MATH 110-02 – Algebra Through History
Discussion – Descartes and Coordinate, or "Analytic" Geometry
December 6 and 9, 2019

Background

In 1637, the French philosopher and mathematician René Descartes published a pamphlet called *La Géometrie* as one of a series of discussions about particular sciences intended apparently as illustrations of his general ideas expressed in his work *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences* (in English: Discourse on the method of rightly directing one's reason and searching for truth in the sciences). This work introduced other mathematicians to a new way of dealing with geometry. Via the introduction (at first only in a rather indirect way) of numerical coordinates to describe points in the plane, Descartes showed that it was possible to *define geometric objects by means of algebraic equations* and to apply techniques from *algebra* to deduce geometric properties of those objects.

He illustrated his new methods first by considering a problem first studied by the ancient Greeks after the time of Euclid:

Given three (or four) lines in the plane, find the locus of points P that satisfy the relation that the square of the distance from P to the first line (or the product of the distances from P to the first two of the lines) is equal to the product of the distances from P to the other two lines.

The resulting curves are called *three-line* or *four-line* loci depending on which case we are considering.

For instance, the locus of points satisfying the condition above for the four lines

$$\begin{aligned} x+2 &= 0, \\ x-1 &= 0, \\ y+1 &= 0, \\ y-1 &= 0 \end{aligned}$$

is shown in red in Figure 1 (on the next page). The point shown in black is $P = (1,1)$. Note that it satisfies the defining condition *the product of the distances from P to the first two of the lines is equal to the product of the distances from P to the other two lines* since it lies on the line $x-1=0$ so the first product is zero, and it lies on the line $y-1=0$, so the second product is also zero.

The Greek mathematician Apollonius (ca. 262 BCE - ca. 190 BCE) considered this problem in Book III of his masterwork, the *Conics*. He showed via *extremely elaborate* "synthetic" (i.e. Euclid-style) proofs that *both the three- and four-line loci are conic sections – the circles, ellipses, hyperbolas, and parabolas that are obtained as plane sections of a cone.* The lines in Figure 1 give a *hyperbola*; the other conics (circles, ellipses, parabolas) can also

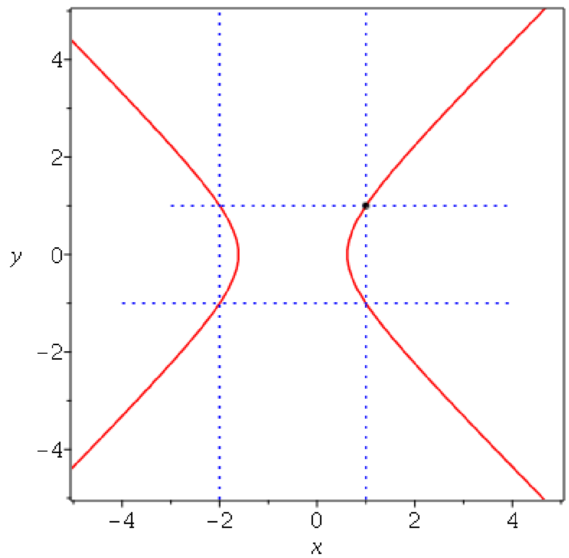

Figure 1: A four-line locus

be obtained in this way by varying the positions of the lines, making them meet at non-right angles, etc.

Descartes actually learned about this work by reading an account given in the *Mathematical Collection* of Pappus (ca. 290 - 350 CE). As we have said, Pappus' *Mathematical Collection* preserved much of the earlier work of Greek mathematicians and earlier thinking about the complementary roles of *analysis* and *synthesis* in mathematics. This work by Pappus was translated into Latin in the 16th century and reintroduced much Greek advanced mathematics to Europe around the time of F. Viète, and slightly later, R. Descartes. With his "analytic" geometry in the plane, Descartes was able to derive Apollonius's results in a much easier way, and he also showed how to solve analogous problems when the locus was described by any number ≥ 3 of lines.

Today, using our knowledge of coordinate geometry, we want to understand what Descartes did, why it was such an advance, and why it essentially (re)-united algebra and geometry.

Questions

(A) First, what are the coordinate equations of the "standard" circles and ellipses if we take the center at $(0, 0)$? What is the coordinate equation of a "standard" parabola? What do "standard" equations of hyperbolas look like? You may look this up online if you need to. Please give the URL for the site where you found your information if you do this(!) Show that the equations of all these curves can be put into the form

$$Dx^2 + Exy + Fy^2 + Gx + Hy + I = 0. \tag{1}$$

for appropriate choices of the numerical coefficents D, E, F, G, H, I. Every curve in the plane defined by an equation of this form is a conic section (or a pair of lines, such as the curve defined by $x^2 - y^2 = (x+y)(x-y) = 0$, or a "line with multiplicity 2" such as the curve defined by $x^2 = 0$).

(B) A line L in the coordinate plane is given by the equation $Ax + By + C = 0$. Let $Q = (x_0, y_0)$ be a point. (Note: The subscript 0's just tell us that these are particular numbers rather than the x- and y-coordinates of a general point, which we think of as variables.) The (perpendicular) distance from L to Q is given by

$$d(L,Q) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Using this formula, determine the distance from the point

$$Q = \left(\frac{-1+\sqrt{5}}{2}, 0\right)$$

to each of the lines given above (the ones that produce the plot at the top of the page). Deduce that Q lies on the four-line locus for these lines.

(C) Use part B to prove Apollonius's result that starting from any three or four lines

$$L_i : A_i x + B_i y + C_i = 0, i = 1, 2, 3, 4$$

the locus of points $P = (x, y)$ such that *the square of the distance from P to line L_1 is equal to the product of the distances from P to L_2 and L_3* is defined by an algebraic equation of total degree 2 in the coordinates of the point P – that is, an equation of the form (1) from part (A) for some real numbers D, E, F, G, H, I (*after possibly ignoring the absolute values*). Do the same for the locus of points $P = (x, y)$ such that *the product of the distances from P to L_1 and to L_2 is equal to the product of the distances from P to L_3 and to L_4*.

(D) Now suppose you had *six* lines $L_1, \ldots, L_6$ in the plane and you wanted to understand the locus of points $P = (x, y)$ that satisfy the condition

$$d(L_1, P)d(L_2, P)d(L_3, P) = d(L_4, P)d(L_5, P)d(L_6, P).$$

What would the coordinate equation of this locus look like? (Note: This was far beyond anything considered by Apollonius or any of the other ancient Greeks! With Descartes' insights, though, it's really not any more difficult than the cases they looked at!)