is dropped, and the diameter BI is drawn. When continued, the line BI meets AΓ at the point E. Apollonius proves that the latus rectum 2p' corresponding to the diameter BI is equal to the sum of the latus rectum 2p corresponding to the axis and 4BE.

Let the parabola be determined by equation (0.3) in the rectangular coordinate system and by analogous equation $y'^2 = 2p'x'$ in oblique coordinate system whose axis 0x' is the diameter BI. If the rectangular coordinates of the point B are x_o and y_o, the equation of the diameter BI is $y = y_o$.

The oblique coordinates x' and y' of points of the parabola with rectangular coordinates x and y can be obtained as follows: the ordinate y' is the hypotenuse of the rectangular triangle with the vertical cathetus $y - y_o$. This hypotenuse is parallel to the straight line tangent to the parabola at the point B. The equation of this straight line is $yy_o = p(x + x_o)$. Therefore the ratio of the length of the vertical cathetus of this triangle to the length of its horizontal cathetus is equal to p/y_o and the length of the horizontal cathetus is equal to $(y - y_o)y_o/p = (yy_o - 2px_o)/p = yy_o/p - 2x_o$. Therefore $x' = x - x_o - yy_o/p + 2x_o = x + x_o - yy_o/p$, $y'^2 = (y - y_o)^2 + (yy_o/p + 2x_o)^2 = y^2 - 2yy_o + y_o^2 + y^2y_o^2/p^2 + 4yy_ox_o/p + 4x_o^2 = 2px - 2yy_o + 2px_o + 4pxx_o/p + 4yy_ox_o/p + 4x_o^2 = (2p + x_o)(x + x_o - yy_o/p)$.

Hence the latus rectum $2p'$ corresponding to the diameter BI is equal to

$$2p' = 2p + 4x_o , \qquad (7.3)$$

where x_o = BE. Since coordinates x_o, y_o of the point B satisfy equation (0.3), they are connected by the condition $y_o^2 = 2px_o$, therefore $x_o = y_o^2/2p$ and the latus rectum $2p'$ corresponding to the diameter $y = y_o$ is equal to $2p + 2y_o^2/p$.

Propositions VII.6 - VII.31 on conjugate diameters of conics

10. In Prop. VII.6, a hyperbola AB with the latus transversum $A\Gamma = 2a$, the axis ΓE, and the center Θ is considered. From an arbitrary point B of the hyperbola, the ordinate BE to the axis, the tangent straight line $B\Delta$ meeting the axis at point Δ, and the diameter $B\Theta K$ are drawn. Through the point Θ, the diameter $Z\Theta H$ of the conjugate hyperbola conjugate with the diameter $B\Theta K$ is drawn. From the point A, a line $A\Lambda$, parallel to diameter $Z\Theta H$, and meeting the hyperbola at a point Λ is drawn. From the point Λ of the hyperbola, the ordinate ΛM to its axis is dropped. On the line $A\Gamma$ points N and Ξ are taken, such that AN = CX = $2ap/(a + p)$, and AX = CN = $2a^2/(a + p)$ being "gomologous segments" (see Note 5 on this book). Apollonius proves that

$$BK^2/ZH^2 = \Xi M/MN \qquad (7.4)$$

holds. If the hyperbola is determined by equation (1.46) in rectangular coordinate system, conjugate diameters BK and ZH can be obtained from its axes by hyperbolic turn (1.95). Therefore if the coordinates of the vertex of the hyperbola are $x = a$, $y = 0$, the coordinates of the end of the diameter BK are equal to $x' = a\ ch\varphi$, $y' = b\ sh\varphi$, and the square of the length of half the diameter BK is equal to

$$a'^2 = a^2 ch^2\varphi + b^2 sh^2\varphi \ . \quad (7.5)$$

If the coordinates of the vertex of the conjugate hyperbola are equal to $x = 0$, $y = b$, the coordinates of the end of the diameter ZH are $x' = a\ sh\varphi$, $y' = b\ ch\varphi$ and the square of the length of half the diameter ZH is equal to

$$b'^2 = a^2\ sh^2\varphi + b^2\ ch^2\varphi \ . \quad (7.6)$$

Therefore the left hand side of equality (7.4) can be rewritten in the form

$$BK^2/ZH^2 = a'^2/b'^2 = (a^2 ch^2\varphi + b^2 sh^2\varphi)/(a^2 sh^2\varphi + b^2 ch^2\varphi). \quad (7.7)$$

Since the straight lines ΘH and AΛ are parallel, these straight lines are hypotenuses of two similar rectangular triangles. The horizontal and vertical catheti of the first of these triangles are, respectively, equal to $x' = a\ sh\varphi$ and $y' = b\ sh\varphi$. The horizontal and vertical catheti of the second triangle are straight lines AM and ΛM . Corresponding catheti of these triangles are proportional, that is $a\ sh\varphi = k\ AM$, $b\ sh\varphi = k\ LM$, and equality (7.7) can be rewritten in the form

$$BK^2/ ZH^2 = ((a^2/b^2)\Lambda M^2 + (b^2/a^2)AM^2)/(AM^2 + \Lambda M^2). \quad (7.8)$$

The lines AM and ΛM can be regarded as the abscissa and ordinate of the point Λ of hyperbola (0.10), and if we denote $AM = z$, then $\Lambda M^2 = 2pz + (p/a)z^2$. Therefore equality (7.8) can be rewritten in form

$$BK^2/\Gamma H^2 = ((a^2/b^2)(2pz + (p/a)z^2) + (b^2/a^2)z^2)/(z^2 + 2pz + (p/a)z^2), \quad (7.9)$$

or since $b^2/a^2 = p/a$,

$$BK^2/\Gamma H^2 = ((a/p)(2p + (p/a)z)) + (p/a)z)/(z + 2p + (p/a)z) =$$
$$(2a + z + (p/a)z)/(2p + z + (p/a)z = (2a^2/(a + p) + z)/(2pa/(a + p) + z). \quad (7.10)$$

Equality (7.10) is equivalent to equality (7.4).In the case where the point Λ coincides with the vertex A, the line AΛ is tangent to the hyperbola at the point A, AM = 0, the diameters BK and ZH coincide with the axes of the hyperbola and both hand sides of equality (7.7) are equal to $a^2/b^2 = a/p$.

11. In Prop. VII.7 an ellipse AB with an axis AΓ and the center Θ is considered. From an arbitrary point B of the ellipse ordinate BE to axis AΓ, the tangent straight line BΔ meeting the continued axis at a point Δ, and the diameter BΘK are drawn. Through the point Θ the diameter ZΘH conjugate with the diameter BΘK is drawn. Through the point A a line AΛ parallel to ΓΘH and meeting the ellipse at a point Λ is drawn. From the point Λ of the ellipse the ordinate ΛM to the axis is dropped. On continued straight line AΓ, points N and Ξ are taken, in the case when AΓ is the major axis. such that AN = XΞ = 2pa/(a - p), AΞ = XN = $2a^2/(a - p)$ being "homologous segments". Apollonius proves that equality (7.4) holds. If the ellipse is determined by equation (1.45) in a rectangular coordinate system, conjugate diameters BK and ZH can be obtained from its axes 2a and 2b by elliptic turn (1.94). Therefore if the coordinates of the end of the major axis are x = a , y = 0, the coordinates of the end of the diameter BK are x' = a cosφ , y' = b sinφ and the square of the length of half the diameter BK is equal to

$$a'^2 = a^2\cos^2\varphi + b^2 \sin^2\varphi \quad , (7.11)$$

if the coordinates of an end of the minor axis of the ellipse are x = 0, y = b, the coordinates of an end of diameter ZH are x' = -a sinφ , y' =b cosφ and the square of the length of half the diameter ZH is equal to

$$b'^2 = a^2 \sin^2\varphi + b^2 \cos^2\varphi \quad . (7.12)$$

The proof of Prop. VII.7 is analogous to the proof of Prop. VII.6.

In the case when AX is the minor axis points N and Ξ are taken such that AN = XΞ = $2b^2/(q - b)$, AΞ = XN = 2bq/(q - b), and the proof is analogous, as in the case when AX is the major axis.

12. In the case where the point Λ is the end of the minor axis of the ellipse, the angular coefficient of the line AΛ is equal to b/a. Since in the case of the ellipse (1.45), angular coefficients k_1 and k_2 of two conjugate diameters are connected by the correlation $k_1k_2 = -b^2/a^2$, and the angular coefficient of the diameter ΓH is equal to b/a, then the angular coefficient of the conjugate to its

diameter BK is equal to -b/a. Therefore the diameters BK and ZH are symmetric with respect to the minor axis, and the equality of the lengths of these diameters follows.

In the case of hyperbola (1.46), angular coefficients $k_1$and k_2 of two conjugate diameters are connected by the correlation $k_1k_2 = b^2/a^2$, straight lines with angular coefficients b/a and -b/a passing through the center are the asymptotes. Therefore diameters with finite length cannot be equal to conjugate to them diameters.

13. In Prop. VII.8 the same hyperbola and ellipse as in Prop. VII.6 and VII.7 are considered. Apollonius proves the proportion

$$A\Gamma^2/(BK + ZH)^2 = N\Gamma.M\Xi / (M\Xi + (MN.M\Xi)^{1/2})^2. \quad (7.13)$$

14. In Prop. VII.9 the same hyperbola and ellipse as in Prop. VII.6 and VII.7 are considered. Apollonius proves that proportion

$$A\Gamma^2/(BK - ZH)^2 = N\Gamma.M\Xi / (M\Xi - (MN.M\Xi)^{1/2})^2 \quad (7.14)$$

holds

15. In Prop. VII.10 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that proportion

$$A\Gamma^2/BK.ZH = N\Gamma/ (MN.M\Xi)^{1/2} \quad (7.15)$$

holds

16. In Prop. VII.11 the same hyperbola as in Prop VII.6 is considered.

Apollonius proves that proportion

$$A\Gamma^2/ (BK^2 + ZH^2) = N\Gamma/(NM + M\Xi) \quad (7.16)$$

holds.

17. In Prop. VII.12 an ellipse $AB\Gamma\Delta$ with the major axis $A\Gamma$ = 2a, the minor axis 2b, and two arbitrary conjugate diameters BK = 2a' and ZH = 2b' are is considered.

Apollonius proves that

$$BK^2+ZH^2 = (2a')^2 + (2b')^2 = (2a)^2 +(2b)^2. \quad (7.17)$$

Correlation (7.17) follows from formulas (7.11) and (7.12).

18. In Prop. VII.13 a hyperbola with the axis $A\Gamma = 2a$, the conjugate hyperbola with axis 2b, and two arbitrary conjugate diameters $BK = 2a'$ and $ZH = 2b'$ of these hyperbolas are considered. Apollonius proves that

$$| BK^2 - ZH^2 | = | (2a')^2 - (2b')^2 | = | (2a)^2 - (2b)^2 | . \quad (7.18)$$

The correlation (7.18) follows from formulas (7.5) and (7.6).

19. In Prop. VII.14 the same ellipse as in Prop VII.7 is considered. Apollonius proves that

$$A\Gamma/| BK - ZH| = A\Gamma/2M\Theta , \quad (7.19)$$

where the point Θ is the center of the ellipse.

Prop. VII.14 is the analogue of Prop. VII.11.

20. In Prop. VII.15 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that proportion

$$A\Gamma^2/(2p')^2 = N\Gamma.M\Xi / MN^2 \quad (7.20)$$

holds. Here latus rectum 2p' corresponds to the diameter BK = 2a'.

21. In Prop. VII.16 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that

$$A\Gamma^2/ (BK - 2p')^2 = N\Gamma.M\Xi/(MN - M\Xi)^2 . \quad (7.21)$$

Here latus rectum 2p' also corresponds to the diameter $BK = 2a'$.

22. In Prop. VII.17 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that

$$A\Gamma^2/ (BK +2p')^2 = N\Gamma.M\Xi/ (MN + M\Xi)^2 . \quad (7.22)$$

Here latus rectum 2p' also corresponds to the diameter BK = 2a'.

23. In Prop. VII.18 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that

$$A\Gamma^2/BK.2p' = N\Gamma/ MN \quad (7.23)$$

Here latus rectum 2p' also corresponds to the diameter $BK = 2a'$.

24. In Prop. VII.19 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that

$$\mathrm{A\Gamma}^2/(\mathrm{BK}^2 + (2p')^2 = \mathrm{N\Gamma.M\Xi}/(\mathrm{MN}^2 + \mathrm{M\Xi}^2) \quad (7.24)$$

Here latus rectum 2p' also corresponds to the diameter BK = 2a'.

25. In Prop. VII.20 the same hyperbola and ellipse as in Prop VII.6 and VII.7 are considered. Apollonius proves that

$$\mathrm{A\Gamma}^2/|\,\mathrm{BK}^2 - (2p')^2| = \mathrm{N\Gamma.M\Xi}/|\,\mathrm{MN}^2 - \mathrm{M\Xi}^2\,| \quad (7.25)$$

Here latus rectum 2p' also corresponds to the diameter BK = 2a'.

26. In Prop. VII.21 a hyperbola AB with the latus transversum AΓ = 2a on its axis, and the center Θ is considered. Through the point Θ axis of the conjugate hyperbola with the latus transversum OI = 2b, and two diameters BK and ZH of both hyperbolas are drawn.

Apollonius proves that if AΓ > OI, that is a > b, the diameters BK and ZH are greater than the diameters conjugate to them, the ratio AΓ/OI = a/b is greater than the ratios of the diameters BK and ZH to the diameters conjugate to them, and if the point B is between the points A and Z, the ratio of BK to the diameter conjugate to it is greater than the ratio of the diameter ZH to the diameter conjugate to it.

The assertions of this proposition follow from equalities

$$y_1/b = x_o/a, \quad y_o/b = x_1/a \quad (7.26)$$

where x_o, y_o and x_1, y_1 are coordinates of the ends of two conjugate diameters of the conjugate hyperbolas determined by equations (1.46) and (1.96) in rectangular coordinate system.

27. Prop. VII.22 is the analogue of Prop. VII.21 for the case where AC < OI, that is a < b.

28. Prop. VII.23 is the analogue of Prop. VII.21 for the case where AC = OI, that is a = b.

29. Prop. VII.24 is the analogue of Prop. VII.21 and VII.22 for ellipses.

Assertion of this proposition follows from equality (7.26) where x_o, y_o and x_1, y_1 are the coordinates of the ends of two conjugate diameters of the ellipse determined by equation (1.45) in a rectangular coordinate system.

30. In Corollary 2 to Prop. VII.24, expression “line forming with diameter eidos of ellipse” means the latus rectum of the eidos corresponding to this diameter. Since the major axis of an ellipse is the maximal of its diameters and minor axis is the minimal of them, when a diameter 2a' rotates from the major

axis to the minor one, its length becomes smaller, the diameter 2b' conjugate to it rotates from the minor axis to the major one, and its length becomes greater, and the latus rectum $2p' = 2b'^2/a'$ corresponding to the diameter 2a' also becomes greater.

If a diameter of a hyperbola rotates, the diameter conjugate to it rotates in the opposite direction, and the asymptotes can be regarded as autoconjugate diameters.

31. In Prop. VII.25 a hyperbola AB with the center Φ, and the latus transversum AX = 2a on its axis and conjugate to it hyperbola with the latus transversum 2b on its axis are considered. Through the point Φ two conjugate diameters ΕΓ = 2a' and HK = 2b' are drawn. Apollonius proves that for any conjugate diameters of these hyperbolas inequality

$$\mathrm{E\Gamma} + \mathrm{HK} = 2a' + 2b' > 2a + 2b \quad (7.27)$$

holds. This inequality follows from the fact that in any hyperbola the latus transversum on its axis is smaller than the latus transversum on any other diameter, and therefore $2a < 2a'$ and $2b < 2b'$.

These two inequalities imply that for a hyperbola inequality

$$2a.2b < 2a'.2b' . \quad (7.28)$$

holds.

32. In Prop. VII.26 an ellipse ΑΧΒΔ with the center Φ, the major axis AB = 2a, and the minor axis ΧΔ = 2b is considered. Through the point Φ two conjugate diameters ΕΓ = 2a' and HK = 2b' are drawn. Apollonius proves that for any pair of conjugate diameters 2a' and 2b' of the ellipse, inequality (7.27) holds.

This inequality for the ellipse follows from the fact that the major axis of an ellipse is its maximal diameter, and the minor axis is the minimal diameter. Inequalities $2a > 2a'$ and $2b < 2b'$ imply that for an ellipse inequality (7.28) hold.

The inequality (7.28) and equality (7.17) imply that $(2a)^2 + 2.2a.2b + (2b)^2 = (2a + 2b)^2 < (2a')^2 + 2.2a'.2b' + (2b')^2 = (2a'+2b')$, which is equivalent to inequality (7.27).

33. In Prop. VII.27 the same hyperbola and ellipse as in Prop. VII.25 and VII.26 are considered. Apollonius proves that if the axes 2a and 2b of these conics are not equal, then for any pair of conjugate diameters 2a' and 2b' of these conic inequality

$$| E\Gamma - HK | = |2a' - 2b'| < | 2a - 2b | \quad (7.29)$$

holds. This inequality for an ellipse follows from the inequalities 2a > 2a' and 2b < 2b', and equality (7.17); and for hyperbola it follows from inequality (7.27) and equality (7.18).

In the case of an ellipse, 2a is always greater than 2b, and in formula (7.29) all absolute values of magnitudes can be replaced by these magnitudes.

34. In Prop. VII.28 the same hyperbola and ellipse as in Prop. VII.25 and VII.26 are considered. Apollonius proves that in any ellipse and hyperbola inequality (7.28) holds.

This assertion for hyperbolas was proven in Note 31, and for ellipses - in Note 32 of this book.

Apollonius does not mention that inequalities (7.28), which follow for hyperbolas from the inequalities 2a <2a' and 2b <2b', and for ellipses from the equalities 2a > 2a' and 2b < 2b', are valid not only where the diameters 2a' and 2b' are conjugate, but also where these two diameters are arbitrary.

Apollonius also does not mention that inequality (7.27) for hyperbolas and inequality (7.29) for ellipses is valid for any diameters 2a' and 2b'. Note that inequalities (7.27) for ellipses and (7.29) for hyperbolas, in whose proofs equalities (7.17) and (7.18) were used, are valid only for conjugate diameters 2a' and 2b'.

35. In Prop. VII.29 Apollonius proves that in any pair of conjugate opposite hyperbolas any two diameters $2a_1$ and $2a_2$ and the diameters $2b_1$ and $2b_2$ conjugate to them are connected by the correlation

$$| (2a_1)^2 -(2b_1)^2 | = | (2a_2)^2 - (2b_2)^2 |. \quad (7.30)$$

Equality (7.30) follows from equality (7.18).

36. In Prop. VII.30 Apollonius proves that in any ellipse any two diameters $2a_1$ and $2a_2$ and the diameters $2b_1$ and $2b_2$ conjugate to them are connected by correlation

$$(2a_1)^2 +(2b_1)^2 = (2a_2)^2 + (2b_2)^2. \quad (7.31)$$

Equality (7.31) follows from equality (7.17).

37. In Prop. VII.31 Apollonius proves that in any ellipse and in any pair of conjugate opposite hyperbolas, the areas of parallelograms built on conjugate diameters are equal to the areas of rectangles built on the axes of these conics.

This proposition follows from the fact that the rectangles built on the axes of an ellipse or a hyperbola are mapped to the mentioned parallelograms by elliptic or hyperbolic turns which are equiaffine transformations and do not change areas of polygons.

Since the area of a parallelogram built on lines 2a' and 2b' is equal to product $(2a')(2b')\sin\omega$, where ω is the acute angle of the parallelogram, for any ellipse and hyperbola with the axes 2a and 2b and the conjugate diameters 2a' and 2b' the equality

$$2a.2b = 2a'.2b'.\sin\omega \quad (7.32)$$

holds. Equality (7.32) also implies inequality (7.28) for hyperbolas and ellipses.

Apollonius considers the parallelogram ΞΖΘΗ, bounded by halves ΘΞ = a' and ΘΖ =b' of two conjugate diameters of the conics and tangents to the conic at ends Ξ and Ζ of these diameters, and proves that the area of this parallelogram is equal to the area of the built rectangle on semiaxes a and b of the conic. The parallelograms and rectangular plane in the formulation of the proposition are equal to the quadruple parallelograms and rectangular plane by the equality of areas proved by Apollonius.

38. Corollary 1 to Prop. VII.31 is the assertion that in any hyperbola $(2a)^2+(2b)^2 < (2a')^2+(2b')^2$. This assertion follows from the inequalities $2a < 2a'$ and $2b < 2b'$.

39. Corollary 2 to Prop. VII.31 is the assertion that in any ellipse $(2a)^2-(2b)^2 > (2a')^2-(2b')^2$. This assertion follows from the inequalities $2a > 2a'$ and $2b < 2b'$.

40. Corollary 3 to Prop. VII.31 is the assertion that if the latus transversum 2a of the eidos of hyperbola (1.46) corresponding to its axis is greater than the latus rectum $2p = 2b^2/a$ of the same eidos, then the latus transversum 2a' of the eidos corresponding to any other diameter is greater than the latus rectum 2p' of the same eidos. This assertion follows from equalities (7.5) and (7.6), since, in this case, the diameter 2a' is greater than the conjugate to it diameter 2b' and therefore $2a' > 2p' = 2b'^2/a'$.

41. Corollary 4 to Prop. VII.31 is the assertion that if the latus transversum 2a of the eidos of hyperbola (1.46) corresponding to its axis is smaller than the latus rectum $2p = 2b^2/a$ of the same eidos, then the latus transversum 2a' of the eidos corresponding to any other diameter is less than the latus rectum 2p' of the same eidos. This assertion follows from equalities (7.5) and (7.6), since, in this case, the diameter 2a' is smaller than the conjugate to it diameter 2b' and therefore $2a' < 2p' = 2b'^2/a'$.

42. Corollary 5 to Prop. VII.31 is the assertion that if the latus transversum 2a of the eidos of hyperbola (1.46) corresponding to its axis is equal to the latus rectum $2p = 2b^2/a$ of the same eidos, that is if this eidos is a square and

the hyperbola is equilateral, then the latus transversum 2a' of the eidos corresponding to any other diameter is equal to the latus rectum 2p' of the same eidos. This assertion follows from equalities (7.5) and (7.6), since, in this case, any diameter 2a' is equal to the conjugate to it diameter 2b' and therefore $2a' = 2p' = 2b'^2/a'$.

43. Corollary 6 to Prop. VII.31 is the assertion that in each ellipse (1.45) $2a > 2b$ and therefore $2a > 2p = 2b^2/a$, and in the case where diameter 2a' of ellipse is drawn between the major axis and diameter which is equal to the diameter conjugate to it, equalities (7.11) and (7.12) imply that $2a' > 2b'$ and therefore $2a' > 2p' = 2b'^2/a'$.

44. Corollary 7 to Prop. VII.31 is the assertion that, in the case where a diameter 2a' of an ellipse is drawn between the minor axis and the diameter which is equal to the diameter conjugate to it, equalities (7.11) and (7.12) imply that $2a' < 2b'$ and therefore $2a' < 2p' = 2b'^2/a'$.

Propositions VII.32 - VII.51 on latera recta and transversa of *eidoi* of conics

45. In Prop. VII.32 Apollonius proves that in parabola (0.3) in a rectangular coordinate system the latus rectum 2p corresponding to the axis is smaller than latera recta 2p' corresponding to other diameters, and for diameters $y = y_1$ and $y = y_2$, where $y_1 < y_2$, the latera recta $2p_1$ and $2p_2$ corresponding to these diameters are connected by the correlation

$2p_1 < 2p_2$.

The assertions of this proposition follow from correlation (7.3).

46. In Prop. VII.33 a hyperbola with the center Θ, and the latus transversum $A\Gamma = 2a > 2p$, and points B and Y of the hyperbola , where B is between A and Y, is considered.

Apollonius proves that the latus rectum 2p corresponding to the axis is smaller than latera recta 2p' corresponding to other diameters, and the latus rectum $2p_1$ corresponding to the diameter $BK = 2a_1$ is smaller than the latus rectum $2p_2$ corresponding to the diameter $YT = 2a_2$.

47. Prop. VII.34 is the analogue of Prop. VII.33 for a hyperbola whose latus transversum $A\Gamma = 2a$ satisfies the inequalities $p < 2a < 2p$.

Apollonius proves that the latus rectum 2p corresponding to the axis is smaller than latera recta 2p' corresponding to other diameters, and finds the dependence of latera recta 2p' on the position of the corresponding diameters 2a'.

48. Prop. VII.35 is the analogue of Prop. VII.33 for a hyperbola whose latus transversum ΑΓ = 2a satisfies the inequality $2a < p$.

Apollonius finds two diameters symmetric with respect to the axis such that the latus rectum corresponding to each of them is equal to doubled this diameter and proves that latera recta corresponding to these diameters are minimal, and finds the dependence of latus rectum 2p' corresponding to the diameters 2a' on the position on this diameter.

49. In Prop VII.36, hyperbola (1.46) with unequal axes 2a and 2b is considered. Apollonius finds the dependence of the difference $|2p' - 2a'|$ between two sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter, and proves that this difference is maximal in the case where a diameter is the axis of the hyperbola.

50. In Prop. VII.37, ellipse (1.45) is considered. Apollonius finds the dependence of the difference $|2a' - 2p'|$ between two sides of the eidos corresponding to a diameter 2a' on the position of this diameter and finds that difference $2a' - 2p'$ is maximal in the case where a diameter is major axis, and difference $2p' - 2a'$ is maximal in the case where a diameter is minor axis.

51. In Prop. VII.38, a hyperbola whose latus transversum 2a and latus rectum 2p of the eidos corresponding to its axis satisfy the inequality $2a \geq 2p$, that is $2a \geq 2b$, is considered.

Apollonius finds the dependence of the sum $4a' + 4p'$ of the four sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter and proves that this sum is minimal in the case where a diameter of the hyperbola is its axis.

52. Prop. VII.39 is the analogue of Prop. VII.38 for a hyperbola whose sides 2a and 2p of the eidos corresponding to its axis satisfy the inequalities $2p/3 \leq 2a < 2p$, that is $2b/3^{1/2} \leq 2a < 2b$. Apollonius finds the dependence of the sum $4a' + 4p'$ of the four sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter and proves that this sum also is minimal in the case where a diameter of the hyperbola is its axis.

53. Prop. VII.40 is the analogue of Prop. VII.38 and VII.39 for hyperbola whose sides 2a and 2p of the eidos corresponding to its axis satisfy the inequality $2a < 2p/3$, that is $2a < 2b^2 / 3a$ or $b > 3^{1/2}a$.

Apollonius finds the dependence of the sum $4a' + 4p'$ of the four sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter and proves that in this case this sum is minimal for those two diameters each of which is equal to one third of the latus rectum of the eidos corresponding to this diameter.

54. In Prop. VII.41, ellipse (1.45) with the major axis 2a and latus rectum

2p of the eidos corresponding to the major axis is considered. Apollonius finds the dependence of the sum 4a'+4p' of four sides of the eidos corresponding to a diameter 2a' of the ellipse on the position of this diameter and proves that this sum is minimal in the case where a diameter of the ellipse is its major axis, and is maximal in the case where a diameter of the ellipse is its minor axis.

55. In Prop. VII.42, Apollonius finds the dependence of the area of the eidos of hyperbola (1.46) corresponding to its diameter 2a' on the position of this diameter.

Since the area of the eidos of a hyperbola corresponding to its diameter 2a' is equal to $4b'^2$, that is the square of the diameter 2b' conjugate to the diameter 2a', the dependence of the area of the eidos corresponding to the diameter 2a', whose position is determined by the argument φ, on this argument, is expressed by the correlation

$$2a'.2p'/(2a')^2 = p'/a' = 4b'^2/4a'^2 = b'^2/a'^2 = \qquad (7.34)$$
$$(a^2 \operatorname{sh}^2\varphi + b^2\operatorname{ch}^2\varphi)/(a^2\operatorname{ch}^2\varphi + b^2\operatorname{sh}^2\varphi) =$$
$$(p/a + \operatorname{th}^2\varphi)/(1 + (p/a)\operatorname{th}^2\varphi) . \quad (7.33)$$

Formula (7.33) follows from correlations (7.5) and (7.6).

In the case where $\varphi = 0$, a'=a, b'=b, and therefore p'=p .

56. Prop. VII.43 is the analogue of Prop. VII.42 for an ellipse.

Since the area of the eidos of an ellipse corresponding to its diameter 2a' is equal to $4b'^2$, that is the square of the diameter 2b' conjugate to the diameter 2a', the dependence of the area of the eidos corresponding to the diameter 2a' intersecting the major axis under an angle φ, on this angle is expressed by the correlation

$$2a'.2p'/(2a')^2 = p'/a' = 4b'^2/4a'^2 = b'^2/a'^2 = (a^2\sin^2\varphi + b^2\cos^2\varphi)/(a^2\cos^2\varphi + b^2\sin^2\varphi) = (p/a + \tan^2\varphi)/(1 + (p/a)\tan^2\varphi) . \quad (7.34)$$

Formula (7.34) follows from correlations (7.11) and (7.12)

In the case where $\varphi = 0$, a'=a, b'=b, and therefore p'=p .

In the case where $\varphi = \pi/2$, a'=b, b'=a and therefore p'=q

57. In Prop. VII.44 hyperbola (1.46) whose sides 2a and 2p of the eidos corresponding to its axis satisfy inequality $2a \geq 2p$ is considered. Apollonius finds the dependence of the sum $(2a')^2+(2p')^2$ of the squares of two sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter, and proves that this sum is minimal also where a diameter is the axis of the hyperbola.

58. Prop. VII.45 is the analogue of Prop. VII.44 for a hyperbola whose sides 2a and 2p of the eidos corresponding to its axis satisfy inequalities

$2a < 2p$ and $(2a)^2 \geq (2p - 2a)^2/2$.Apollonius finds the dependence of the sum $(2a')^2+(2p')^2$ of the squares of two sides of the eidos corresponding to a diameter 2a' of the hyperbola on the position of this diameter and proves that this sum is minimal also where a diameter is the axis of the hyperbola.

59. In the proof of Prop. VII.45, after inequality $AM:A\Xi < 2(NM + A\Xi)AM:(AN^2 + A\Xi^2)$ there is a gap in the Greek text, which Halley fills in as follows.

60. Prop. VII.46 is the analogue of Prop. VII.44 and VII.45 for a hyperbola whose sides 2a and 2p of the eidos corresponding to its axis satisfy the inequalities $2a < 2p$ and $(2a)^2 < (2p - 2a)^2/2$. Apollonius finds the dependence of the sum $(2a')^2 +(2p')^2$ on the position of a diameter 2a' and proves that this sum is minimal where $(2a')^2 =(2a' - 2p')^2/2$.

61. In Prop. VII.47, ellipse (1.45) whose sides 2a and 2p of the eidos corresponding to its major axis satisfy the inequality $(2a)^2 \leq (2a +2p)^2/2$ is considered. Apollonius finds the dependence of the sum $(2a')^2+(2p')^2$ of the squares of two sides of the eidos corresponding to a diameter 2a' of the ellipse on the position of this diameter and proves that the sum is minimal where a diameter of the ellipse is its major axis and is maximal where a diameter is its minor axis.

62. Prop. VII.48 is the analogue of Prop. VII.47 for an ellipse whose sides 2a and 2p of the eidos corresponding to its major axis satisfy the inequality $(2a)^2 > (2a + 2p)^2/2$. Apollonius finds the dependence of the sum $(2a')^2 +(2p')^2$ of the squares of two sides of the eidos corresponding to a diameter 2a' of the ellipse on the position of this diameter and proves that this sum is minimal where $(2a')^2 = (2a' + 2p')^2/2$.

63. In Prop. VII.49, hyperbola (1.46) whose sides 2a and 2p of the eidos corresponding to its axis satisfy the inequality $2a > 2p$ is considered. Apollonius finds the dependence of the difference $(2a')^2 - (2p')^2$ between the squares of two sides of the eidos of the hyperbola corresponding to its diameter 2a' on the position of this diameter and proves that this difference is minimal where a diameter 2a' is its axis and the difference $(2a')^2 - (2p')^2$ for any diameter 2a' which is not the axis of the hyperbola is greater than difference between the $(2a)^2$ and the eidos corresponding to the axis of the hyperbola, but is smaller than doubled this difference, that is Apollonius proves inequalities $4a^2 - 4ap < 4a'^2 - 4p'^2 < 8a^2 - 8ap$.

64. Prop. VII.50 is the analogue of Prop.VII.49 for a hyperbola whose sides 2a and 2p of the eidos corresponding to its axis satisfy the inequality $2a < 2p$. Apollonius finds the dependence of the difference $(2p')^2 - (2a')^2$ between the squares of two sides of the eidos of the hyperbola corresponding to

its diameter 2a' on the position of this diameter and proves that this difference is maximal where a diameter 2a' is the axis of the hyperbola and this difference is greater than doubled difference between the square $(2a)^2$ and the eidos corresponding to the axis of the hyperbola, that is Apollonius proves inequality $4p'^2 - 4a'^2 > 8a^2 - 8ap$.

In the case of equilateral hyperbola for which 2a =2b =2p (see Note 42 on this book), all differences $(2a')^2 - (2p')^2$ are equal to 0.

65. Prop. VII.51is the analogue of Prop. VII.49 and VII.50 for ellipse (1.45). Apollonius finds the dependence of the difference $|(2a')^2 - (2p')^2|$ between the squares of two sides of the eidos of the ellipse corresponding to its diameter 2a' on the position of this diameter. Apollonius proves that for diameters of the ellipse which are greater than latera recta corresponding to them difference $(2a')^2 - (2p')^2$ is maximal where a diameter of the ellipse is its major axis and for diameters of the ellipse which are smaller than latera recta corresponding to them difference $(2p')^2 - (2a')^2$ is maximal where a diameter of the ellipse is its minor axis .

Note that $(2a')^2 > (2p')^2$ for diameters of the ellipse drawn between its major axis and the diameter whose length is equal to the length of the diameter conjugate to it, $(2a')^2 < (2p')^2$ for diameters of the ellipse drawn between its minor axis and the diameter whose length is equal to the length of the diameter conjugate to it, and $(2a')^2 = (2p')^2$ for the diameter whose length is equal to the length of the diameter conjugate to it.

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TABLE OF CONTENTS