Two “case studies” on Islamic mathematics

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Outline

Getting here from there

1. Try to get some evidence regarding the claims of historians like Morris Kline that Islamic mathematics only "transmitted" Greek knowledge to Europe without adding anything very important or original
2. Look at a particular family of questions in number theory
3. Trace history from Euclid through the contributions of several Islamic mathematicians; see what we know now
4. (More briefly) to consider some work of Umar al-Khayyami (Omar Khayyam) on solving cubic equations

Back with the Pythagoreans

1. In early Greek mathematics the Pythagorean brotherhood
2. very much “into” number symbolism
3. the “tetraktys,” triangular numbers, squares, pentagonal numbers, ... , “perfect numbers,” “amicable pairs of numbers,” etc.
4. part of their philosophy of the way numbers and mathematical relationships were part of the order underlying the cosmos

A bit of number theory from the *Elements*

1. In Proposition 36 of Book IX of the *Elements*, Euclid studied *perfect numbers*

Definition

A number n is said to be perfect if it equals the sum of all of its proper divisors ("aliquot parts" – the whole numbers less than n that divide n evenly)

2. For example, $n = 6$ is a perfect number (the smallest one) since the proper divisors of 6 are $1, 2, 3$ and $1 + 2 + 3 = 6$
3. Similarly $n = 28$ is perfect since its proper divisors are $1, 2, 4, 7, 14$ and $1 + 2 + 4 + 7 + 14 = 28$

Euclid's result

Theorem (*Elements* IX, 36)

A number $n = 2^{p-1} \times (2^p - 1)$ *is perfect if and only if* $2^p - 1$ *is prime.*

1. Recall, primes are numbers p whose only divisors are 1 and the number itself: $p = 2, 3, 5, 7, 11, 13, 17, 19, ...$
2. Euclid had shown in Book IX, 20 that there are infinitely many primes (one of the jewels of this section of the *Elements*)
3. Not too hard to see that $2^p - 1$ can only be prime if p itself is prime

Perfect numbers

1. For example, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$ are prime
2. But $2^{11} - 1 = 2047 = 23 \cdot 89$ is not prime
3. So by Euclid IX, 36, for instance $n = 2^{3-1} \times (2^3 - 1) = 28$, $n = 2^{5-1} \times (2^5 - 1) = 496$, $n = 2^{7-1} \times (2^7 - 1) = 8128$ are perfect numbers
4. But $2^{11-1} \times (2^{11} - 1) = 2096128$ is not perfect

Ibn al-Haytham and Thabit ibn-Qurra

1. Another Islamic mathematician named Ibn al-Haytham (ca. 1000 CE) (known as "Alhazen" in medieval and later Europe through a mangling of his real name!) studied this and conjectured that the *only time n* is an even perfect number is when n has Euclid's form $n = 2^{p-1} \times (2^p - 1)$
2. But he was not able to prove this completely
3. Thabit ibn-Qurra (836 - 901 CE), whom we met briefly before, was interested in perfect numbers too, and other related questions

Definition

Two numbers m, n are said to be an amicable pair if the sum of the proper divisors of m is n and, vice versa, the sum of the proper divisors of n is m.

The smallest example

1. $m = 220$ and $n = 284$ are the smallest amicable pair:
2. The proper divisors of $m = 220 = 2^2 \times 5 \times 11$ are $1, 2, 4, 5, 10, 20, 11, 22, 44, 55, 110$ and

   $$1 + 2 + 4 + 5 + 10 + 20 + 11 + 22 + 44 + 55 + 110 = 284,$$

   while
3. the proper divisors of $n = 284 = 2^2 \times 71$ are $1, 2, 4, 71, 142$ and

   $$1 + 2 + 4 + 71 + 142 = 220.$$

Thabit's theorem

Thabit was interested in determining whether there was a systematic way to generate amicable pairs – something along the lines of Euclid's formula for perfect numbers.

Theorem

Suppose that for some integer $k \geq 1$, $p = 3 \times 2^{k-1} - 1$, $q = 3 \times 2^k - 1$ *and* $r = 9 \times 2^{2k-1} - 1$ *are all primes. Then* $m = 2^k \times p \times q$ *and* $n = 2^k \times r$ *are an amicable pair.*

Example: $k = 2$ gives $p = 5, q = 11, r = 71$ all prime. So $m = 4 \times 5 \times 11 = 220$ and $n = 4 \times 71 = 284$ are amicable(!)

A more balanced view(?)

1. The Islamic tradition widened the range of mathematics by enlarging the role of numerical and algebraic computation and introducing the number system we still use – not just geometry
2. But these mathematicians also studied the foundations of Euclidean geometry and attempted to resolve some of the outstanding questions
3. Extended work of Archimedes and other Greeks in several ways

Leonhard Euler

1. Alhazen's conjecture about Euclid's formula for perfect numbers was eventually proved true by Leonhard Euler (1707 - 1783, CE) a Swiss mathematician, one of the most prolific and original mathematicians of the modern era, built on work of Pierre de Fermat, Mersenne that *rediscovered* the results of Thabit ibn-Qurra, and others.
2. Might they actually have had access to Thabit's work??
3. Euler also generalized Thabit ibn-Qurra's method for generating amicable pairs and showed it could be used to get lots of them!

Mathematics is not "finished!"

1. Students encountering mathematics in school may be tempted to think that the subject is "finished" and that everything that can be known about it is already known
2. But this is *far from true*!!!
3. The very questions we are talking about here include some very famous *unsolved problems* in mathematics – cases where no one knows the answer and where, if one of you were to be able to develop an answer, your name would join the ranks of famous mathematicians.

For instance, ...

1. In Euclid's formula, $n = 2^{p-1} \times (2^p - 1)$, natural to ask: Are there infinitely many primes p for which $2^p - 1$ is also a prime number? Or does this generate only finitely many different perfect numbers?
2. Answer: No one knows! Such primes are called "Mersenne primes" after a later French monk and mathematician Marin Mersenne (1588 - 1648, CE) who studied them
3. At the current time (as of November, 2019), exactly 51 Mersenne primes are known to exist. The largest is $2^{82,589,933} - 1$
4. Found via extensive computer calculations including massive computations on distributed networks of computers over the internet – GIMPS project: can donate your computer's free time to the search if you want!

For instance, ...

1. In Euclid's formula for perfect numbers, $n = 2^{p-1} \times (2^p - 1)$, is always *even*, so when n is perfect, we only get even perfect numbers this way.
2. Question: Are there any *odd* perfect numbers?
3. Answer: Nobody knows!!
4. Some indications that the answer is probably "no," but nothing definitive yet. Any progress here would basically make a mathematician's career!

For instance, ...

1. Similarly, one can ask: Are there infinitely many amicable pairs?
2. Answer: Nobody knows!!
3. By using Thabit's formulas, Euler's generalizations, lots of computing, many such pairs are known (in the millions!) but again, nothing definitive.

Umar al-Khayyami and cubic equations

1. Umar made a systematic study of classes of cubic equations
2. Very similar in spirit to what Al-Khwarizmi had done for quadratic equations
3. E.g. "A cube plus sides equals numbers" was one case
4. In modern notation, $x^3 + cx = d$
5. Still assuming $c, d > 0$ here, so for instance, al-Khayyami would treat $x^3 = cx + d$ as a different type of equation.

al-Khayyami's solution

1. Heavily based on properties of conic sections – work of Apollonius
2. Apollonius' *Conics* had been translated from Greek to Arabic some time before and was a standard reference for mathematicians at this time.
3. The solution of the cubic equation $x^3 + cx = d$ was phrased in *geometric terms*.
4. In modern language:

The solution (in modern terms), continued

1. Take the number $\frac{d}{2c}$ and construct a circle with that radius and center at $\left(\frac{d}{2c}, 0\right)$.
2. The equation of that circle is $x^2 - \frac{d}{c}x + y^2 = 0$
3. Also construct the parabola with equation $y = \frac{1}{\sqrt{c}}x^2$.
4. Then substituting for y in the equation of the circle,

$$x^2 - \frac{d}{c}x + \frac{1}{c}x^4 = 0$$

so rearranging and canceling one power of x,

$$x^3 + cx = d \qquad (\textit{Pretty amazing}!)$$

Conclusions

1. The other cases are treated using some of the same ideas.
2. This was definitely an advance over what the Greeks had done before,
3. and over what others were doing at this time in Europe.