Some high points of Greek mathematics after Euclid

Algebra Through History

October 2019

Outline

Who was Archimedes?

- Lived ca. 287 - 212 BCE, mostly in Greek city of Syracuse in Sicily
- Studied many topics in what we would call mathematics, physics, engineering (less distinction between them at the time)
- We don't know much about his actual life; much of his later reputation was based on somewhat dubious anecdotes, e.g. the "eureka moment," inventions he was said to have produced to aid in defense of Syracuse during Roman siege in which he was killed, etc.
- Perhaps most telling: we do know he designed a tombstone for himself illustrating the discovery he wanted most to be remembered for (discussed by Plutarch, Cicero)

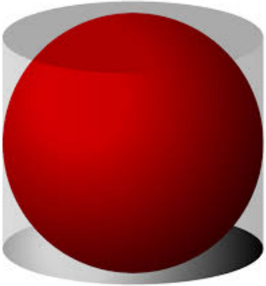


Figure: Sphere inscribed in cylinder of equal radius

$3V_{sphere} = 2V_{cyl}$ and $A_{sphere} = A_{cyl}$ (lateral area)

Surviving works

- *On the Equilibrium of Planes* (2 books)
- *On Floating Bodies* (2 books)
- *Measurement of a Circle*
- *On Conoids and Spheroids*
- *On Spirals*
- *On the Sphere and Cylinder* (2 books)

Surviving works, cont.

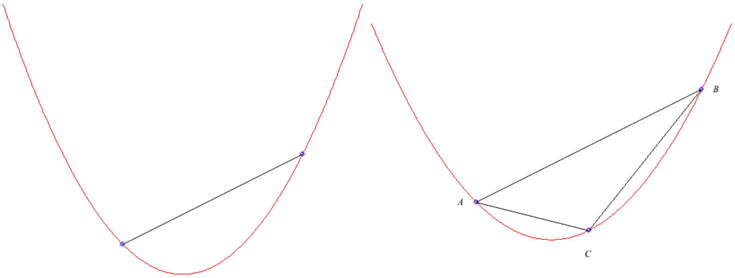
- *Book of Lemmas*, *The Sand-Reckoner*, *The Cattle-Problem*, the *Stomachion*
- *Quadrature of the Parabola*
- *The Method of Mechanical Theorems*
- *The Method* was long known only from references in other works
- 1906 – a *palimpsest* prayerbook created about 1229 CE (a reused manuscript) was found by to contain substantial portions (a 10th century CE copy from older sources)

A page from the palimpsest

Scientific context

- Archimedes flourished at height of Hellenistic period; Greek kingdoms (fragments of Alexander's empire) in Egypt, Syria, Syracuse strong patrons of science and mathematics
- State of the art in mathematics: 2nd or 3rd generation *after* Euclid, rough contemporary of Apollonius (best known for work on conic sections)
- Part of an active scientific community in fairly close communication; for instance several of the works cited above start with letters to colleagues
- Example: Letter to Eratosthenes (in Alexandria) at start of *The Method*

What is the area of a parabolic segment?



- Let C have same x-coordinate as the midpoint of AB
- The area of the parabolic segment $= \frac{4}{3}\text{area}(\Delta ABC)$

"Quadrature of the Parabola"

- In the work in the title of this slide, Archimedes gives two different arguments for this statement,
- One using the "method of exhaustion" – part of standard repertoire of Euclidean mathematics (see Book 12 of *Elements*), based on earlier work of Eudoxus
- A recurring theme in his mathematical work (used it as a standard method to obtain areas and volumes, approximate the value of π, etc.)
- One by the novel method also described in *The Method*
- an application of mechanics that anticipated some ideas in integral calculus(!)

Who was Apollonius?

- Lived ca. 262–190 BCE, born in Perga (south coast of present-day Turkey)
- Active roughly 75 - 100 years after time of Euclid (ca. 300 BCE); slightly younger than Archimedes (ca. 287–212 BCE)
- Studied with successors of Euclid at *Museum* in Alexandria
- Have lists of his works from later commentaries, but most have not survived
- Know he did astronomy as well, work incorporated into Claudius Ptolemy's (90 - 168 CE) geocentric model of solar system with epicycles, etc.

From Taliaferro's translation – start of Book I

A few historical details can be gleaned from prefatory letters at the start of several books of the *Conics*, e.g. from letter to Eudemus at head of Book I:

... I worked out the plan for these conics at the request of Naucrates, the geometer, at the time he was with us in Alexandria lecturing, and ... on arranging them in eight books, we immediately communicated them in great haste because of his near departure, not revising them but putting down whatever came to us ... it happened that some others among those frequenting us got acquainted with the first and second books before the revision ...

Previous work on conics

- Some historians attribute the first work on conics to Menaechmus (ca. 380 - 320 BCE; in Plato's circle)
- But conics only feature in his solutions of the "Delian problem" – duplication of the cube – using parabolas and/or hyperbolas
- (More) systematic work on conics by Aristaeus (before the time of Euclid) and Euclid (ca. 300 BCE) himself
- Those older works are known now only through comments made by Pappus of Alexandria (ca. 300 - 350 CE) in his *Collection* – a sort of summary and encyclopedia of much of the Greek mathematics of the classical and Hellenistic periods – and the commentaries on Euclid by the Neoplatonist philosopher Proclus (412 - 485 CE)

Theory of conic sections before Apollonius

- On the basis of some details preserved, it's thought that the earlier work dealt only with *right cones* generated by rotating a right triangle about one of its legs, and slicing by planes perpendicular to the other leg.
- Note that then the *vertex angle* of the cone determines the type of the section obtained:
- In later terminology: acute angles – ellipses; right angles – parabolas; obtuse angles – hyperbolas
- Apollonius discusses some aspects of this earlier work in rather disparaging ways at several points; judging from the tone, he seems (to me at least) to have had a “prickly” streak(!)

Archimedes and conics

- Archimedes is probably the best-known of the Greek mathematicians following Euclid (certainly more commonly read than Apollonius – more accessible in several ways!)
- His work also involves conic sections to a large degree, especially the *Quadrature of the Parabola*
- He does not use the Apollonian terminology
- calls a parabola a “section of a right-angled cone”

Plan of Apollonius' *Conics*

- Books I, II, III, IV – "Elements of conics" – definitions and basic properties; properties of asymptotes of hyperbolas; tangents; intersections of conics
- The above survive in original Greek versions. The following are only known through later Arabic translations:
- Book V, VI, VII – "Researches on conics" – normals to conics, maximum and minimum distances from a point, equality and similarity of conics, "limiting properties"
- Book VIII – ? (lost – several attempts at "reconstruction" including one famous one by E. Halley, 1710 CE)

Apollonius' framework is more general

- Definition 1 from Book I: *If, from an arbitrary point, a line is drawn to the perimeter of a circle which does not lie in the same plane with the point and extended indefinitely in both directions, and with the point remaining fixed, the line is moved around the circle back to its starting position, then the surface described by the line, which consists of two surfaces both containing the fixed point, ... I will call a conic surface; I will call the fixed point the vertex of the conic surface and the line through the vertex and the center of the circle I will call the axis.* (translation by JBL, close to literal)
- A *cone* is the figure bounded by a nappe of the conic surface and a plane parallel to the circle's plane.

Apollonius' Definition 4 – anachronism?

- Apollonius defines a *diameter* of a plane curve in his Definition 4: *a straight line that bisects all the straight lines between pairs of points of the curve, drawn parallel to some given straight line, is called a diameter of the curve*; endpoints of a diameter are *vertices* of the curve.
- Some standard English translations of Apollonius (e.g. Heath, Taliaferro, ...) say those parallels have been drawn *"ordinatewise"* to the diameter.
- The actual Greek phrase is used repeatedly in the *Conics*, so deserves special consideration: literally means something more like *"lined-up"* or *"in order"*, or drawn *"in an orderly fashion"*.

The next propositions

- Apollonius classifies the conic sections according to the way the cone is sectioned
- Derives from that their *sumptomata* ("fundamental properties")
- For future reference – these *sumptomata* are expressed in each case as a relation between a given square and a given rectangle
- Constructed from an arbitrary point on the curve, together with an auxiliary fixed line segment (the *orthia pleura*, "*latus rectum*," upright side)

Proposition 11 – incorporates definition of the parabola (Taliaferro’s translation)

If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if, further, the diameter of the section is parallel to one side of the axial triangle, and if any straight line is drawn from the section of the cone to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone’s base, then this straight line to the diameter will equal in square the rectangle contained by (a) the straight line from the section’s vertex to where the straight line to the diameter cuts it off and (b) another straight line which has the same ratio to the straight line between the angle of the cone and the vertex of the section as the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola.

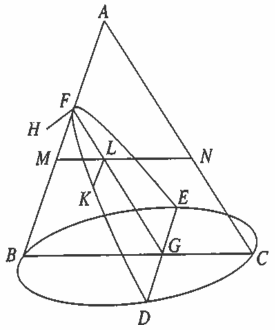
Some tendentious comments

- As you might guess from this sample, Apollonius' prose is verbose, complicated in syntax, *and mathematically dense* – a "hard slog" (in the original Greek, or in translation)!
- J. Kepler in response to criticism of his own works: *If anyone thinks that the obscurity of this presentation arises from the perplexity of my mind, ... I urge any such person to read the Conics of Apollonius. He will see that there are some matters which no mind, however gifted, can present in such a way as to be understood in a cursory reading. There is need of meditation, and a close thinking through of what is said.*
- Do we still have the patience and persistence to do that?

Structure of a Euclidean or Apollonian proposition

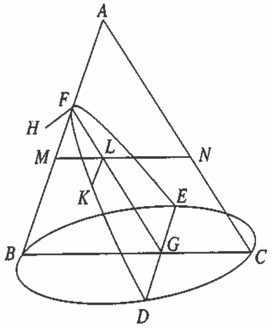
- Probably time for a digression on the structure of the propositions in Apollonius's text (follows Euclidean model closely)
- Typically, there are 6 "parts" – *protasis*, [diagram and] *ekthesis*, *diorismos*, *kataskeuē*, *apodeixis*, *sumperasma*
- The supremely complex and convoluted first sentence on the previous slide is the *protasis* of this proposition – the "statement"
- The *ekthesis* then "lays out" the statement by means of a figure and the usual sort of labeling of important points with letters.

Ekthesis of Proposition 11, beginning (condensed translation by JBL)



Let A be the vertex, ΔABC the axial triangle, and let the other plane cut the plane of the base in DE perpendicular to BC. The section is the curve DFE, with diameter FG parallel to AC.

Conclusion of *ekthesis* and *diorismos* of Proposition 11



Let H be "contrived so that"

$$(*) \quad sq.BC : rect.BA, AC :: FH : FA$$

Finally let K be taken at random on the section and let KL be parallel to DE.

I say that $sq.KL = rect.HF, FL$.

A word on terminology and notation

- The notation here is Taliaferro's modern attempt to capture what Apollonius actually says in a (more) readable way
- Apollonius' Greek is highly conventionalized and abbreviated, but entirely expressed in words
- Here $sq.XY$ means (the area of) the square with side XY (Apollonius in fact just says literally "the from XY")
- $rect.XY, YZ$ stands for (the area of) the rectangle with sides XY and YZ (Apollonius in fact just says literally "the by XYZ")
- The $:$ and $::$ are standard notation for comparing ratios (Books V, X of Euclid contain an exposition of Eudoxus' theory of these)

An interesting question

- Is the equality $sq.KL = rect.HF, FL$ a forerunner of the *coordinate equation* of a “sideways” parabola in the form $y^2 = cx$?
- Katz and Parshall say (p. 53) “Although it is now generally held that Apollonius did not use algebra, it is frequently ... quite straightforward to translate some of his properties, as well as his derivations and proofs into algebraic forms.”
- Our previous acquaintance Sabetai Unguru would have a “conniption fit,” though, at any claim that Apollonius was doing anything other than geometry with areas of rectangles and squares(!)