MATH 110-02 – Algebra Through History
Sample Midterm Exam Questions – Solutions

Disclaimer: The following questions indicate the rough length of the coming midterm and the "mix" of different kinds of questions I might ask. The actual exam might be set up somewhat differently and it might draw on other topics from the review sheet.

I. Refer to the Old Babylonian tablet YBC 7289 in Fig 1 (at top of next page).

A) The base-60 form of the number written directly over the diagonal of the square has just one base-60 digit to the left of the semicolon separating the whole number part from the fractional part. Give all the base-60 digits, separated by the semicolon and commas.

Solution: In our representation of the base-60 system, it is

$$1; 24, 51, 10$$

B) Express the whole number from part A in base 10. (It's OK to leave this as a sum of fractions if you don't have a calculator.)

Solution: The equivalent base-10 number is

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \doteq 1.4142$$

(rounding to 4 decimal places).

II.

A) What does it mean to say that that the diagonal of a square and the side of that square are *incommensurable*?

Solution: It means that there are no positive integers m, n such that

$$m \cdot (\text{diagonal}) = n \cdot (\text{side}),$$

or equivalently that the ratio

$$\frac{\text{diagonal}}{\text{side}}$$

is not a rational number (a fraction with integer numerator and denominator).

B) Give the proof the ancient Greeks found for the fact in part A.

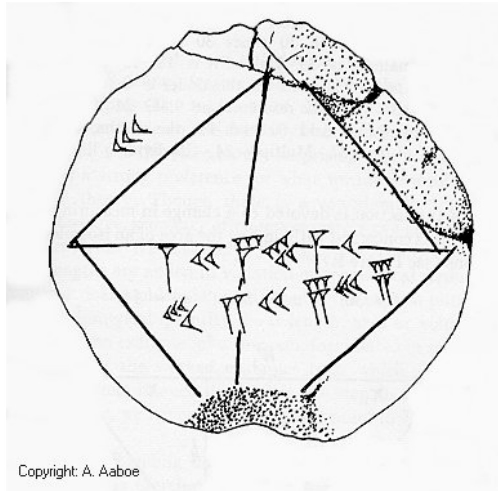

Figure 1: Drawing of the YBC 7289 tablet by Asger Aaboe. I do not have his permission to use this, but he died in 2007, so I think the copyright issue is effectively moot(!)

Solution: Suppose the side is 1. Then the diagonal has length $\sqrt{2}$ by the Pythagorean theorem. If $\sqrt{2} = \frac{m}{n}$ for some integers m, n, then we can take the fraction in *lowest terms*; that is, we can assume that m, n have no common factors. Squaring, we get $m^2 = 2n^2$. This shows m^2 is even and it follows that m is even too, since the square of an odd number is odd. Hence $m = 2k$ for some integer k. Substituting, we get $4k^2 = 2n^2$. So then $n^2 = 2k^2$, and as before n^2 must be even, so n is also even. But this is a contradiction since we assumed m, n had no common factors, and we have shown that they must both be divisible by 2. Hence $\sqrt{2}$ and 1 are incommensurable. (The same is true in any square since the fraction $\frac{\text{diagonal}}{\text{side}}$ is also equal to $\sqrt{2}$.)

C) What consequences did the discovery of incommensurable magnitudes have for the way the Greeks thought of (pure) mathematics?

Solution: Especially in Euclid's *Elements*, we see that numbers and magnitudes like lengths were treated entirely separately. Euclid never uses a numerical measure of a length (or an angle, or an area, or a volume) in the geometric books I - VI and X - XIII. The properties of numbers are treated separately in Books VII-IX.

III. Short answer. Answer any four of the following. If you answer more than four, the best four will be used.

A) What is the main feature that makes the presentation of the number theoretic problems in Diophantus' *Arithmetica* different from earlier mathematics and a forerunner of what we do now.

Solution: Diophantus was the first mathematician to use *symbolic* representations of algebraic expressions and techniques for manipulating those symbolic expressions to solve equations.

B) What is the "algebraic form" of the fact proved in Proposition II.5 in Euclid's *Elements*? (The statement: If a straight line is cut into equal and unequal pieces then the rectangle contained by the unequal pieces of the whole line, plus the square on the difference between the equal and unequal pieces is equal to the square on half of the line.)

Solution: One possible algebraic form is obtained by letting the unequal pieces of the straight line be represented by $x > y$. Then the equal pieces have length $\frac{x+y}{2}$, and the statement can be translated as

$$xy + \left(\frac{x+y}{2} - y\right)^2 = \left(\frac{x+y}{2}\right)^2$$

(The difference between the equal and unequal pieces is the

$$\frac{x+y}{2} - y$$

on the left side. Since we assume $x > y$, we have

$$x > \frac{x+y}{2} > y,$$

and the difference is between the equal pieces and the *smaller* of the unequal pieces.)

C) What geometric fact did Archimedes have inscribed on his tombstone?

Solution: Let a sphere be inscribed in a cylinder whose height is equal to twice the radius. The statements are that the volume of the cylinder is 3/2 the volume of the inscribed sphere, and the lateral surface area of the cylinder is equal to the surface area of the inscribed sphere. (Note: this says the volumes of the sphere and the cylinder *are* commensurable in the language from question II above.)

D) What are the approximate dates of the YBC 6967 and YBC 7289 tablets? About when did Euclid live?

Solution: The Babylonian tablets are from somewhere between 2000BCE and 1600BCE, probably toward the middle of that range. Euclid lived some time around 300BCE.

E) What does the $M^o 20\varsigma 1$ in Diophantus' Proposition I.4 mean?

Solution: It means 20 units plus one unknown, or as we might write, $20 + x$, where x is the unknown.

G) What is "Fermat's Last Theorem" and what is its connection with Diophantos? When and by whom was this finally solved?

Solution: "Fermat's Last Theorem" is the statement that the equation $x^n + y^n = z^n$ has no integer solutions with x, y, z all nonzero when $n \geq 3$. The connection with Diophantus is that Pierre Fermat wrote this in his copy of a Latin translation of Diophantus' *Arithmetica*. He said he had found a "marvelous proof" but that the margin of the page was too small to contain it. It is now thought that he probably did not have a complete proof. This was finally proved by Andrew Wiles (and Richard Taylor) and announced to the world in 1993.

H) How would the Old Babylonian scribes have approached a large calculation such as multiplying $23,41;16$ by $17,28;30$? (You don't need to do the calculation.) Compare this with what we need to know to do multiplication in our base-10 positional number system.

Solution: They would have used essentially the same method we do for multiplication, but in base-60. They would have had a base-60 multiplication table (giving all the products of one base-60 digit times another) to use for the digit-by-digit multiplications. They would have "carried" into the next digit the way we do when the product was larger than 60.

IV. Essay. You have the choice of responding to *either prompt 1 or 2*. State which one you have chosen at the start of your essay.

1) A certain older history of mathematics says, flatly, that "the distinguishing feature of Babylonian mathematics is its algebraic character." Of the historians we have mentioned, who would agree with this claim and who would disagree? Explain using the the interpretations your historians would give for the YBC 6967 problem of (what we would phrase as) solving the equation $x = 60/x + 7$.

2) George G. Joseph, the author of another book on the non-European roots of modern mathematics called *The Crest of the Peacock*, offers this overall evaluation of the ultimate impact of Greek geometry: "There is no denying that the Greek approach to mathematics produced remarkable results, but it also *hampered* the subsequent development of the subject. ... Great minds such as Pythagoras, Euclid, and Apollonius spent much of their time creating what were essentially abstract idealized constructs; how they arrived at a conclusion was in some way more important than any practical significance." First, what does the last sentence mean? Would this criticism seem to be apt for Diophantos' *Arithmetica* as well? Is it necessary for all the mathematics we learn and do to have practical usefulness or significance?