College of the Holy Cross, Fall Semester, 2018
MATH 351, Modern Algebra I – Review for Final Exam
November 28

General Information, Format and Groundrules

The final exam for this course will be given during the regular scheduled period for 8:00am MWF classes: 8:00am to 10:30am on Friday, December 14. This will be an *closed book* exam. The exam will be written so that if you are well prepared and work steadily, you should take about 100 minutes hours to complete it, but you will have the full exam period (150 minutes) to work on it.

As I indicated to the class after the first midterm, I will compute a subscore (out of 100 possible points) on questions covering the material from the first midterm. If that subscore is higher than your grade on the first midterm, I will use it to replace the first midterm grade in computing your overall course average.

Topics

The final exam will be a *comprehensive exam.* It will cover all of the material we have discussed since the beginning of the semester, giving roughly equal attention (roughly 2/5 of the possible points) to each of the sections of the class corresponding to the two midterms and roughly 1/5 of the possible points on the material on the symmetric and alternating groups, the class equation, and the Sylow Theorems after the second midterm. (Of course, many things from the second midterm and the new material also depend on earlier material!)

Like the midterm exams, this one will focus mainly on the key definitions and theorems we have studied. There will also be a few problems or parts of problems where you will need to apply some of those definitions and theorems to new situations.

Some more specific information:

1. All "prove Theorem X"-type questions will be ones from the lists of proofs to know from the review sheets for the two midterms.

2. The new topics to be covered, *in addition to what was included on the two midterms*, are:

   (a) The disjoint cycle decomposition of permutations in S_n; the decomposition of permutations into transpositions (2-cycles), the subgroup $A_n \subset S_n$. Know which permutations are in A_n, how to compute orders of permutations, and what the conjugacy classes in S_n are. *According to the first point above, I will not ask you to show that A_5 is a simple group.*

   (b) The Sylow theorems and the background needed for their proof – the conjugacy relation on a group, the class equation of G and the role of the center $Z(G)$, centralizers of $a \in G$, the formula $|C_a| = [G : C(a)] = |G|/|C(a)|$ (that is: the order of the conjugacy class of a is equal to the index of the centralizer of a), normalizers of subgroups, and so forth. *As noted in the first point above, I will not ask you for the proofs, but you should know the statements of the three Sylow theorems.*

(c) Applications of the Sylow theorems to structure of finite groups.

Here's a good "computational" review problem that covers a lot of what we did this semester. The parts covering material from the first midterm are marked with (*). (In fact, I'm considering putting something like this on the final itself!) Consider the following operation table:

$\cdot$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
g_2	g_2	g_1	g_4	g_3	g_6	g_5	g_8	g_7	g_{10}	g_9	g_{12}	g_{11}
g_3	g_3	g_4	g_7	g_8	g_{11}	g_{12}	g_1	g_2	g_5	g_6	g_9	g_{10}
g_4	g_4	g_3	g_8	g_7	g_{12}	g_{11}	g_2	g_1	g_6	g_5	g_{10}	g_9
g_5	g_5	g_6	g_9	g_{10}	g_1	g_2	g_{11}	g_{12}	g_3	g_4	g_7	g_8
g_6	g_6	g_5	g_{10}	g_9	g_2	g_1	g_{12}	g_{11}	g_4	g_3	g_8	g_7
g_7	g_7	g_8	g_1	g_2	g_9	g_{10}	g_3	g_4	g_{11}	g_{12}	g_5	g_6
g_8	g_8	g_7	g_2	g_1	g_{10}	g_9	g_4	g_3	g_{12}	g_{11}	g_6	g_5
g_9	g_9	g_{10}	g_{11}	g_{12}	g_7	g_8	g_5	g_6	g_1	g_2	g_3	g_4
g_{10}	g_{10}	g_9	g_{12}	g_{11}	g_8	g_7	g_6	g_5	g_2	g_1	g_4	g_3
g_{11}	g_{11}	g_{12}	g_5	g_6	g_3	g_4	g_9	g_{10}	g_7	g_8	g_1	g_2
g_{12}	g_{12}	g_{11}	g_6	g_5	g_4	g_3	g_{10}	g_9	g_8	g_7	g_2	g_1

You may assume without proof that this is the operation table for a group G of order 12.

a. (*) Which is the identity element in this group? Which element is the inverse of each element? Is G abelian?

b. (*) What are the orders of each element in this group?

c. (*) Why is $H = \{g_1, g_2, g_{11}, g_{12}\}$ a subgroup of G?

d. To which "standard" group of order 4 is H isomorphic?

e. (*) Why is $J = \{g_1, g_3, g_5, g_7, g_9, g_{11}\}$ a subgroup of G?

f. To which "standard" group of order 6 is J isomorphic?

g. (*) Why is $K = \{g_1, g_3, g_7\}$ a subgroup of G?

h. (*) Which of the subgroups H, J, K from the previous parts is normal in G?

i. For each subgroup that is normal in G from the previous part, construct the factor group. (That is construct G/H if H is normal, G/J if J is normal in G, and G/K if K is normal).

j. Construct a group homomorphism $\alpha : G \to L$ for some group L to make $\ker(\alpha) = \{g_1, g_2\}$. To which "standard" group of order 6 is L isomorphic? (Hint: To see how to define α, you might note the way the whole table for G breaks up into 2×2 blocks according to the cosets of the subgroup $\{g_1, g_2\}$!)

k. Show that G is the internal direct product of its subgroups $N = \{g_1, g_2\}$ and $J = \{g_1, g_3, g_5, g_7, g_9, g_{11}\}$.

l. What is $Z(G)$, the center of G? What is the centralizer of the element g_2?

m. What are the conjugacy classes in G?

n. How many different Sylow 2-subgroups are there in G? How many different Sylow 3-subgroups? Verify that the statement of Sylow III holds here.

For more review problems, also consult the review sheets for the two midterms and:

- 6.21, 6.23, 6.25
- 7.1, 7.7, 7.9, 7.11, 7.17, 7.21, 7.29, 7.31, 7.37