College of the Holy Cross, Fall Semester, 2018
MATH 351, Modern Algebra I – Solution for Review Problem for Final Exam
November 29

Consider the following operation table:

$\cdot$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
g_2	g_2	g_1	g_4	g_3	g_6	g_5	g_8	g_7	g_{10}	g_9	g_{12}	g_{11}
g_3	g_3	g_4	g_7	g_8	g_{11}	g_{12}	g_1	g_2	g_5	g_6	g_9	g_{10}
g_4	g_4	g_3	g_8	g_7	g_{12}	g_{11}	g_2	g_1	g_6	g_5	g_{10}	g_9
g_5	g_5	g_6	g_9	g_{10}	g_1	g_2	g_{11}	g_{12}	g_3	g_4	g_7	g_8
g_6	g_6	g_5	g_{10}	g_9	g_2	g_1	g_{12}	g_{11}	g_4	g_3	g_8	g_7
g_7	g_7	g_8	g_1	g_2	g_9	g_{10}	g_3	g_4	g_{11}	g_{12}	g_5	g_6
g_8	g_8	g_7	g_2	g_1	g_{10}	g_9	g_4	g_3	g_{12}	g_{11}	g_6	g_5
g_9	g_9	g_{10}	g_{11}	g_{12}	g_7	g_8	g_5	g_6	g_1	g_2	g_3	g_4
g_{10}	g_{10}	g_9	g_{12}	g_{11}	g_8	g_7	g_6	g_5	g_2	g_1	g_4	g_3
g_{11}	g_{11}	g_{12}	g_5	g_6	g_3	g_4	g_9	g_{10}	g_7	g_8	g_1	g_2
g_{12}	g_{12}	g_{11}	g_6	g_5	g_4	g_3	g_{10}	g_9	g_8	g_7	g_2	g_1

You may assume without proof that this is the operation table for a group G of order 12.

a. (*) Which is the identity element in this group? Which element is the inverse of each element? Is G abelian?

Solution: g_1 is the identity element since $g_1 \cdot g_j = g_j \cdot g_1 = g_j$ for all $j = 1, \ldots, 12$. We have

$$\begin{array}{cccc} g_1^{-1} = g_1, & g_2^{-1} = g_2, & g_3^{-1} = g_7, & g_4^{-1} = g_8 \\ g_5^{-1} = g_5, & g_6^{-1} = g_6, & g_7^{-1} = g_3, & g_8^{-1} = g_4 \\ g_9^{-1} = g_9, & g_{10}^{-1} = g_{10}, & g_{11}^{-1} = g_{11}, & g_{12}^{-1} = g_{12}. \end{array}$$

G is *not abelian*, since for example $g_4 \cdot g_9 = g_6$, but $g_9 \cdot g_4 = g_{12}$.

b. (*) What are the orders of each element in this group?

Solution: We have

$$\begin{array}{cccc} |g_1| = 1, & |g_2| = 2, & |g_3| = 3, & |g_4| = 6 \\ |g_5| = 2, & |g_6| = 2, & |g_7| = 3, & |g_8| = 6 \\ |g_9| = 2, & |g_{10}| = 2, & |g_{11}| = 2, & |g_{12}| = 2. \end{array}$$

For example, to see $|g_4| = 6$, compute from the table:

$$g_4^2 = g_7, g_4^3 = g_4 \cdot g_7 = g_2, g_4^4 = g_4 \cdot g_2 = g_3,$$

then

$$g_4^5 = g_4 \cdot g_3 = g_8, \quad \text{and} \quad g_4^6 = g_4 \cdot g_8 = g_1 = e.$$

Hence the cyclic subgroup $\langle g_4 \rangle = \{g_1, g_4, g_7, g_2, g_3, g_8\}$, which has order 6.

c. (*) Why is $H = \{g_1, g_2, g_{11}, g_{12}\}$ a subgroup of G?

Solution: Probably the best way to see this is to pick out the entries from the table for G in the rows and columns corresponding to these four elements. We find:

$\cdot$	g_1	g_2	g_{11}	g_{12}
g_1	g_1	g_2	g_{11}	g_{12}
g_2	g_2	g_1	g_{12}	g_{11}
g_{11}	g_{11}	g_{12}	g_1	g_2
g_{12}	g_{12}	g_{11}	g_2	g_1

This shows that $e = g_1 \in H$, H is closed under products (only elements from H appear in these entries in the table), and H is closed under taking inverses (since every element in H is its own inverse). Hence H is a subgroup of G.

d. To which "standard" group of order 4 is H isomorphic?

Solution: H is a non-cyclic group of order 4, hence isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

e. (*) Why is $J = \{g_1, g_3, g_5, g_7, g_9, g_{11}\}$ a subgroup of G?

Solution: Proceed as in part c. Selecting the rows and columns corresponding to the elements of J, we obtain:

$\cdot$	g_1	g_3	g_5	g_7	g_9	g_{11}
g_1	g_1	g_3	g_5	g_7	g_9	g_{11}
g_3	g_3	g_7	g_{11}	g_1	g_5	g_9
g_5	g_5	g_9	g_1	g_{11}	g_3	g_7
g_7	g_7	g_1	g_9	g_3	g_{11}	g_5
g_9	g_9	g_{11}	g_7	g_5	g_1	g_3
g_{11}	g_{11}	g_5	g_3	g_9	g_7	g_1

As in part c, the form of the table shows that J is a subgroup of G.

f. To which "standard" group of order 6 is J isomorphic?

Solution: J is nonabelian of order 6, since, for instance $g_5 \cdot g_{11} = g_7$ but $g_{11} \cdot g_5 = g_3$. Hence J is isomorphic to D_6 (which is also isomorphic to S_3). Recall that any group of order $2p$, where p is an odd prime, is isomorphic to either $\mathbb{Z}_{2p}$ (abelian), or to D_{2p} (nonabelian). That applies here since $6 = 2 \cdot 3$.

g. (*) Why is $K = \{g_1, g_3, g_7\}$ a subgroup of G?

Solution: Same idea as parts c and e. Alternately, by the calculations done above, you can see that $K = \langle g_3 \rangle = \langle g_7 \rangle$, so it's the cyclic subgroup generated by an element.

h. (*) Which of the subgroups H, J, K from the previous parts is normal in G?

Solution: J must be normal in G, since it has $|J| = 6 = \frac{1}{2}|G|$. H is *not normal* in G. We can see that because, for instance

$$g_5 \cdot g_{11} \cdot g_5^{-1} = g_7 \cdot g_5 = g_9 \notin H.$$

Finally, we claim K is *normal* in G. Here's another way to check that, by looking at the distinct left and right cosets.

$$\begin{aligned} g_1K = g_3K = g_7K = K = Kg_1 = Kg_3 = Kg_7 \\ g_2K = g_4K = g_8K = \{g_2, g_4, g_8\} = Kg_2 = Kg_4 = Kg_8 \\ g_5K = g_9K = g_{11}K = \{g_5, g_9, g_{11}\} = Kg_5 = Kg_9 = Kg_{11} \\ g_6K = g_{10}K = g_{12}K = \{g_6, g_{10}, g_{12}\} = Kg_6 = Kg_{10} = Kg_{12}. \end{aligned}$$

i. For each subgroup that is normal in G from the previous part, construct the factor group. (That is construct G/H if H is normal, G/J if J is normal in G, and G/K if K is normal).

Solution: The normal subgroups are J and K. We have G/J is a group of order 2 consisting of the cosets J, g_2J. The table looks like

$\cdot$	J	g_2J
J	J	g_2J
g_2J	g_2J	J

Note that because of the way the coset product is defined,

$$(g_2J) \cdot (g_2J) = (g_2 \cdot g_2)J = g_1J = J.$$

For the factor group G/K, let's label the left cosets as K, g_2K, g_5K, g_6K. Then the table for the factor group looks like this:

$\cdot$	K	g_2K	g_5K	g_6K
K	K	g_2K	g_5K	g_6K
g_2K	g_2K	K	g_6K	g_5K
g_5K	g_5K	g_6K	K	g_2K
g_6K	g_6K	g_5K	g_2K	K

(This is the non-cyclic group of order 4.)

j. Construct a group homomorphism $\alpha : G \to L$ for some group L to make $\ker(\alpha) = \{g_1, g_2\}$. To which "standard" group of order 6 is L isomorphic? (Hint: To see how to define α, you might note the way the whole table for G breaks up into 2×2 blocks according to the cosets of the subgroup $\{g_1, g_2\}$!)

Solution: This can be phrased in multiple ways. The "slickest" is to note how the table for G breaks up into 2×2 blocks according to the cosets of the subgroup $N = \{g_1, g_2\}$. Those cosets are:

$$N,\ g_3N = \{g_3, g_4\} = Ng_3,\ \ g_5N = \{g_5, g_6\} = Ng_5,$$
$$g_7N = \{g_7, g_8\} = Ng_7,\ g_9N = \{g_9, g_{10}\} = Ng_9,\ \ g_{11}N = \{g_{11}, g_{12}\} = Ng_{11}.$$

This shows that N is a normal subgroup of G. Hence the factor group $L = G/N$ is defined and the map $\alpha : G \to G/N$ taking $g_i \in G$ to $g_iN \in L$ is a mapping as required with $\ker(\alpha) = N$. (Note that if i is even, then g_iN is equal to one of the cosets above. For instance $g_8N = \{g_8, g_7\} = g_7N$.)

k. Show that G is the internal direct product of its subgroups $L = \{g_1, g_2\}$ and $J = \{g_1, g_3, g_5, g_7, g_9, g_{11}\}$.

Solution: We know J is normal in G from part h above and we saw in part j that N is normal in G. Moreover $J \cap N = \{g_1\} = \{e\}$. Therefore the 12 products of elements of J and N are all distinct and give all the elements in G: $J \cdot N = G$. Therefore, by definition G is the internal direct product of J and N. Recall that this means G is also isomorphic to the external direct product $J \times N \cong D_6 \times \mathbb{Z}_2$.

l. What is $Z(G)$, the center of G? What is the centralizer of the element g_2?

Solution: The center of G is the collection of elements commuting with all elements in G. This means that in the table we're looking for g_i such that if you read across the row for g_i, then the elements are listed *in the same order* as they appear in the column for g_i. The elements of G for which this is true are g_1, g_2. Hence $Z(G) = \{g_1, g_2\}$. This implies that the centralizer of g_2 (the subgroup

$$C(g_2) = \{g \in G \mid g \cdot g_2 = g_2 \cdot g\})$$

is all of G: $C(g_2) = G$.

m. What are the conjugacy classes in G?

Solution: The elements of the center appear alone in separate conjugacy classes. Elements in any conjugacy class must have the same order, so the best way to organize this is to look one order at a time. The two elements of order 6 (g_4 and g_8) are conjugate, since for instance

$$g_{11} \cdot g_4 \cdot g_{11}^{-1} = g_6 \cdot g_{11} = g_8.$$

The two elements of order 3 (g_3 and g_7) are also conjugate, since for instance

$$g_{11} \cdot g_3 \cdot g_{11}^{-1} = g_5 \cdot g_{11} = g_7.$$

The remaining 6 elements of the group all have order 2. But they are not all conjugate since for instance if we look at the element g_{12}, then its centralizer is $C(g_{12}) = \{g_1, g_2, g_{11}, g_{12}\}$ which has order 4. This means that the conjugacy class of g_{12} contains only $[G : C(g_{12})] = 12/4 = 3$ distinct elements:

$$\{g_{12}, g_6, g_{10}\}$$

Similarly the class of g_{11} contains only 3 distinct elements (g_{11} has the same centralizer as g_{12}):

$$\{g_{11}, g_5, g_9\}.$$

To summarize, G has

- Two classes of size 1 from the elements of the center: $\{g_1\}, \{g_2\}$
- One class consisting of the elements of order 6: $\{g_4, g_8\}$
- One class consisting of the elements of order 3: $\{g_3, g_7\}$, and
- Two classes of noncentral elements of order 2: $\{g_5, g_9, g_{11}\}$ and $\{g_6, g_{10}, g_{12}\}$.

n. How many different Sylow 2-subgroups are there in G? How many different Sylow 3-subgroups? Verify that the statement of Sylow III holds here.

Solution: Since $12 = 2^2 \cdot 3$, any Sylow 3-subgroup must have order 3, hence be isomorphic to $\mathbb{Z}_3$. Since G contains only two elements of order 3 (g_3 and g_7), there is exactly one Sylow 3-subgroup, namely the subgroup $K = \{g_1, g_3, g_7\}$ studied above. (Note that it must be normal since there is only one!) The Sylow 2-subgroups must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ since G does not contain any elements of order 4. Each such subgroup must contain 3 of the elements of order 2 and the ones in each subgroup must commute among themselves. Note that the only pairs of the noncentral elements of order 2 that do commute with each other are g_5 and g_6, g_9 and g_{10}, and g_{11} and g_{12}. Hence we get 3 Sylow 2-subgroups

$$\{g_1, g_2, g_5, g_6\}, \{g_1, g_2, g_9, g_{10}\}, \{g_1, g_2, g_{11}, g_{12}\}.$$

This all agrees with Sylow III since that theorem says the number of Sylow 3-subgroups must be $\equiv 1 \bmod 3$ and divide 4, while the number of Sylow 2-subgroups must be $\equiv 1 \bmod 2$ and divide 3.