MATH 110-2 – Algebra Through History
Greek Thinking about Commensurability, Concepts of Magnitude and Number,
from Plato (ca. 424 - 348 BCE) and Euclid, *Elements* (ca. 300 BCE)[1]
September, 2019

Definitions from Book V

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.[2]
2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.
3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.
4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.[3]
5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.[4]
6. And let magnitudes having the same ratio be called proportional.

Definitions 2,3 from Book VII

2. A number is a multitude composed of units.
3. A number is a part of a number, the lesser of the greater, when it measures the greater.

Definition 1 from Book X

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.[5]

[1] Quotations taken from the online version of the Heiberg edition of Euclid with English translation by Richard Fitzpatrick

[2] In other words, α is said to be a part of β if $\beta = m\alpha$ (for some integer m, comment added by JL)

[3] In other words α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$ (JL: for some integers m, n)

[4] In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenver $m\gamma > n\delta$, and $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenver $m\gamma < n\delta$. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, etc. are irrational.

[5] By JL: In Proposition 5 of Book X, Euclid shows that if two magnitudes α, β are commensurable then $\alpha : \beta :: m : n$ for some integers m, n. Then Proposition 6 shows that the implication goes the other way too, so this definition is actually equivalent to saying $\alpha : \beta :: m : n$ for some integers m, n.

Proposition. Let it be proposed to show that the diagonal of a square is incommensurable with its side.[6]

Proof: Let the square be $ABCD$ whose diagonal is AC. I say that the diagonal AC is incommensurable with AB, its side.

Suppose that these magnitudes were commensurable. I say their ratio is then the same as $a : b$ be where a, b are integers, and let this ratio have the smallest numbers among all with the same ratio [JL: that is, in our language, the fraction a/b is in "lowest terms"]. Then $a > b$ and necessarily $a > 1$ because $AC > AB$. It is clear the square on AC is twice the square on AB in area [JL: by the form of the Pythagorean Theorem from *Elements*, Book I, Proposition 47] and hence $a^2 = 2b^2$. Therefore a^2 is even and a is even, since if a number is odd, so is its square. Since $a : b$ is in lowest terms, the numbers a and b cannot have any common factors and it follows that b must be odd. Put $a = 2c$. Therefore $4c^2 = 2b^2$ and hence $b^2 = 2c^2$, so b and hence b must be even. But b was also odd which is impossible. Therefore the diagonal of a square is incommensurable with its side–what was to be proved.

A Related Passage from Plato (ca. 424 - 348 BCE) *Theaetetus*[7]

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Theaetetus: Theodorus here was proving to us something about square roots, namely, that the sides [or roots] of squares representing three square feet and five square feet are not commensurable in length with the line representing one foot, and he went on in this way, taking all the separate cases up to the root of seventeen square feet. There for some reason he stopped. The idea occurred to us, seeing that these square roots were evidently infinite in number, to try to arrive at a single collective term by which we could designate all these roots.

Socrates: And did you find one?

Theaetetus: I think so, but I should like your opinion.

Socrates: Go on.

Theaetetus: We divided number in general into two classes. Any number which is the product of a number multiplied by itself we likened to a square figure, and we called such a number 'square' or 'equilateral.'

Socrates: Well done!

Theaetetus: Any intermediate number, such as three or five, or any number that cannot be

[6]by JL: This proposition and the following proof were included in Euclid, *Elements*, Book X, as Proposition 117 in some versions. However, modern scholarship has identified it as a later interpolation from a commentary on Aristotle's *Prior Analytics*, where Aristotle discusses the logical structure of the proof – a *proof by contradiction*. Many modern editions of Euclid do not include it for that reason. The standard edition of the Greek by J.L.Heiberg includes it in an appendix to the main text, but this was not translated by Fitzpatrick. Non-literal, reorganized and slightly modernized translation from the Greek by JL.

[7]Plato, The Collected Dialogues, Hamilton and Cairns, eds., pp. 852-853

obtained by multiplying a number by itself, but has one factor either greater or less than the other, so that the sides containing the corresponding figure are always unequal, we likened to the oblong figure, and we called it an oblong number.

Socrates: Excellent. And what next?

Theaetetus: All the lines which form the four equal sides of the plane figure representing the equilateral number we defined as *length*, while those which form the sides of squares equal in area to the oblongs we called *roots* [surds], as not being commensurable with the others in length, but only in the plane areas to which their squares are equal. And there is another distinction of the same sort in solids.

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Comment by JL: The general theme of this Platonic dialogue is the question, "what is knowledge?" The point made by Theaetetus here seems to be that simply having *names for things* can be knowledge of a sort. But as is true in many of the Platonic dialogues, Socrates is eventually going to get him to admit that neither this, nor any of the other possible definitions of knowledge that they consider is satisfactory – knowledge is not just having a name for something; it is not just *perception*, it is not *true judgment*; and it's not even *true judgment with an argument in support of the judgment*. There is no definitive answer to the main question at the end of the conversation(!)