MATH 110-02 – Algebra Through History
Problem Set 4
due: 5:00pm on Friday, November 22

Background

As we said in class, Viète's work on algebra was, in some ways, a step backward from the work of Islamic mathematicians like Al-Khwarizmi and others. The Islamic mathematicians had what looks mostly like a more modern understanding of quadratic and higher-degree equations because they did not insist on the geometric interpretations of terms and they did require the *homogeneity condition* that was a key idea for Viète. *On the other hand,* Viète's work did represent a real advance in other ways, in particular in the use of symbolic parameters (written as letters) in algebraic equations. In addition to the *Introduction to the Analytic Art* that we are reading, he published many other works including one called the *Zetetics* (after the "zetetic" phase of *analysis* in studing a mathematical problem). In this work, he reconsidered many of the Propositions proved by Diophantus and showed (in effect) that Diophantus' methods were actually general, even though Diophantus gave only a single numerical example. In this problem set, you will look at two of those reworkings and then derive what are now called Viète's equations relating the roots and the coefficients of a polynomial.

I. From the First Book of Viète's *Zetetics*: [Compare with Diophantus I.1 (see previous handout)] Given a difference of two sides and their sum, find the sides. Let the difference of the two sides be given as B and the sum given as D. It is required to find the sides. Let the lesser side be A. The greater will then be $A + B$ and the sum of the sides is $2A + B$ and this equals D. By antithesis, $2A = D - B$, and one side is $A = (D - B)/2$ and the larger one is $A + B = (D + B)/2$. ... *Indeed: The half sum of the sides minus the half difference is equal to the lesser side and the greater is the lesser plus the difference.*

A) How does the statement in italics relate to what we said when we discussed Diophantus I.1? Discuss the similarities and the differences between this and what Diophantus does.

B) What are the sides if the difference is 80 and the sum is 131? (It seems that Viète is not bothered at all by fractions(!))

II. We discussed the following example from the Fourth Book of Viète's *Zetetics* in class on November 18: [also compare with Diophantus, II.8 (see handout on Diophantus from the course homepage)]. *To find two squares that sum to a given square.* Let the given number be F *square*. It is required to find two squares that sum to the given F *square*. Let any right triangle be given with sides B, D and hypotenuse Z. Among the triangles similar to this one, there will be one with hypotenuse F. Therefore the squares from $\frac{B \text{ in } F}{Z}$ and $\frac{D \text{ in } F}{Z}$ will add to equal the given F *square*.

A) In the explanation, Viète says that one way to get the numbers B, D, Z is to use the formulas

$$B = R^2 - S^2$$
$$D = 2RS$$
$$Z = R^2 + S^2$$

for some numbers $R > S$. Show that these formulas always give the lengths of the sides of a right triangle. (Hint: That means you want to show $Z^2 = B^2 + D^2$.)

B) To use the formulas in part A, we would look for R, S to make $Z = R^2 + S^2$ in some "easy" ratio to the F that is given. For instance, Viète gives the example $F\ square = 100^2$ and says to choose $R = 4$, $S = 3$. What are the B, D, Z then? What is the proportionality factor you need to make a similar triangle with hypotenuse 100 and what are the squares that sum to 100^2 you obtain?

III. ("Viète's Formulas").

A) Let $f(x) = x^3 + Ax^2 + Bx + C$ be a cubic polynomial that factors as $f(x) = (x - c_1)(x - c_2)(x - c_3)$ Show (by expanding out the factored form and comparing coefficients after collecting powers of x) that

$$A = -(c_1 + c_2 + c_3)$$
$$B = c_1c_2 + c_1c_3 + c_2c_3$$
$$C = -c_1c_2c_3$$

B) What are the corresponding formulas for the coefficients A, B, C, D in polynomials of degree 4 in terms of the roots c_1, c_2, c_3, c_4:

$$f(x) = x^4 + Ax^3 + Bx^2 + Cx + D = (x - c_1)(x - c_2)(x - c_3)(x - c_4)?$$

C) (Extra Credit) What is the general pattern here? How do you get the coefficient of x^{n-k} in a polynomial of degree n if the roots are $c_1, \ldots, c_n$?