within symbolic mathematics was of essential significance for the development of mathematical *physics*.

346. Corresponding to the Greek term παραβολή (parabola, that which falls alongside); cf., e.g., Zeuthen, *Die Mathematik im Altertum und Mittelalter*, p. 38. Such *applicationes* "nonnisi καταχρηστικῶς Divisiones vocantur" (are not called divisions except by a misuse of language). Note that Vieta, who refuses to recognize the law of homogeneity as valid precisely in the case of division, also uses the term "adplicare" (to apply) for "dividing" (cf. P. 171 and Note 252, furthermore *Isagoge*, Chap. IV, Precept IV, Appendix, P. 335 and P. 334, Note 39).

347. The third case: division of a magnitude (of determinate dimensions) by a (dimensionless) *numerus* "non tam divisio est . . . quam multiplicatio (nempe quantitas A non tam dividitur per 2, quam multiplicatur per $\frac{1}{2}$) quippe non quaeritur, quoties *numerus* 2 contineatur in *magnitudine* A (quod absurdum esset) sed datae quantitati A, alia in data ratione quaeritur; quod *Multiplicationis* est, potius quam *Divisionis* opus; quippe quae in Multiplicatione Ratio datur, in Divisione quaeritur" (is not so much division as multiplication, for clearly the quantity *A* is not so much divided by 2 as multiplied by $\frac{1}{2}$; since what is sought is, of course, not how many times the "number" 2 is contained in the "magnitude" *A* — which is absurd — but rather for a given quantity *A*, another in a given ratio is sought; and this is the business of "multiplication" rather than "division," for the ratio given in multiplication is the very one sought in division—pp. 135–136).

348. The immediate continuation of this passage is: "totumque Euclidis Elementum quintum Arithmeticum esse, utut speciatim de Magnitudinibus efferantur propositiones, quae interim non minus recte de Quantitatibus simpliciter quibusvis efferi possent, quo sensu apud Euclidem μεγέθη intelligenda sunt" (and the whole fifth book of Euclid is arithmetic, however specifically, as if concerned with [geometric] magnitudes, propositions may be presented, propositions which could meanwhile be carried out just as correctly for any quantities desired in general, and this is the sense in which the word "magnitudes" must be understood in Euclid).

Appendix

INTRODUCTION TO THE ANALYTICAL ART

by François Viète (Vieta)

To the Illustrious Princess Mélusine,
Catherine of Parthenay,
Most Pious Mother of the Lords of Rohan,
I, François Viète of Fontenay,
Pledge Honor and Obedience.[1]

O *Princess Mélusine,*[2] *most pious mother of the lords of Rohan, the Bretons extol the noble family and ancient ancestry of the house of Rohan, which I do not think could be matched on the whole earth by any other more ancient and illustrious on account of more legitimate possessions or more authentic monuments. They will acknowledge your children as the original stock and as the descendants of the royal blood of Conan, as those who by God's will escaped the yoke of the invader Nominhoë; and they will be confident that this noble race will last as long as they, while going about the quarries, woods, and ponds of your domain of Salles, see engraved in marbles, oaks, and scales of fish the insignia of the golden rhomboids which it*

[1] This translation is based primarily on the text of the *Isagoge* as republished with annotations in the Francisci Vietae *Opera Mathematica,* ed. F. van Schooten (Leyden, 1646) pp. 1–12, and as much as possible of its style has been preserved. The original edition was also consulted; its full title is: F. Vietae, *In Artem Analyticem* [sic!] *Isagoge*, Seorsim excussa ab *opere restitutae Mathematicae Analyseos, seu,* Algebra Nova (*Introduction to the Analytical Art,* excerpted as a separate piece from the *opus* of the restored Mathematical Analysis, or *The New Algebra* [Tours, 1591]; cf. Pp. 151 and 153).

The passages in small italics are the editor's annotations printed in the 1646 edition. The passages in square brackets, as well as the footnotes, have been added by the translator. This translation was made in 1955 at St. John's College in Annapolis — J. Winfree Smith.

[2] Catherine of Parthenay (1554–1631) was an ardent Huguenot. After her first husband was killed in the Massacre of St. Bartholomew, she married René of Rohan in Brittany and had by him five children, the eldest of whom, Henri of Rohan, became the famous leader of the

wears.[3] *For by their own religious lore (cabalâ), the Bretons will testify that as it was granted of His sole favor by God most great and most good to the prayers of St. Mériadec, a former prince of the family, so also now it is granted to me, who*

Huguenots. Vieta had supervised her education and remained her friend and adviser all his life.

Catherine herself was descended from the family of Lusignan, whose ancestral seat was the château of Lusignan, fifteen miles from Poitiers. The legendary ancestress of the family was a fairy named Mélusine. The name was originally Mère des Lusignans, then became Mère Lusigne, afterwards Merlusine, and finally Mélusine. Mélusine had the remarkable ability to turn the lower part of her body into a serpent every Saturday. When she married Raymond, it was on the condition that he would never see her on Saturday. He broke the agreement, whereupon she turned completely into a serpent, escaped by the window, and disappeared, only to reappear on the occasion of the death of the lords of Lusignan, when she would utter strange cries of grief. Mélusine was a beneficent fairy and, according to the legend, built the château of Lusignan and many others for her husband.

A Hugh of Lusignan went on the crusade of 1100–1101. Another member of the family, Hugh the Brown, went as a pilgrim to the Holy Land in 1164. In the last quarter of the twelfth century his son Guy became king of Jerusalem and ruler of Cyprus, where his brother's descendants reigned as kings until 1475. In the middle of the fourteenth century, some of the Lusignans of Cyprus went off and made themselves rulers of Armenia, where they held sway from 1342 to 1375. A branch of the family continued in Poitou during the thirteenth century and ruled La Marche until 1303. Hugh of La Marche, whose betrothed wife, Isabel of Angoulême, was seized by King John of England and made his queen, was a nephew of Guy of Lusignan. After John's death Hugh married her and had by her a number of sons who were, therefore, half-brothers of Henry III of England.

The family of René of Rohan owned extensive domains in Brittany, including those of Porhoët and Léon. They were descended from the ancient kings of Brittany the first of whom was Conan Mériadec (409). Judicaël, Eudon, and Erech, whom Vieta mentions, were all kings of Brittany. St. Mériadec, a descendant of Conan, lived in the seventh century and was bishop of Vannes.

[3] A reference to the coat of arms of the Rohan family.

marvel at few things, to marvel time and again at the strange warblings of birds and other remarkable things around the sanctuary, which long ago was his, constructed in the midst of woods and pleasant groves. I, of Fontenay in Poitou, a regular inhabitant of the banks of the Vendée, cherish the name (nomen) and the majesty (numen) of Mélusine and her descendants of the castle constructed long ago by the divine Mélusine, of whom by Raymond you are the blessed progeny. And I also add a prophecy (omen). However, I do not for this purpose oppose to the Judicaëls, the Eudons, and the Erechs of the house of Rohan your Guys, Godfreds, Hughs, and Bruns; nor to their Breton kings, princes in Léon, counts in Porhoët, do I oppose your kings of Cyprus, your princes of Antioch and Armenia, your counts of Angoulême and La Marche nor to their Isabel, daughter of the Scot, nor to Isabel of Navarre, do I oppose your Isabel, mother of English kings and of your ancestors of Lusignan. But rather I piously recall and judge that it happened auspiciously and as if by decree of destiny that the goddess Mélusine in gratitude for the help received from René of Rohan, since he had strenuously defended her castle of Lusignan when it was besieged at the instigation of the Guises, forthwith bestowed on him you, her own and Raymond's offspring and heir, and the rule of the family of Rohan. Raymond himself, to be sure, was descended from the family of Rohan, and now the offspring of Raymond and Mélusine were returned to that source from which they first began; thus it will hardly perish, for this circle is a true and truly physical symbol of perpetuity. But even less will your virtues perish in this cyclical restitution of the beginning. And just as our ancestors, in their own idiom, which was then being adopted, called your ancestress "Fairy Mélusine" because of her venerable appearance and her rare and remarkable gifts of mind, so posterity *will invoke you as heavenly goddess (δῖαν θεάων) and will address you as queen (πότνιαν), as trustworthy ruler (κεδνήν), and with a more worthy epithet, if any occurs.*[4] *And may the fruits of our nightly labor be pleasing to her, so that she may credit them where they are owed, to you and to your most dear sister Françoise of Rohan, duchess of Nîmes and of Loudinois. For the benefits which you and she bestowed on me in most unhappy times are infinite. How can I adequately commemorate that you delivered me from brigand's chains and from the jaws of death and that, in a word, you helped me with your solicitude and generosity as often as my needs and misfortunes prompted you? I owe my life, or if there is anything dearer to me than life, entirely to you; and now, O divine Mélusine, I owe to you especially the whole study of Mathematics, to which I have been spurred on both by your love for it and by the very great skill you have in that art, nay more, the comprehensive knowledge in all sciences (Encyclopaedia) which can never be sufficiently admired in one of your sex who is of so royal and noble a race. O princess most to be revered, those things which are new are wont in the beginning to be set forth rudely and formlessly and must then be polished and perfected in succeeding centuries. Behold, the art which I present is new, but in truth so old, so spoiled and defiled by the barbarians, that I considered it necessary, in order to introduce an entirely new form into it, to think out and publish a new vocabulary, having gotten rid of all its pseudo-technical terms (pseudo-categorematis) lest it should retain its filth and continue to stink in the old way, but since till now ears have been little accustomed to it, it will be hardly avoidable that*

[4] These are all Homeric epithets, applied in Homer to gods and heroes. Cf. *Iliad* XVIII, 388; XIX, 6; *Odyssey* V, 215; XIII, 291; XX, 11; XIV, 170.

many will be offended and frightened away at the very threshold. And yet underneath the Algebra or Almucabala which they lauded and called "the great art," all Mathematicians recognized that incomparable gold lay hidden, though they used to find very little. There were those who vowed hecatombs and made sacrifices to the Muses and Apollo if any one would solve some one problem or other of the order of such problems as we solve freely by the score, since our art is the surest finder of all things mathematical.[5] *Now that the thing has come to pass,* will they be bound by their vows? *However, it would be right for me not now to commend my own wares, but in all moderation yours and those which have been acquired and renewed through your beneficence, to bear witness to my desire that whatever glory is due on accout of the felicity of your rule should not be snatched away. For it is not the same in mathematics as in other studies, that everyone's opinion is free and free his judgment. Here things are done by rule and effort, and neither the persuasions of rhetoricians nor the pleadings of lawyers are of use. The metal which I bring forth yields the kind of gold which they wanted for so long a time. Either that gold is alchemical and faked or it is genuine and true. If it is alchemical, it will surely vanish into smoke, or certainly by the royal touchstone. If on the contrary it is genuine, as it surely is (for I am not one who fights against nature [φυσιομάχος]), I yet do not accuse of deceit those who, with every expectation of seeing their work rewarded, enticed others into digging that gold out of mines hitherto inaccessible and barred by the watchful custody of flame-spouting dragons and*

[5] According to legend, Pythagoras sacrificed an ox upon the discovery of the famous Pythagorean theorem. Vieta introduces Theorem III of Chapter IX of his *Ad Problema Adriani Romani Responsum* with the words "Moved by the beauty of this discovery, O divine Mélusine, I have sacrificed to you a hundred sheep in place of one Pythagorean ox."

other poisonous and deadly serpents, but I fairly ask and expect that they should at least not refuse the support of their authority (which I esteem) against the ignorance or impudence of men who calumniate and detract from another's praise. Therefore, my princess, hold your own work dear and bless it with your blessedness, having referred everything to the supreme ruler of rulers whom you most religiously reverence in soul and in truth (ἐν ψυχῇ καὶ ἀληθείᾳ), with the praise and glory of all praises. From the marshes of the Isles de Mont of your most dear sister, in the second year of our most Christian and august King Henry IV, *most zealous and most just punisher of the enemies of the state and the murderers of* Christ *(χριστοκτόνων).*

Chapter I

On the definition and division of analysis and those things which are of use to zetetics

In mathematics there is a certain way of seeking the truth, a way which Plato is said first to have discovered,[6] and which was called "analysis" by Theon and was defined by him as "taking the thing sought as granted and proceeding by means of what follows to a truth that is uncontested"; so, on the other hand, "synthesis" is "taking the thing that is granted and proceeding by means of what follows to the conclusion and comprehension of the thing sought."[7] And although the ancients set forth a twofold analysis,[8] the zetetic *(ζητητική)* and the poristic *(ποριστική)*, to which Theon's definition particularly refers, it is nevertheless fitting that there be established also a third kind, which may be called rhetic or exegetic *(ῥητικὴ ἢ ἐξηγητική)*, so that there

[6] See Note 218.
[7] See Note 217.
[8] See P. 155.

is a zetetic art by which is found the equation or proportion between the magnitude that is being sought and those that are given, a poristic art by which from the equation or proportion the truth of the theorem set up is investigated, and an exegetic art by which from the equation set up or the proportion there is produced the magnitude itself which is being sought. And thus, the whole threefold analytical art, claiming for itself this office, may be defined as the science of right finding in mathematics. Now what truly pertains to the zetetic art is established by the art of logic through syllogisms and enthymemes, the foundations of which are those very stipulations (symbola)[9] by which equations and proportions are arrived at, which stipulations must be derived from common notions as well as from theorems that are demonstrated by the power of analysis itself. In the zetetic art, however, the form of proceeding is peculiar to the art itself, inasmuch as the zetetic art does not employ its logic on numbers — which was the tediousness of the ancient analysts — but uses its logic through a logistic which in a new way has to do with species.[10] This logistic is much

[9] See Note 226.

[10] See P. 165. Although Diophantus seems the most likely source for Vieta's use of the word "species," the Reverend John Wallis in his *A Treatise of Algebra* (London, 1685), p. 66, advances another theory: "The name of Specious Arithmetick is given to it (I presume) with respect to a sense wherein the civilians use the word *Species*; for whereas it is usual with our Common Lawyers to put *Cases* in the name of John-an-Oaks and John-a-Stiles or John-a-Down, and the like (by which names they mean any person indefinitely who may be so concern'd) and of later times (for brevity sake) of J.O. and J.S. or J.D. (or yet more shortly) of A, B, C, etc. In like manner, the Civilians make use of the Names of Titus, Sempronius, Caius, and Mevius or the like, to represent indefinitely, any person in such circumstances. And cases so propounded they call *Species.* Now with respect hereunto, Vieta (accustomed to the language of the Civil Law) did give, I suppose, the Name of *Species* to the letters A, B, C, etc., made use of by him to represent indefinitely any Number or Quantity, so circumstanced as the occasion required. And accordingly, the accommodation of Arithmetical Operations to Numbers or other

more successful and powerful than the numerical one for comparing magnitudes[11] with one another in equations, once the law of homogeneity has been established; and hence there has been set up for that purpose a series or ladder, hallowed by custom, of magnitudes ascending or descending by their own nature from genus to genus, by which ladder the degrees and genera of magnitudes in equations may be designated and distinguished.

Chapter II
On the stipulations (symbola) governing equations and proportions

The Analytical Art assumes as manifest the better known stipulations governing equations and proportions which are to be found in the *Elements*, such as are:[12]

1. The whole is equal to its parts.
2. Things which are equal to the same thing are equal to each other.
3. If equals are added to equals, the sums are equal.
4. If equals are subtracted from equals, the remainders are equal.
5. If equals are multiplied by equals, the products are equal.

Quantities thus designed by *Symbols* or *Species*, was called *Arithmetica Speciosa* or *Specious Arithmetick*; the word *Species* signifying what we otherwise call *Notes*, *Marks*, *Symbols*, or *Characters*, made use of for the compendious expressing or designation of Numbers or other Quantities."

Wallis' theory derives credence from the fact that Vieta was a jurist. It may be, of course, that the word "species" as used by Vieta is meant to contain something of the meaning of the Diophantine *eide* and also something of the juridical meaning. (Cf. Note 275, P. 281.)

[11] See Pp. 156 and 157.

[12] See Note 226.

6. If equals are divided by equals, the results are equal.
7. If any magnitudes are proportional directly, they are proportional inversely and alternately [i.e., if $a:b::c:d$, then $b:a::d:c$ and $a:c::b:d$].
8. If like proportionals are added to like proportionals, the sums are proportional [i.e., if $a:b::c:d$, then $a+c:b+d::a:b$].
9. If like proportionals are subtracted from like proportionals, the remainders are proportional [i.e., if $a:b::c:d$, then $a-c:b-d::a:b$].
10. If proportionals are multiplied by proportionals, the products are proportional [i.e., if $a:b::c:d$ and $e:f::g:h$, then $ae:bf::cg:dh$].

For when proportionals are multiplied by proportionals, the same ratios are being compounded. Now it was commonly received by the ancient geometers that ratios which are compounded of the same ratios are the same with each other, as is seen everywhere in Apollonius, Pappus, and the other geometers. But the compounding of ratios is effected by the multiplication of the antecedents and the consequents, respectively, as is clear from those things that Euclid shows in the twenty-third proposition of the sixth book and the fifth proposition of the eighth book of the Elements.

11. If proportionals are divided by proportionals, the results are proportional [i.e., if $a:b::c:d$ and $e:f::g:h$, then $a/e:b/f::c/g:d/h$].

For when proportionals are divided by proportionals from the same ratios other same ratios are separated, and just as by multiplication ratios are compounded together, so by division one ratio is separated from another; for division undoes what multiplication, as shown, does.

12. The equation or ratio is not changed by a common multiplier or divisor [i.e., $ma:mb::a:b$ and $a/m:b/m::a:b$].
13. Products under the several segments are equal to the product under the whole [i.e., $ab+ac=a(b+c)$].
14. Products obtained by a succession of magnitudes, or quotients obtained by a succession of divisors, are equal, no

matter in what order the multiplication or division is done [i.e., $a.b=b.a$ and $(a/b)/c=(a/c)/b$].

But the paramount stipulation governing equations and proportions and the one that is all-important in analysis is:

15. If there be three or four magnitudes and the result of the multiplication of the extreme terms is equal to the result of the multiplication of the mean by itself or to the product of the means, then those magnitudes are proportional [i.e., if $ab=cd$, then $a:c::d:b$; or if $ab=c^2$, then $a:c::c:b$]. And conversely.

16. If there be three or four magnitudes, and as the first is to the second, so that second, or else some third, is to another, the product of the extreme terms will be equal to the product of the means [i.e., if $a:b::c:d$, then $ad=bc$; and if $a:b::b:c$, then $ac=b^2$].

And so, a proportion can be called the composition (constitutio) of an equation, an equation the resolution (resolutio) of a proportion.

Chapter III

Concerning the law of homogeneity and the degrees and genera of the magnitudes that are compared[13] (comparatarum)

The supreme and everlasting law of equations or proportions, which is called the law of homogeneity because it is conceived with respect to homogeneous magnitudes, is this:

1. Only homogeneous magnitudes are to be compared (comparari) with one another.

[13] Comparison (comparatio) means, on the one hand, adding and subtracting magnitudes to form algebraic expressions and, on the other, equating magnitudes or expressions with one another. Cf. Descartes, *Rules for the Direction of the Mind*, eds. Haldane and Ross (Dover, 1955), p. 55, Rule XIV.

For, as Adrastus[14] said, it is impossible to know how heterogeneous magnitudes may be conjoined.

And so, if a magnitude is added to a magnitude, it is homogeneous with it.

If a magnitude is multiplied by a magnitude, the product is heterogeneous in relation to both.

If a magnitude is divided by a magnitude, it is heterogeneous in relation to it.

Not to have considered these things was the cause of the darkness and blindness of the ancient analysts.

2. Magnitudes which by their own nature ascend and descend proportionally from genus to genus may be called "ladder-rungs."[15]

3. The first of the ladder magnitudes is "side" (latus) or "root" (radix).

The second is "square" (quadratum).

The third is "cube" (cubus).

The fourth is "squared-square" (quadrato-quadratum).

The fifth is "squared-cube" (quadrato-cubus).

The sixth is "cubed-cube" (cubo-cubus).

The seventh is "squared-squared-cube" (quadrato-quadrato-cubus).

The eighth is "squared-cubed-cube" (quadrato-cubo-cubus).

The ninth is "cubed-cubed-cube" (cubo-cubo-cubus).

And those remaining may be denominated from these by this series and method.

4. The genera of the compared (comparatarum) magnitudes, so that they may be equated in an orderly way to the ladder magnitudes, are:[16]

[14] See P. 173 and Notes 253, 254.

[15] See Note 248.

[16] The equated magnitudes would be not simply known magnitudes which we nowadays would designate by such letters as a, b, or c and which by the law of homogeneity would have to be understood as "lengths" or "planes" or "solids" according as they are equated with x

The first, "length" (longitudo) or "breadth" (latitudo),
The second, "plane" (planum),
The third, "solid" (solidum),
The fourth, "plane-plane" (plano-planum),
The fifth, "plane-solid" (plano-solidum),
The sixth, "solid-solid" (solido-solidum),
The seventh, "plane-plane-solid" (plano-plano-solidum),
The eighth, "plane-solid-solid" (plano-solido-solidum),
The ninth, "solid-solid-solid" (solido-solido-solidum).

And the remaining ones may be denominated from these by this series and method.

5. Of ladder magnitudes, the higher degree in relation to the "side" (latus), as the lowest and that to which the compared magnitude corresponds, is called the "power" (potestas). The other, lower, ladder magnitudes are called degrees "on the way (parodici) to the power" [translated simply by "lower"].

6. The power is pure when it is free from "conjoined" magnitudes. If the power is joined with a magnitude which is the product of a lower rung and a coefficient, it is a "conjoined" power [x^5 is a pure power; x^5+ax^4 is a "conjoined" power].

Pure powers are: "square," "cube," "squared-square," "squared-cube," cubed-cube," etc.

Conjoined powers are:

At the second rung: a "square" together with a "plane" which is the product of a "side" and a "length" or "breadth" [x^2+ax]*;*

At the third rung:

(i) a "cube" together with a "solid" which is the product of a "square" and a "length" or "breadth" [x^3+ax^2],

(ii) a "cube" together with a "solid" which is the product of a "side" and a "plane" [x^3+bx, *where* b *is understood as a "plane" magnitude*],

(iii) *a "cube" together with a "solid" which may be either the product of a "square" and a "length" or "breadth" or the product of a "side" and a "plane"* [x^3+c, *where* c *is understood as a "solid" magnitude;* c *can equal* mx^2 *or* dx, *where* d *is a "plane" magnitude*]*;*

At the fourth rung:

(i) a "squared-square" together with a "plane-plane" which is the product of a "cube" and a "length" or "breadth" [x^4+ax^3],

(ii) a "squared-square" together with a "plane-plane" which is the product of a "square" and a "plane" [x^4+bx^2, *where* b *is a "plane"*],

(iii) a "squared-square" together with a "plane-plane" which is the product of a "side" and a "solid" [x^4+cx, *where* c *is a solid"*],

(iv) a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth" or the product of a "square" and a "plane" [x^4+c, *where* c *is a "plane-plane";* c *can be equal to* mx^3, *or to* dx^2, *where* d *is a "plane"*],

(v) a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth" or the product of a "side" and a "solid" [x^4+c, *where* c *is a "plane-plane";* c *can equal* mx^3 *or* dx *where* d *is a "solid"*],

(vi) a "squared-square" together with a "plane-plane" which is either the product of a "square" and a "plane" or of a "side" and a "solid" [x^4+c, *where* c *is a "plane-plane";* c *can equal* mx^2 *where* m *is a "plane" or* dx, *where* d *is a "solid"*],

(vii) a "squared-square" together with a "plane-plane" which is either the product of a "cube" and a "length" or "breadth," or of a *"square" and a "plane," or of a "side" and a "solid"* [x^4+c, *where* c *is a "plane-plane";* c *can equal* mx^3 *or* dx^2, *where* d *is a "plane," or* ex, *where* e *is a "solid"*].

In the same order the conjoined powers at the remaining rungs of the ladder may be found. But if we want to know how many genera of conjoined powers are at each rung, let there be taken a number less by unity than that term which is produced by geometric progression from unity in the double ratio [1:2::2:4::4:8, *etc.*] *and which has the same ordinal position as the power under consideration. Thus, if one wants to know how many conjoined powers are at the rung of the "squared-square," i.e., at the fourth rung, the fourth term of the geometric progression, namely* 8, *must be taken, from which, when unit has been*

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or x^2 or x^3, but also, as appears in the sequel, products of known and unknown magnitudes. Thus, ax^2 would be a product of a "length" and a "square." It would itself be a "solid" and might be equated with x^3. See Note 249 and P. 172.

taken away, 7 remains. And so, there are at the fourth rung as many conjoined powers as we have just enumerated. By this procedure it will be found that at the rung of the "squared-cube," i.e., the fifth rung, there are fifteen genera of conjoined powers.

7. Coefficient[17] magnitudes which multiply ladder magnitudes that are lower in relation to a certain power and thus produce a homogeneous magnitude to be added to that power shall be called "subrungs."

The "subrungs" are "lengths" or "breadths," "plane," "solid," "plane-plane," etc. Thus, if there be a "squared-square" to which there is joined a "plane-plane" which is the product of a "side" and a "solid," the "solid" magnitude will be the "sub-rung"; and in relation to the "squared-square" the "side" will be a lower ladder magnitude. [In the expression x^4+cx *it is apparent that* x^4 *is a "squared-square";* cx *is a "plane-plane," being the product of the "side"* x *and the "solid"* c; c, *then, is the subrung, and* x *is in relation to* x^4 *a lower ladder magnitude.] Or if there be a "squared-square" together with a "plane-plane," which is either the product of a "square" and a "plane" or the product of a "side" and a "solid," the "plane" and the "solid" will be "subrung" magnitudes; and in relation to the "squared-square" the "square" and the "side" will be lower ladder magnitudes.* [Thus *we may have* x^4+cx *as above or* x^4+cx^2, *where* cx^2 *is a "plane-plane" and* c *is understood as a "plane." Then the "plane"* c *is the "subrung," and the "square"* x^2 *is a lower ladder magnitude in relation to* x^4.]

Chapter IV

On the precepts of the reckoning by species

The numeral reckoning (logistice numerosa) operates with numbers; the reckoning by species (logistice speciosa) operates with species or forms of things,[18] as, for example, with the letters of the alphabet.

[17] See Note 262.

[18] See Pp. 166 and 171.

Diophantus has handled the numerical reckoning in the thirteen books of the Arithmetic, *of which only the first six are extant, but which are now available in Greek and Latin and elucidated by the very learned commentaries*[19] *of a most illustrious man, Claude Bachet [de Méziriac]. But Vieta has produced the reckoning by species in the five books of the* Zetetics, *which he has arranged chiefly from selected problems of Diophantus, some of which he solves by a method peculiar to himself. Wherefore, if you wish to discern profitably the distinction between the two kinds of reckoning, you must consult Diophantus and Vieta together, and the zetetics of the latter must by viewed along with the arithmetical problems of the former; it is in order that I may lighten for you the labor of this task that I shall briefly note the zetetics which have been taken from the problems of Diophantus* (see P. 330).

There are four canonical precepts for reckoning by species (logistices speciosae).

Precept I

To add a magnitude to a magnitude

Let there be two magnitudes *A* and *B*. It is required to add the one to the other.

But, since heterogeneous magnitudes cannot be conjoined, those which are proposed to be added to one another are two homogeneous magnitudes. That one of them is greater or less than the other does not imply that they are of different genera. Therefore, they may be fittingly added[20] by means of the sign for coupling or addition; and, put together, they will be *A* "plus" *B*, if they are simple "lengths" or "breadths."

But if they stand higher on the aforesaid ladder or if they share a genus with those that stand higher, they will be designated by the appropriate denominations, as, for instance, we may say, "*A* square 'plus' *B* plane" or "*A* cube 'plus' *B* solid," and similarly in other cases.

The analysts, however, are accustomed to indicate the performance of addition by the symbol +.[21],

[19] See Pp. 176–178.

[20] See Pp. 176 ff.

[21] See Note 153.

Diophantus		Vieta	
Book of the Arithmetic	*Problem*[22]	*Book of the Zetetics*	*Problem*[23]
I	1	I	1
	4		2
	2		3
	7		4
	9		5
	5		7
	6		8
II	8, 9	IV	1
	10		2, 3
	11		6
	12		7
	13		8
	14		9
V	8		11
III	7, 8	V	1
	9		3
	10		4
	11		5
	12		7
	13		8
V	9		9
IV	34		13

[22] This problem (Tannery, p. 16) is reproduced here as an example. For Diophantus' signs see Pp. 141–147.

To divide a given number into two numbers with a given difference.

So, let the given number be $\bar{\rho}$ [one hundred], and let the difference be $\mathring{M}\bar{\mu}$ (forty units).

To find the numbers.

Let the less be taken as $\varsigma\bar{\alpha}$ [one unknown]. Then the greater will be $\varsigma\bar{\alpha}\mathring{M}\bar{\mu}$ [one unknown and forty units]. Then both together become $\varsigma\beta\mathring{M}\bar{\mu}$ [two unknowns and forty units]. But they have been given as $\mathring{M}\bar{\rho}$ [one hundred units.]

$\mathring{M}\bar{\rho}$ [one hundred units], then, are equal to $\varsigma\beta\mathring{M}\bar{\mu}$ [two unknowns and forty units].

Precept II

To subtract a magnitude from a magnitude

Let there be two magnitudes A and B, and let the former be greater than the latter. It is required to subtract the less from the greater.

Since, then, a magnitude is to be subtracted from a magnitude, but heterogeneous magnitudes cannot be conjoined, those which are proposed are two homogeneous magnitudes. That one of them is greater and the other less does not imply that they are of different genera. Therefore, subtraction may be fittingly effected by means of the sign of the disjoining or removal[24] of the less

[24] "Removal" here translates a juridical term "multa" which means a fine, and is preserved in the English word "mulct."

And, taking like things from like: I take $\mathring{M}\bar{\mu}$ [forty units] from the $\bar{\rho}$ [one hundred] and likewise $\bar{\mu}$ [forty] from the β [two] numbers and $\bar{\mu}$ [forty] units. The $\varsigma\beta$ [two unknowns] are left equal to $\mathring{M}\bar{\xi}$ [sixty units]. Then, each ς [unknown] becomes $\mathring{M}\bar{\lambda}$ [thirty units].

As to the actual numbers required: the less will be $\mathring{M}\bar{\lambda}$ [thirty units] and the greater $\mathring{M}\bar{o}$ [seventy units], and the proof is clear.

[23] This problem (*Opera Mathematica*, p. 42) is reproduced here as an example.

Given the difference of two "sides" and their sum, to find the "sides."

Let the difference B of the two "sides" be given, and also let their sum D be given.

It is required to find the "sides."

Let the less "side" be A; then the greater will be $A+B$. Therefore, the sum of the "sides" will be $A2+B$. But the same sum is given as D. Wherefore, $A2+B$ is equal to D. And, by antithesis, $A2$ will be equal to $D-B$, and if they are all halved, A will be equal to $D\frac{1}{2}-B\frac{1}{2}$.

Or, let the greater "side" be E. Then the less will be $E-B$. Therefore, the sum of the "sides" will be $E2-B$. But the same sum is given as D. Therefore, $E2-B$ will be equal to D, and by antithesis, $E2$ will be equal to $D+B$, and if they are all halved, E will be equal to $D\frac{1}{2}+B\frac{1}{2}$.

Therefore, with the difference of two "sides" given and their sum, the "sides" are found.

For, indeed, half the sum of the "sides" minus half their difference is equal to the less "side," and half their sum plus half their difference is equal to the greater.

Which very thing the zetesis shows.

Let B *be* 40 *and* D 100. *Then* A *becomes* 30 *and* E *becomes* 70.

from the greater; and disjoined, they will be *A* "minus" *B*, if they are simple "lengths" or "breadths."

But if they stand higher on the aforesaid ladder or if they share a genus with those that stand higher, they will be designated by the appropriate denominations, as, for example, we may say "*A* square 'minus' *B* plane" or "*A* cube 'minus' *B* solid," and similarly in the other cases.

Nor will it be done differently if the magnitude which is subtracted is itself conjoined with some magnitude, since the whole and the parts are not to be judged by separate laws; thus, if "*B* 'plus' *D*" is to be subtracted from *A*, the remainder will be "*A* 'minus' *B*, 'minus' *D*," the magnitudes *B* and *D* having been subtracted one by one.

But if *D* is already subtracted from *B* and "*B* 'minus' *D*" is to be subtracted from *A*, the result will be "*A* 'minus' *B* 'plus' *D*," because in the subtraction of the whole magnitude *B* that which is subtracted exceeds by the magnitude *D* what was to have been subtracted. Therefore, it must be made up by the addition of that magnitude *D*.

The analysts, however, are accustomed to indicate the performance of the removal by means of the symbol —. And this is "defect" *(λεῖψις)* in Diophantus, as the performance of addition is "presence" *(ὕπαρξις)*.

But when it is not said which magnitude is greater or less, and yet the subtraction must be made, the sign of the difference is: $=$, i.e., when the less is undetermined; as, if "*A* square" and "*B* plane" are the proposed magnitudes, the difference will be: "*A* square$=$*B* plane," or "*B* plane$=$*A* square."[25]

Precept III

To multiply a magnitude by a magnitude

Let there be two magnitudes *A* and *B*. It is required to multiply the one by the other.

Since, then, a magnitude is to be multiplied by a magnitude,

[25] The introduction of negative quantities makes it unnecessary for us to distinguish the two minus signs. The second of these signs, which is now used to signify equality, was so used as early as 1557 by Robert Recorde in his *The Whetstone of Witte*. Vieta has no symbol for equality.

they will by their multiplication produce a magnitude heterogeneous in relation to each of them; and therefore, their product will rightly be designated by the word "*in*" or "*sub*," as, for example, "*A* in B," by which it will be signified that the one has been multiplied by the other; or as others say, that a magnitude is produced "under" *A* and *B*, and that simply, if *A* and *B* are simple "lengths" or "breadths."[26]

But if they stand higher on the ladder or if they share in genus with magnitudes that stand higher, it is agreed to add the names themselves of the ladder magnitudes or of those that share in their genus, as, for example, "*A* square in *B*" or "*A* square in *B* plane" or "*A* square in *B* solid," and similarly in the other cases.

If, however, the magnitudes to be multiplied, or one of them, be of two or more names, nothing different happens in the operation.[27] Since the whole is equal to its parts, therefore also the products under the segments of some magnitude are equal to the product under the whole. And when the positive name[28] (nomen adfirmatum) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (nomen negatum), the product will be negative.

From which precept it also follows that by the multiplication of negative names by each other a positive product is produced, as when "$A=B$" is multiplied by "$D=G$" [giving $DA-DB-GA+GB$], since the product of the positive *A* and the negative *G* is negative, which means that too much is removed or taken away, inasmuch as *A* is, inaccurately, brought forward (producta) as a magnitude to be multiplied [as a whole, i.e., in the factor $(-AG)$], and since, similarly, the product of the negative *B* and the positive *D* is negative, which again means that too much is removed, inasmuch as *D* is, inaccurately, brought forward as a magnitude to be multiplied [i.e., in $(-DB)$]. Therefore, by way of compensation, when the negative *B* is multiplied by the negative G, the product is positive.

[26] See P. 171.

[27] That is, $a(u+v)+a(x+y)=au+av+ax+ay$; or $a(x+y+z)=ax+ay+az$ (cf. Euclid II, 1).

[28] The "names," i.e., the signs themselves, are multiplied together *as if* they were particular numbers.

The denominations of products made by magnitudes ascending proportionally from genus to genus are related to one another in precisely the following way:

A "side" multiplied by itself produces a "square."
A "side" multiplied by a "square" produces" a "cube."
A "side" multiplied by a "cube" produces a "squared-square."
A "side" multiplied by a "squared-square" produces a "squared-cube."
And interchangeably, i.e., a "square" multiplied by a "side" produces a "cube," a "cube" multiplied by a "side" produces a "squared-square," etc.
Again,
A "square" multiplied by itself produces a "squared-square."
A "square" multiplied by a "cube" produces a "squared-cube."
A "square" multiplied by a "squared-square" produces a "cubed-cube."
And interchangeably.
Again,
A "cube" multiplied by itself produces a "cubed-cube."
A "cube" multiplied by a "squared-square" produces a "squared-squared-cube."
A "cube" multiplied by a "squared-cube" produces a "squared-cubed-cube."
A "cube multiplied by a "cubed-cube" produces a "cubed-cubed-cube."
And interchangeably, and so on in that order.
In like manner, among the homogeneous magnitudes,
A "breadth" multiplied by a "length" produces a "plane."
A "breadth" multiplied by a "plane" produces a "solid."
A "breadth" multiplied by a "solid" produces a "plane-plane."

A "breadth" multiplied by a "plane-plane" produces a "plane-solid."
A "breadth" multiplied by a "plane-solid" produces a "solid-solid."
And interchangeably.
A "plane" multiplied by a "plane" produces a "plane-plane."
A "plane" multiplied by a "solid" produces a "plane-solid."
A "plane" multiplied by a "plane-plane" produces a "solid-solid."
And interchangeably.
A "solid" multiplied by a "solid" produces a "solid-solid."
A "solid" multiplied by a "plane-plane" produces a "plane-plane-solid."
A "solid" multiplied by a "plane-solid" produces a "plane-solid-solid."
A "solid" multiplied by a "solid-solid" produces a "solid-solid-solid."
And interchangeably, and so on in that order.

Precept IV

To divide a magnitude by a magnitude

Let there be two magnitudes *A* and *B*. It is required to divide the one by the other.

Since, then, a magnitude is to be divided by a magnitude, namely higher ones by lower ones, i.e., magnitudes of one kind by magnitudes of another kind, the proposed magnitudes are different in kind. Let *A*, if you will, be a "length" and *B* "plane." And then let a horizontal line appropriately stand between the higher magnitude *B* which is being divided and the lower *A* by which the division is made.

But the magnitudes themselves, i.e. the resultant quotients, will be denominated in accordance with their own rungs at which they are fixed or to which they have been reduced in the ladder of proportional or homogeneous magnitudes, as, for example:

(B plane)/A, by which symbol the "length" which results from the division of "B plane" by "A length" may be signified.

And if B is given as a "cube" and A as a "plane," the result will be (B cube)/A, by which symbol the "length" which results from the division of "B cube" by "A plane" may be signified.

And if B is assumed to be a "cube" and A a "length," the result will be (B cube)/A, by which symbol the "plane" which arises from the division of "B cube" by A may be signified, and so on in that order, *in infinitum*.

Nor will anything different be observed among binomial or polynomial magnitudes.

The denominations of the magnitudes that arise from dividing by magnitudes that ascend proportionally by degrees from genus to genus are related to one another in precisely the following way:

A "square" divided by a "side" gives a "side."
A "cube" divided by a "side" gives a "square."
A "squared-square" divided by a "side" gives a "cube."
A "squared-cube" divided by a "side" gives a "squared-square."
A "cubed-cube" divided by a "side" gives a "squared-cube."
And interchangeably, i.e., a "cube" divided by a "square" gives a "side," a "squared-square" divided by a "cube" gives a "side," etc.
Again,
A "squared-square" divided by a "square" gives a "square."
A "squared-cube" divided by a "square" gives a "cube."
A "cubed-cube" divided by a "square" gives a "squared-square."
And interchangeably.
Again,
A "cubed-cube" divided by a "cube" gives a "squared-square" [sic!].

A "squared-cubed-cube" divided by a "cube" gives a "squared-cube."
A "cubed-cubed-cube" divided by a "cube" gives a "cubed-cube."
And interchangeably, and so on in that order.
In like manner, among the homogeneous magnitudes,
A "plane" divided by a "breadth" gives a "length."
A "solid" divided by a "breadth" gives a "plane."
A "plane-plane" divided by a "breadth" gives a "solid."
A "plane-solid" divided by a "breadth" gives a "plane-plane."
A "solid-solid" divided by a "breadth" gives a "plane-solid."
And interchangeably.
A "plane-plane" divided by a "plane" gives a "plane."
A "plane-solid" divided by a "plane" gives a "solid."
A "solid-solid" divided by a "plane" gives a "plane-plane."
And interchangeably.
A "solid-solid" divided by a "solid" gives a "solid."
A "plane-plane-solid" divided by a "solid" gives a "plane-plane."
A "plane-solid-solid" divided by a "solid" gives a "plane-solid."
A "solid-solid-solid" divided by a "solid" gives a "solid-solid."
And interchangeably, and so on in that order.

Moreover, if the magnitude that is being divided be the sum, difference, product, or quotient of other magnitudes, nothing prevents the aforesaid precepts from applying to the division, it being noted that, when the magnitude that is being divided, whatever may be its rung, is the product of some magnitude and a magnitude that is the same as the divisor, nothing either in genus or value is added to or taken away from the factor that is not the same as the divisor and that also arises from the division, since what multiplication does division undoes: for example, (B in A)/B is A, and (B in A plane)/B is "A plane."

And thus, in the case of additions, let it be required to add Z to *A* plane/*B*, The sum will be

$$\frac{(A \text{ plane}) + (Z \text{ in } B)}{B}$$

[i.e., $a^2/b + z = (a^2 + zb)/b$].

Or let it be required to add (*Z* square)/*G* to (*A* plane)/*B*. The sum will be

$$\frac{(G \text{ in } A \text{ plane}) + (B \text{ in } Z \text{ square})}{B \text{ in } G}.$$

In the case of subtractions, let it be required to subtract *Z* from (*A* plane)/*B*. The remainder will be

$$\frac{(A \text{ plane}) - (Z \text{ in } B)}{B}.$$

Or, let it be required to subtract (*Z* square)/*G* from (*A* plane)/*B* The remainder will be

$$\frac{(A \text{ plane in } G) - (Z \text{ square in } B)}{B \text{ in } G}.$$

In the case of multiplications, let it be required to multiply (*A* plane)/*B* by *B*. The result will be *A* plane.

Or let it be required to multiply (*A* plane)/*B* by *Z*. The result will be (*A* plane in *Z*)/*B*.

Or, finally, let it be required to multiply (*A* plane)/*B* by (*Z* square)/*G*. The result will be [(*A* plane)/(*B* in *G*)] in *Z* square.

In the case of division, let it be required to divide (*A* cube)/*B* by *D*. Each magnitude having been multiplied by *B*, the result will be (*A* cube)/(*B* in *D*) [i.e., $(a^3/b)/d = (a^3b/b)/bd = a^3/bd$].

Or let it be required to divide *B* in *G* by (*A* plane)/*D*. Each magnitude having been multiplied by *D*, the result will be (*B* in *G* in *D*)/*A* plane.

Or, finally, let it be required to divide (*B* cube)/*Z* by (*A* cube)/(*D* plane). The result will be (*B* cube in *D* plane)/(*Z* in *A* cube).

Chapter V
Concerning the laws of zetetics

The way to do zetetics is, in general, encompassed in the following laws:

1. If it is a "length" that is being sought, but the equation or proportion is hidden under the wrappings[29] of what is given in the problem, let the unknown which is being sought be a "side."

2. If it is a "plane" that is being sought, but the equation or proportion is hidden under the wrappings of what is given in the problem, let the unknown which is being sought be a "square."

3. If it is a "solid" that is being sought, but the equation or proportion is hidden under the wrappings of what is given in the problem, let the unknown which is being sought be a "cube."

Accordingly, that magnitude which is being sought will by its own nature ascend or descend through the several rungs of the magnitudes that are compared or equated with it.

4. Let the magnitudes that are given, as well as those that are being sought, be assimilated and compared (in accordance with the condition dictated by the problem) by adding, subtracting, multiplying, and dividing, the constant law of homogeneity being everywhere observed.

Accordingly, it is clear that finally something will be found which is equal to the magnitude that is being sought or to the power to which it ascends and that that will consist either entirely of given magnitudes or partly of given magnitudes

[29] In solving a problem by algebra the equation may not emerge immediately from the given conditions. It may take some reflection before one sees the equation that satisfies the conditions. This is what Vieta means when he speaks of the equation as "hidden under the wrappings of what is given in the problem" (cf. Descartes, *Geometry*, [Dover, 1954], pp. 6–8).

and partly of the unknown which is being sought or of magnitudes lower than it on the ladder.[30]

5. In order that this work may be assisted by some art, let the given magnitudes be distinguished from the undetermined unknowns by a constant, everlasting and very clear symbol, as, for intance, by designating the unknown magnitude by means of the letter *A* or some other vowel *E*, *I*, *O*, *U*, or *Y*, and the given magnitudes by means of the letters *B*, *G*, and *D* or the other consonants.[31]

6. Products composed entirely of given magnitudes may be added to one another, or subtracted from one another, according to the sign of their conjunction, and may merge into one product, which shall be the homogeneous element of the equation, i.e., the element under a given measure; and it shall constitute one side of the equation.[32]

7. In like manner, products composed of given magnitudes and of the same lower ladder magnitude may be added one

[30] In the equation $x^2=ab$, x^2 is the magnitude which is being sought; ab is equal to it, and is a product entirely of given magnitudes. In the equation $x^3=ax^2$, x^3 is the unknown which is being sought; ax^2 is equal to it and is a product partly of the given magnitude a and partly of a magnitude lower than x^3 on the ladder, namely x^2.

[31] This, of course, differs from the convention of present-day algebra, according to which the letters at the end of the alphabet (x, y, z, . . .) are used to represent unknowns and the letters at the beginning (a, b, c, . . .) represent knowns. Thomas Harriot in his *Artis analyticae praxis* (1631) followed Vieta in using vowels for unknowns and consonants for known quantities, except that he substituted small letters for Vieta's capitals. Descartes in his *Geometry* (1637) introduced the system we use; cf. Note 275.

[32] In Vieta's symbols, "*B* in *C*" and "*D* in *F*" would be products composed entirely of given magnitudes, which products may be added to or subtracted from one another by means of the plus sign or the minus sign. When so added or subtracted, they become "*B* in *C*+*D* in *F*" or "*B* in *C*−*D* in *F*." If we were then to form the equation "*A* square is equal to *B* in *C*+*D* in *F*," "*B* in *C*+*D* in *F*" would be homogeneous with "*A* square." Since "*B* in *C*" and "*D* in *F*" belong to the rank of "planes," the unit measure is given as a "plane" unit. In modern notation this would be $x^2=ab+cd$.

to another, or subtracted one from another, according to the sign of their conjunction, and may merge into one product which shall be the element homogeneous in conjunction, or the element under the rung of the lower ladder magnitude.[33]

8. Elements which are homogeneous under the rungs of lower ladder magnitudes shall accompany the power with which they are conjoined, and, along with that power, shall constitute one side of the equation. And thus, the element that is homogeneous under a given measure will be equated to a power designated in its own genus or order; simply, if that power is free from all conjunction with other magnitudes, but if magnitudes homogeneous in conjunction accompany it, which magnitudes are indicated both by the symbol of the conjunction and by the rung of the lower ladder magnitudes, then the magnitude homogeneous under a given measure will be equated not only to it, but to it along with the magnitudes that are products of rungs and coefficient magnitudes.[34]

[33] For example, "*A* square in *B*" and "*A* square in *C*" would be products composed of the given magnitudes *B* and *C* and of the same lower ladder magnitude "*A* square." They may be conjoined by means of the plus sign or the minus sign, and then we get either "*A* square in *B*+*A* square in *C*" or "*A* square in *B*−*A* square in *C*." Each is a product under the lower ladder magnitude "*A* square," which is lower in relation to the rung of the product. "*A* square in *B*+*A* square in *C*" or "*A* square in *B*−*A* square in *C*" is called the element homogeneous in conjunction because it is regarded as something to be "conjoined" with a pure power "*A* cube" with which it is homogeneous.

In modern notation this would be ax^2+bx^2 or ax^2-bx^2, either of which binomials would be the element homogeneous in conjunction because it would be considered as "conjoined" with x^3 to make $x^3+ax^2+bx^2$, $x^3+ax^2-bx^2$, etc.

[34] "*A* square in *G*" and "*A* square in *H*" are elements homogeneous under the rung "*A* square." They accompany the power "*A* cube," are conjoined with it by addition or subtraction, and with it constitute one side of the equation. "*B* plane in *C*+*D* solid" is an element homogeneous under a given measure. It might be equated to "*A* cube" simply, in which case we would have the equation: "*A* cube is equal to *B* plane in *C*+*D* solid"; or "*A* cube" might be accompanied by "*A* square in

9. And, therefore, if the element that is homogeneous under a given measure happens to be mingled with the element that is homogeneous in conjunction, there shall be antithesis.[35]

There is antithesis when positively or negatively conjoined magnitudes cross from one side of the equation to the other under the opposite signs of conjunction, by which operation the equation is not changed. But that must now be demonstrated.

Proposition I

An equation is not changed by antithesis

Let it be given that "*A* square 'minus' *D* plane" is equal to "*G* square 'minus' *B* in *A*."

I say that "*A* square 'plus' *B* in *A*" is equal to "G square 'plus' *D* plane" and that by this transposition under opposite signs of conjunction the equation is not changed. For since "*A* square 'minus' *D* plane" is equal to "*G* square 'minus' *B* in *A*," let there be added to both sides "*D* plane 'plus' B in A." Therefore, from the common notion, "A square 'minus' D plane 'plus' *D* plane 'plus' *B* in *A*" is equal to "*G* square 'minus' *B* in *A* 'plus' *D* plane 'plus' *B* in A." Now the negative conjunction on the same side of an equation cancels the positive. On the one side, the conjunction of "*D* plane" vanishes; on the other, the conjunction of

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G−*A* square in *H*," a magnitude homogeneous in conjunction, in which case we might have this equation: "*A* cube+*A* square in *G*−*A* square in *H* is equal to *B* plane in *C*+*D* solid."

In modern notation the last equation would be $x^3+ax^2-bx^2=cd+e$, where c is understood as a "plane," d as a "length," and e as a "solid."

35 "Antithesis" means the transposition of terms from one side of the equation to the other, with accompanying change of sign. Thus we might have the equation: "*A* cube+*A* square in *G*−*A* square in *H*−*B* plane in *C*+*F* plane in *K* is equal to *D* solid." By antithesis we could infer the equation: "*A* cube+*A* square in *G*−*A* square in *H* is equal to *B* plane in *C*−*F* plane in *K*+*D* solid."

In modern notation, from $x^3+ax^2-bx^2-cd+ef=g$ we get by antithesis $x^3+ax^2-bx^2=cd-ef+g$. We understand c and e as "planes" and g as a "solid."

"*B* in *A*," and there will remain: "*A* square 'plus' *B* in *A*" is equal to "*G* square 'plus' *D* plane."[36]

10. And if it happens that all the magnitudes have as a factor a certain rung, and therefore that the homogeneous element determined by the over-all measure does not immediately appear, there shall be a hypobibasm.[37]

Hypobibasm is the like lowering of the power and of the lower ladder magnitudes, the order of the ladder being observed, until the homogeneous element determined by the lower rung coincides with the over-all homogeneity according to which the magnitudes that remain are equated, by which operation the equation is not changed. But that must now be demonstrated.

The operation of hypobibasm differs from parabolism only in this, that in the case of hypobibasm each side of the equation is divided by an unknown quantity, but in the case of parabolism each side is divided by a known quantity, as is clear from the examples presented by the author.

Proposition II

An equation is not changed by hypobibasm

Let it be given that "*A* cube 'plus' *B* in *A* square" is equal to "*Z* plane in *A*."

I say that, by hypobibasm, "*A* square 'plus' *B* in *A*" is equal to "*Z* plane."

For that means to have divided all the "solids" by a common divisor, by which it is certain that the equation is not changed.[38]

11. And if it happens that the higher rung to which the unknown magnitude ascends does not subsist by itself but is

36 In modern notation, we are given that $x^2-d=y^2-bx$. We want to show that $x^2+bx=y^2+d$. We add to both sides $d+bx$ and get $x^2-d+d+bx=y^2-bx+d+bx$ or $x^2+bx=y^2+d$. Here d, of course, is understood as "d plane."

37 "Hypobibasm" means dividing both sides of the equation by the unknown. It comes from the verb ὑποβιβάζω, "to lower." Division "lowers" a magnitude from a higher rung to a lower rung.

38 In modern notation: If $x^3+bx^2=cx$, $x^2+bx=c$. Here c is thought of as a "plane" so that cx is a solid; then it may be said that all the "solids" are divided by the common divisor x.

multiplied by some given magnitude, parabolism[39] may be effected.

There is parabolism whenever the homogeneous magnitudes of which an equation is composed are divided by a given magnitude which in the equation appears as multiplied by the higher rung of the unknown magnitude, so that that rung assumes the name of the power, and in that power the final equation remains. But this must now be demonstrated.

Proposition III

An equation is not changed by parabolism

Let it be given that "*B* in *A* square 'plus' *D* plane in *A*" is equal to "*Z* solid."

I say that by parabolism "*A* square 'plus' [(*D* plane)/*B*] in *A*" is equal to "*Z* solid"/*B*. For that means to have divided all the "solids" by the common divisor *B*, by which it is certain that the equation is not changed.[40]

12. And then the equation shall be thought to be expressed clearly and shall be called "well ordered": it must be capable of being referred to a proportion, the following condition (cautio) especially being satisfied: the product of the extremes must correspond to the power together with the conjoined homogeneous elements; the product of the means must correspond to the homogeneous element under the given measure.[41]

13. Whence also an ordered proportion may be defined as a series of three or four magnitudes, so expressed in terms either simple or conjoined that all are given except that

[39] "Parabolism" means dividing both sides of the equation by a known quantity. It comes from the verb παραβάλλω, "to apply," i.e., to divide, as when an area of *a* units is applied to a length of *b* units, the breadth of the figure will give the the quotient a/b. See Note 346.

[40] In modern notation: If $bx^2+cx=d$, $x^2+cx/b=d/b$ where c is thought of as a "plane" and d as a "solid."

[41] Thus, if we have the equation "*A* square $+B$ in *A* is equal to *C* in $D+C$ in *E*," then it follows that *A* is to *C* as "$D+E$" is to "$A+B$."

which is being sought, or else the power and the lower ladder magnitudes.[42]

14. Finally, when the equation has been thus ordered, or the proportion thus ordered, let it be considered that zetetics has performed its function.[43]

Diophantus in those books which concern arithmetic employed zetetics most subtly of all. But he presented it as if established by means of numbers and not also by species (which, nevertheless, he used), in order that his subtlety and skill might be the more admired; inasmuch as those things that seem more subtle and more hidden to him who uses the reckoning by numbers (logistice numerosa) are quite common and immediately obvious to him who uses the reckoning by species (logistice speciosa).[44]

Chapter VI

Concerning the investigation of theorems by means of the poristic art

When the zetesis has been completed, the analyst turns from hypothesis to thesis and presents theorems of his own finding, theorems that obey the regulations of the art and are subject to the laws 'κατὰ παντὸς, καθ' αὑτὸ, καθόλου πρῶτον,'[45] which theorems, although they have from the

[42] That is, an ordered proportion would be one of the type given in the preceding note or one of the type $x^2+ax:b::c:d+x$, which yields the equation $x^3+dx^2+adx+ax^2=bc$. This law and the preceding indicate that an ordered proportion is one that yields an equation in one unknown.

[43] Cf. Ch. I and P. 170.

[44] Cf. Pp. 165, 170, and Notes 244, 245.

[45] Cf. Note 235. These rules, as applied to the propositions of "poristic" (such, for example, as "*A* cube is equal to *B* plane in *C*") would seem to mean: (1) that the predicate must be "true of every instance" to which the subject is understood to refer, (2) that it must be predicated "essentially" of the subject, which would be the same as the law of homogeneity, and (3) that the predicate must be completely convertible with the subject, as is the case when the predicate is "commensurately universal"

zetesis their demonstration and firmness, are subjected to the law of synthesis, which is considered a more logical way of demonstrating; and whenever there is occasion, they are proved through it, yet by the great miracle of the art of finding. And for this reason, the steps of the analysis are retraced, which retracing is itself also analytical; and yet not in virtue of the reckoning by species (logistice speciosa), which has already performed its assigned duty. But if something unfamiliar has been hit upon and is proposed for proof, or if something has been presented by chance the truth of which must be weighed and investigated, then the way of poristic has first to be tried, from which it is easy to return to the synthesis; examples of this have been offered by Theon in the *Elements*, by Apollonius of Perga in the *Conics* and also by Archimedes himself in various books.[46]

Chapter VII
Concerning the function of the rhetic art

When the equation of the magnitude which is being sought has been set in order, the rhetic or exegetic (*ῥητικὴ ἢ ἐξηγητική*) art, which is to be considered as the remaining part of the analytical art and as one which pertains principally to the application of the art (since the two others are concerned more with general patterns than with precepts, as one must by right concede to the logicians), performs its function both in regard to numbers if the problem concerns a magnitude that is to be expressed by number, and in regard to lengths, surfaces, and solids if it is necessary to show the magnitude itself. And, in the latter case, the analyst appears as a geometer by actually carrying out the work in imitation of the like analytical solution; in the former case, he appears

with the subject. Regarding the last rule, we may remark that if one is to form the "synthesis" from the "analysis" by reversing or retracing the steps, each statement must be completely convertible.

[46] See P. 166 and Notes 233, 275.

as a logistician by resolving whatever powers have been presented numerically, whether simple powers or conjoined. Whether it be in arithmetic or geometry, he produces some specimens of his own [analytic] art according to the conditions of the equation that has been found or of the proportion that has been derived in an orderly way from it.

In fact, not every geometrical solution is a neat one, for particular problems have their own elegances. But that solution is preferred to others which does not derive the synthetic operation from the equation, but derives the equation from the synthesis, while the synthesis proves itself. Thus the skillful geometer, though a learned analyst, conceals this fact and presents and explicates his problem as a synthetic one, as if thinking merely about the demonstration that is to be accomplished; then, by way of helping the logisticians, he constructs and proves a theorem having to do with a proposition or an equation perceived in that synthetic problem.[47]

Chapter VIII
The symbolism in equations and and the epilogue to the art

1. In analysis the name equation is understood simply as referring to an equality properly set in order by means of the zetesis.

2. And so, an equation is the coupling (comparatio) of an unknown magnitude with a known.

3. The unknown magnitude is a root or power.

4. Again, a power is either simple or conjoined.

5. Conjunction exists either through subtraction or addition.

6. When an element homogeneous in conjunction is subtracted from a power, the subtraction is direct.[48]

[47] See Pp. 166–168 and Note 234.

[48] For example, "*A* cube $-$ *A* square in $G-A$ square in *H*," or in modern notation, $x^3-(gx^2+hx^2)$.

7. When, on the contrary, the power is subtracted from the element homogeneous in conjunction, the subtraction is inverse.[49]

8. The measuring subrung is the measure itself of the rung of the element homogeneous in conjunction.[50]

9. But it is necessary to designate the rank of the power, the rank of the lower rungs, and also the quality or sign of the conjunction. Also the coefficient subrung magnitudes must be given.

10. The first lower ladder magnitude is the root which is being sought. The last is that which is lower than the power by one rung of the ladder. This is customarily understood by the name "epanaphora."[51]

Thus "square" is the epanaphora of "cube," "cube" of "squared-square," "squared-square" of "squared-cube," and so on in the same series in infinitum.

11. A lower ladder magnitude is the reciprocal of a lower ladder magnitude when a power is produced through the multiplication of one by the other. Thus, the coefficient magnitude is the reciprocal of that rung which it sustains.

As, for example, if there should be a "side" which is a lower ladder magnitude in relation to the "cube," the reciprocal rung will be the "square." But a "plane" multiplied by a "side" will be a reciprocal magnitude in relation to the "side," since the "solid" is produced from the "side" multiplied by the "plane," the "solid" being itself a magnitude of the same rung as the "cube."

[49] For example, "*A* square in G+*A* square in *H*−*A* cube," or in modern notation $(gx^2+hx^2)-x^3$.

[50] This would seem to mean that in the case of "*A* cube+*A* square in G+*A* square in *H*," where "*A* square in G+*A* square in *H*" is the element homogeneous in conjunction, it is "*A* square" which, while being the subrung of "*A* cube," measures the whole element "*A* square in G+*A* square in *H*."

[51] In modern notation, x^2 is the "epanaphora" of x^3, x^3 of x^4, x^4 of x^5, etc. The word "epanaphora" is from ἐπαναφέρω, "to carry up to, refer to."

12. After the root the lower ladder magnitudes progressing by "length" are the same ones that are designated on the ladder.

13. After the root the lower ladder magnitudes progressing by "plane" are:

"Square"	"*Plane.*"
"Squared-square"	"*Square* of the plane."
"Cubed-cube"	"*Cube* of the plane.'

And so on successively in that order.

14. After the root the lower ladder magnitudes progressing by "solid" are:

"Cube"	"*Solid.*"
"Cubed-cube"	"Square of the *solid.*"
"Cubed-cubed-cube"	"Cube of the *solid.*"

15. "Square," "Squared-square," "Squared-cubed cube," and those magnitudes which are produced from these continuously in this order are simple middle powers; the rest are manifold.

Thus, the simple middle powers can also be defined in such a way that they will be those the exponents of which progress in the geometrical subduplicate ratio. So powers of the second degree, of the fourth, of the eighth, of the sixteenth, will be simple middle powers. The remaining powers, standing in the intermediate degrees, are manifold.[52]

16. A known magnitude with which the others are equated is the homogeneous element of the equation.

*As, for example, if "*A *cube+* A *in* B *square is equal to* B *in* Z *plane," "*B *in* Z *plane" will be the homogeneous element of the equation.*

[52] If, of the series of ladder magnitudes we consider x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}, x^{16} . . . and then the exponents of the powers x^2, x^4, x^8, x^{16} . . ., we see that the exponents are in geometrical progression: 2:4::4:8::8:16. They may be said to progress in the subduplicate ratio in that 2:4 is the subduplicate of 2:8, 4:8 is the subduplicate of 4:16, etc.

"A *cube*" *will be the power to which the unknown magnitude which is being sought ascends by its own nature*

"A *in* B *square*" *will be the element homogeneous in conjunction.*

"A" *is the lower ladder magnitude.*

"B *square*" *is a subrung magnitude or* "*parabola.*"[53]

17. In the case of numbers, the homogeneous elements of equations are units.[54]

18. When a "root" that is being sought is, while remaining on its own base, equated to a given homogeneous magnitude, the equation is simple absolutely.[55]

19. When the power of a "root" that is being sought, being free from all conjunction, is equated to a given homogeneous magnitude, the equation is simple ladderwise.[56]

20. If the power of a "root" that is being sought is joined with magnitudes at the designated rung accompanied by their given coefficients and if it is equated to a given magnitude, the equation is polynomial in proportion to the multitude and variety of the conjunction.[57]

21. A power can be involved in as many conjunctions as there are ladder magnitudes lower in relation to that power.

Thus, a "square" can be conjoined with a magnitude at the rung of the "side"; a "cube" with magnitudes at the rungs of the "side" and the "square"; a "squared-square"

[53] "Parabola" here means the result of application or the quotient resulting from division by the unknown.

[54] In the equation "*A* cube is equal to *B* plane in *C*," the homogeneous element of the equation is "*B* plane in *C*" and its unit is a "solid" unit. We could not have "*A* cube is equal to *B* plane," for the unit of "*B* plane" is a "plane" unit. It is different with equations involving numbers rather than species. In the equation "*A* is equal to 9." "*A*" becomes a number, and the units of all numbers are the same in kind as long as the numbers are pure numbers.

[55] "*A* is equal to *B*," or $x=a$ in modern notation.

[56] "*A* cube is equal to *B* solid," or $x^3=a$ in modern notation, *a* being understood as a "solid."

[57] "*A* cube+*B* in *A* square−*C* plane in *A* is equal to *D* solid" or, in modern notation, $x^3+ax^2-bx=c$, where *b* is a "plane" and *c* is a "solid."

with magnitudes at the rungs of the "side," the "square," and the "cube"; a "squared-cube" with magnitudes at the rungs of the "side," the "square," the "cube," and the "squared-square"; and so on in that series *in infinitum.*

22. Proportions are distinguished from one another and receive their nomenclature from the kinds of equations into which they are resolved.

23. With a view to exegetic in arithmetic the trained analyst is taught:

To add a number to a number.
To subtract a number from a number.
To multiply a number by a number.
To divide a number by a number.

The analytical art, furthermore, yields the resolution of all possible powers whether they be pure or (a thing of which both ancients and moderns have been ignorant) conjoined with other magnitudes.[58]

24. With a view to exegetic in geometry the analytical art selects and enumerates more regular procedures by which equations of "sides" and "squares" may be completely interpreted.[59]

25. With a view to "cube" and "squared-square," in order that the deficiency of geometry may be supplied as if by geometry, the analytical art postulates that

[58] This is the program of Vieta's work *De Numerosa Potestatum Purarum, atque Adfectarum ad Exegesin Resolutione Tractatus*, (*Opera Mathematica*, p. 163, cf. Note 210). The "numerical resolution of powers" means the solution of equations that have numerical solutions, such as the equation $x^2=2916$ (Problem I, p. 165, of the first section of the *De Numerosa*, which section has to do with pure powers), or $x^2+7x=60{,}750$ (Problem I, p. 174, of the second section, which has to do with conjoined powers).

[59] This is the program of Vieta's *Effectionum Geometricarum Canonica Recensio* (*Opera Mathematica*, pp. 229 ff.), at the beginning of which he says, "The geometrical procedure by which all equations *which do not exceed the measure of squares* may be rightly interpreted I enumerate as follows. . . ."

A straight line can be drawn from any point across any two lines in such a way that the intercept between these two lines will be equal to a given distance, any possible intercept having been predefined.

This being granted (it is, indeed, a postulate not difficult to fulfil), analysis skilfully solves the more famous problems which have hitherto been called irrational: the mesographicum, the problem of the division of an angle into three equal parts, the finding of the side of the heptagon, and as many others as fall into the formulas of equations in which "cubes" are equated to "solids," "squared-squares" to "plane-planes," whether simply or with some conjunction.[60]

26. Since all magnitudes are lines, surfaces, or solids, what great use could be made in human affairs of proportions involving triplicate or even quadruplicate ratios, if not perhaps in divisions of angles, so that we might obtain the angles from the sides of the figures or the sides from the angles?

27. Therefore, analysis, whether with a view to arithmetic or to geometry, discloses the mystery, known hitherto by no one, of the division of angles, and it teaches how:

When the ratio of the angles is given, to find the ratio of the sides.

To make an angle to be in the same ratio to an angle that a number is to a number.

28. It does not equate a straight line to a curve, because an

[60] This is the program of Vieta's *Supplementum Geometriae* (*Opera Mathematica*, pp. 240 ff.), which begins with a restatement of the postulate about the intercept and which contains Vieta's solutions of the three problems here mentioned. Propositions V–VII of the *Supplementum Geometriae* contain the solution of the problem of the mesographicum; this was the problem of finding two mean proportionals to two given straight lines, and its solution immediately yields the solution of the problem of doubling the cube. Proposition IX contains the solution of the problem of the trisection of an angle. Proposition XXIV contains the solution of the problem of finding the side of the regular heptagon which is to be inscribed in a given circle.

angle is something in between a straight line and a plane figure. Thus, the law of homogeneity seems to oppose it.

29. Finally, the analytical art, having at last been put into the threefold form of zetetic, poristic, and exegetic, appropriates to itself by right the proud problem of problems, which is:

TO LEAVE NO PROBLEM UNSOLVED.[61]

[61] The capital letters are Vieta's; see P. 185.