The Geometry of René Descartes

BOOK I

Problems the Construction of Which Requires Only Straight Lines and Circles

ANY problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.[1] Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers,[2] and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication); or, again, to find a fourth line which is to one of the given lines as unity is to the other (which is equivalent to division); or, finally, to find one, two, or several mean proportionals between unity and some other line (which is the same

[1] Large collections of problems of this nature are contained in the following works: Vincenzo Riccati and Girolamo Saladino, *Institutiones Analyticae*, Bologna, 1765; Maria Gaetana Agnesi, *Istituzioni Analitiche*, Milan, 1748; Claude Rabuel, *Commentaires sur la Géométrie de M. Descartes*, Lyons, 1730 (hereafter referred to as Rabuel); and other books of the same period or earlier.

[2] Van Schooten, in his Latin edition of 1683, has this note: "Per unitatem intellige lineam quandam determinatam, qua ad quamvis reliquarum linearum talem relationem habeat, qualem unitas ad certum aliquem numerum." *Geometria a Renato Des Cartes, una cum notis Florimondi de Beaune, opera atque studio Francisci à Schooten*, Amsterdam, 1683, p. 165 (hereafter referred to as Van Schooten).

In general, the translation runs page for page with the facing original. On account of figures and footnotes, however, this plan is occasionally varied, but not in such a way as to cause the reader any serious inconvenience.

LA GEOMETRIE.

LIVRE PREMIER.

Des problefmes qu'on peut conftruire fans y employer que des cercles & des lignes droites.

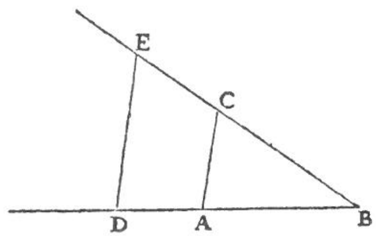
TOus les Problefmes de Geometrie fe peuuent facilement reduire a tels termes, qu'il n'eft befoin par aprés que de connoiftre la longeur de quelques lignes droites, pour les conftruire.

Et comme toute l'Arithmetique n'eft compofée, que de quatre ou cinq operations, qui font l'Addition, la Souftraction, la Multiplication, la Diuifion, & l'Extraction des racines, qu'on peut prendre pour vne efpece de Diuifion: Ainfi n'a-t'on autre chofe a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a eftre connuës, que leur en adioufter d'autres, ou en ofter, Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement eftre prife a difcretion, puis en ayant encore deux autres, en trouuer vne quatriefme, qui foit à l'vne de ces deux, comme l'autre eft a l'vnité, ce qui eft le mefme que la Multiplication; oubien en trouuer vne quatriefme, qui foit a l'vne de ces deux, comme l'vnité

Commét le calcul d'Arithmetique fe rapporte aux operations de Geometrie.

eſt a l'autre, ce qui eſt le meſme que la Diuiſion; ou enfin trouuer vne, ou deux, ou pluſieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui eſt le meſme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligibile.

La Multiplication.

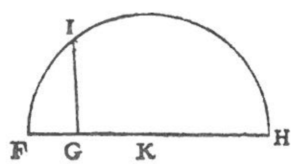


Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diuiſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

l'Extraction de la racine quarrée.



Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Commēt on peut

Mais ſouuent on n'a pas beſoin de tracer ainſi ces ligne

as extracting the square root, cube root, etc., of the given line.[3] And I shall not hesitate to introduce these arithmetical terms into geometry, for the sake of greater clearness.

For example, let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC.

If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE; then BC is the result of the division.

If the square root of GH is desired, I add, along the same straight line, FG equal to unity; then, bisecting FH at K, I describe the circle FIH about K as a center, and draw from G a perpendicular and extend it to I, and GI is the required root. I do not speak here of cube root, or other roots, since I shall speak more conveniently of them later.

Often it is not necessary thus to draw the lines on paper, but it is sufficient to designate each by a single letter. Thus, to add the lines BD and GH, I call one a and the other b, and write $a + b$. Then $a - b$ will indicate that b is subtracted from a; ab that a is multiplied by b; $\frac{a}{b}$ that a is divided by b; aa or a^2 that a is multiplied by itself; a^3 that this result is multiplied by a, and so on, indefinitely.[4] Again, if I wish to extract the square root of a^2+b^2, I write $\sqrt{a^2+b^2}$; if I wish to extract the cube root of $a^3-b^3+ab^2$, I write $\sqrt[3]{a^3-b^3+ab^2}$, and similarly for other roots.[5] Here it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean only simple lines, which, however, I name squares, cubes, etc., so that I may make use of the terms employed in algebra.[6]

[3] While in arithmetic the only exact roots obtainable are those of perfect powers, in geometry a length can be found which will represent exactly the square root of a given line, even though this line be not commensurable with unity. Of other roots, Descartes speaks later.

[4] Descartes uses a^3, a^4, a^5, a^6, and so on, to represent the respective powers of a, but he uses both aa and a^2 without distinction. For example, he often has $aabb$, but he also uses $\frac{3a^2}{4b^2}$.

[5] Descartes writes: $\sqrt{C.a^3 - b^3 + abb}$. See original, page 299, line 9.

[6] At the time this was written, a^2 was commonly considered to mean the surface of a square whose side is a, and b^3 to mean the volume of a cube whose side is b; while b^4, b^5, ... were unintelligible as geometric forms. Descartes here says that a^2 does not have this meaning, but means the line obtained by constructing a third proportional to 1 and a, and so on.

It should also be noted that all parts of a single line should always be expressed by the same number of dimensions, provided unity is not determined by the conditions of the problem. Thus, a^3 contains as many dimensions as ab^2 or b^3, these being the component parts of the line which I have called $\sqrt[3]{a^3-b^3+ab^2}$. It is not, however, the same thing when unity is determined, because unity can always be understood, even where there are too many or too few dimensions; thus, if it be required to extract the cube root of a^2b^2-b, we must consider the quantity a^2b^2 divided once by unity, and the quantity b multiplied twice by unity.[7]

Finally, so that we may be sure to remember the names of these lines, a separate list should always be made as often as names are assigned or changed. For example, we may write, AB=1, that is AB is equal to 1;[8] GH $=a$, BD $=b$, and so on.

If, then, we wish to solve any problem, we first suppose the solution already effected,[9] and give names to all the lines that seem needful for its construction,—to those that are unknown as well as to those that are known.[10] Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most natur-

[7] Descartes seems to say that each term must be of the third degree, and that therefore we must conceive of both a^2b^2 and b as reduced to the proper dimension.

[8] Van Schooten adds "seu unitati," p. 3. Descartes writes, AB ∞ 1. He seems to have been the first to use this symbol. Among the few writers who followed him, was Hudde (1633-1704). It is very commonly supposed that ∞ is a ligature representing the first two letters (or diphthong) of "æquare." See, for example, M. Aubry's note in W. W. R. Ball's *Recréations Mathématiques et Problèmes des Temps Anciens et Modernes*, French edition, Paris, 1909, Part III, p. 164.

[9] This plan, as is well known, goes back to Plato. It appears in the work of Pappus as follows: "In analysis we suppose that which is required to be already obtained, and consider its connections and antecedents, going back until we reach either something already known (given in the hypothesis), or else some fundamental principle (axiom or postulate) of mathematics." *Pappi Alexandrini Collectiones quae supersunt e libris manu scriptis edidit Latina interpellatione et commentariis instruxit Fredericus Hultsch*, Berlin, 1876-1878; vol. II, p. 635 (hereafter referred to as Pappus). See also Commandinus, *Pappi Alexandrini Mathematicae Collectiones*, Bologna, 1588, with later editions.

Pappus of Alexandria was a Greek mathematician who lived about 300 A.D. His most important work is a mathematical treatise in eight books, of which the first and part of the second are lost. This was made known to modern scholars by Commandinus. The work exerted a happy influence on the revival of geometry in the seventeenth century. Pappus was not himself a mathematician of the first rank, but he preserved for the world many extracts or analyses of lost works, and by his commentaries added to their interest.

[10] Rabuel calls attention to the use of $a, b, c, \ldots$ for known, and $x, y, z, \ldots$ for unknown quantities (p. 20).

gnes ſur le papier, & il ſuffiſt de les deſigner par quelques lettres, chaſcune par vne ſeule. Comme pour adiouſter la ligne B D a G H, ie nomme l'vne a & l'autre b, & eſcris $a+b$; Et $a-b$, pour ſouſtraire b d' a; Et ab, pour les multiplier l'vne par l'autre; Et $\frac{a}{b}$, pour diuiſer a par b; Et aa, ou a^2, pour multiplier a par ſoy meſme; Et a^3, pour le multiplier encore vne fois par a, & ainſi a l'infini; Et $\sqrt{a^2+b^2}$, pour tirer la racine quarrée d' a^2+b^2; Et $\sqrt{C.\,a^3-b^3+abb}$, pour tirer la racine cubique d' a^3-b^3+abb, & ainſi des autres. Vſer de chiffres en Geometrie.

Où il eſt a remarquer que par a^2 ou b^3 ou ſemblables, ie ne conçoy ordinairement que des lignes toutes ſimples, encore que pour me ſeruir des noms vſités en l'Algebre, ie les nomme des quarrés ou des cubes, &c.

Il eſt auſſy a remarquer que toutes les parties d'vne meſme ligne, ſe doiuent ordinairement exprimer par autant de dimenſions l'vne que l'autre, lorſque l'vnité n'eſt point determinée en la queſtion, comme icy a^3 en contient autant qu' abb ou b^3 dont ſe compoſe la ligne que i'ay nommée $\sqrt{C.\,a^3-b^3+abb}$: mais que ce n'eſt pas de meſme lorſque l'vnité eſt determinée, a cauſe qu'elle peut eſtre ſouſentendue par tout ou il y a trop ou trop peu de dimenſions: comme s'il faut tirer la racine cubique de $aabb-b$, il faut penſer que la quantité $aabb$ eſt diuiſée vne fois par l'vnité, & que l'autre quantité b eſt multipliée deux fois par la meſme.

Pp 2 Au

Au reſte affin de ne pas manquer a ſe ſouuenir des noms de ces lignes, il en faut touſiours faire vn regiſtre ſeparé, à meſure qu'on les poſe ou qu'on les change, eſcriuant par exemple.

A B ∞ 1, c'eſt a dire, A B eſgal à 1.

G H ∞ a

B D ∞ b, &c.

Commẽt il faut venir aux Equatiõs qui ſeruent a reſoudre les problesmes.

Ainſi voulant reſoudre quelque probleſme, on doit d'abord le conſiderer comme deſia fait, & donner des noms a toutes les lignes, qui ſemblent neceſſaires pour le conſtruire, auſſy bien a celles qui ſont inconnuës, qu'aux autres. Puis ſans conſiderer aucune difference entre ces lignes connuës, & inconnuës, on doit parcourir la difficulté, ſelon l'ordre qui monſtre le plus naturellement de tous en qu'elle ſorte elles dependent mutuellement les vnes des autres, iuſques a ce qu'on ait trouué moyen d'exprimer vne meſme quantité en deux façons: ce qui ſe nomme vne Equation; car les termes de l'vne de ces deux façons ſont eſgaux a ceux de l'autre. Et on doit trouuer autant de telles Equations, qu'on a ſuppoſé de lignes, qui eſtoient inconnuës. Oubien s'il ne s'en trouue pas tant, & que nonobſtant on n'omette rien de ce qui eſt deſiré en la queſtion, cela teſmoigne qu'elle n'eſt pas entierement determinée. Et lors on peut prendre a diſcretion des lignes connuës, pour toutes les inconnuës auſqu'elles ne correſpond aucune Equation. Aprés cela s'il en reſte encore pluſieurs, il ſe faut ſeruir par ordre de chaſcune des Equations qui reſtent auſſy, ſoit en la conſiderant toute ſeule, ſoit en la comparant auec les autres, pour expliquer chaſcune de ces lignes inconnuës; & faire ainſi

ally the relations between these lines, until we find it possible to express a single quantity in two ways.[11] This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.

We must find as many such equations as there are supposed to be unknown lines;[12] but if, after considering everything involved, so many cannot be found, it is evident that the question is not entirely determined. In such a case we may choose arbitrarily lines of known length for each unknown line to which there corresponds no equation.[13]

If there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines; and so we must combine them until there remains a single unknown line[14] which is equal to some known line, or whose square, cube, fourth power, fifth power, sixth power, etc., is equal to the sum or difference of two or more quantities,[15] one of which is known, while the others consist of mean proportionals between unity and this square, or cube, or fourth power, etc., multiplied by other known lines. I may express this as follows:

$$z=b,$$
$$\text{or } z^2=-az+b^2,$$
$$\text{or } z^3=az^2+b^2z-c^3,$$
$$\text{or } z^4=az^3-c^3z+d^4, \text{ etc.}$$

That is, z, which I take for the unknown quantity, is equal to b; or, the square of z is equal to the square of b diminished by a multiplied by z; or, the cube of z is equal to a multiplied by the square of z, plus the square of b multiplied by z, diminished by the cube of c; and similarly for the others.

[11] That is, we must solve the resulting simultaneous equations.

[12] Van Schooten (p. 149) gives two problems to illustrate this statement. Of these, the first is as follows: Given a line segment AB containing any point C, required to produce AB to D so that the rectangle AD.DB shall be equal to the square on CD. He lets $AC=a$, $CB=b$, and $BD=x$. Then $AD=a+b+x$, and $CD=b+x$, whence $ax+bx+x^2=b^2+2bx+x^2$ and $x=\frac{b^2}{a-b}$.

[13] Rabuel adds this note: "We may say that every indeterminate problem is an infinity of determinate problems, or that every problem is determined either by itself or by him who constructs it" (p. 21).

[14] That is, a line represented by x, x^2, x^3, x^4, ...

[15] In the older French, "le quarré, ou le cube, ou le quarré de quarré, ou le sursolide, ou le quarré de cube &c.," as seen on page 11 (original page 302).

Thus, all the unknown quantities can be expressed in terms of a single quantity,[16] whenever the problem can be constructed by means of circles and straight lines, or by conic sections, or even by some other curve of degree not greater than the third or fourth.[17]

But I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind by working over it, which is in my opinion the principal benefit to be derived from this science. Because, I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra, who will consider carefully all that is set forth in this treatise.[18]

[16] See line 20 on the opposite page.

[17] Literally, "Only one or two degrees greater."

[18] In the Introduction to the 1637 edition of *La Géométrie,* Descartes made the following remark: "In my previous writings I have tried to make my meaning clear to everybody; but I doubt if this treatise will be read by anyone not familiar with the books on geometry, and so I have thought it superfluous to repeat demonstrations contained in them." See *Oeuvres de Descartes,* edited by Charles Adam and Paul Tannery, Paris, 1897-1910, vol. VI, p. 368. In a letter written to Mersenne in 1637 Descartes says: "I do not enjoy speaking in praise of myself, but since few people can understand my geometry, and since you wish me to give you my opinion of it, I think it well to say that it is all I could hope for, and that in *La Dioptrique* and *Les Météores,* I have only tried to persuade people that my method is better than the ordinary one. I have proved this in my geometry, for in the beginning I have solved a question which, according to Pappus, could not be solved by any of the ancient geometers.

"Moreover, what I have given in the second book on the nature and properties of curved lines, and the method of examining them, is, it seems to me, as far beyond the treatment in the ordinary geometry, as the rhetoric of Cicero is beyond the a, b, c of children. . . .

"As to the suggestion that what I have written could easily have been gotten from Vieta, the very fact that my treatise is hard to understand is due to my attempt to put nothing in it that I believed to be known either by him or by any one else. . . . I begin the rules of my algebra with what Vieta wrote at the very end of his book, *De emendatione aequationum.* . . . Thus, I begin where he left off." *Oeuvres de Descartes, publiées par Victor Cousin,* Paris, 1824, Vol. VI, p. 294 (hereafter referred to as Cousin).

In another letter to Mersenne, written April 20, 1646, Descartes writes as follows: "I have omitted a number of things that might have made it (the geometry) clearer, but I did this intentionally, and would not have it otherwise. The only suggestions that have been made concerning changes in it are in regard to rendering it clearer to readers, but most of these are so malicious that I am completely disgusted with them." Cousin, Vol. IX, p. 553.

In a letter to the Princess Elizabeth, Descartes says: "In the solution of a geometrical problem I take care, as far as possible, to use as lines of reference parallel lines or lines at right angles; and I use no theorems except those which assert that the sides of similar triangles are proportional, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides. I do not hesitate to introduce several unknown quantities, so as to reduce the question to such terms that it shall depend only on these two theorems." Cousin, Vol. IX, p. 143.

ainsi en les demeslant, qu'il n'en demeure qu'vne seule, esgale a quelque autre, qui soit connuë, oubien dont le quarré, ou le cube, ou le quarré de quarré, ou le sursolide, ou le quarré de cube, &c. soit esgal a ce, qui se produist par l'addition, ou soustraction de deux ou plusieurs autres quantités, dont l'vne soit connuë, & les autres soient composées de quelques moyennes proportionnelles entre l'vnité, & ce quarré, ou cube, ou quarré de quarré, &c. multipliées par d'autres connuës. Ce que i'escris en cete sorte.

$z \infty b$. ou

$z^2 \infty -a z + bb$. ou

$z^3 \infty +a z^2 + bbz - c^3$. ou

$z^4 \infty a z^3 - c^3 z + d^4$. &c.

C'est a dire, z, que ie prens pour la quantité inconnuë, est esgalé a b, ou le quarré de z est esgal au quarré de b moins a multiplié par z. ou le cube de z est esgal à a multiplié par le quarre de z plus le quarré de b multiplié par z moins le cube de c. & ainsi des autres.

Et on peut tousiours reduire ainsi toutes les quantités inconnuës à vne seule, lorsque le Problesme se peut construire par des cercles & des lignes droites, ou aussy par des sections coniques, ou mesme par quelque autre ligne qui ne soit que d'vn ou deux degrés plus composée. Mais ie ne m'areste point a expliquer cecy plus en detail, a cause que ie vous osterois le plaisir de l'apprendre de vous mesme, & l'vtilité de cultiuer vostre esprit en vous y exerceant, qui est a mon auis la principale, qu'on puisse

Pp 3 tirer

tirer de cete ſcience. Auſſy que ie n y remarque rien de ſi difficile, que ceux qui ſeront vn peu verſés en la Geometrie commune, & en l'Algebre, & qui prendront garde a tout ce qui eſt en ce traité, ne puiſſent trouuer.

C'eſt pourquoy ie me contenteray icy de vous auertir, que pourvû qu'en demeſlant ces Equations on ne manque point a ſe ſeruir de toutes les diuiſions, qui ſeront poſſibles, on aura infalliblement les plus ſimples termes, auſquels la queſtion puiſſe eſtre reduite.

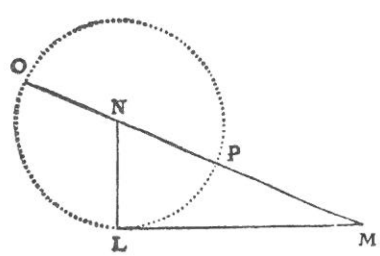
Quels ſont les probleſmes plans

Et que ſi elle peut eſtre reſolue par la Geometrie ordinaire, c'eſt a dire, en ne ſe ſeruant que de lignes droites & circulaires tracées ſur vne ſuperficie plate, lorſque la derniere Equation aura eſté entierement démeſlée, il n'y reſtera tout au plus qu'vn quarré inconnu, eſgal a ce qui ſe produiſt de l'Addition, ou ſouſtraction de ſa racine multipliée par quelque quantité connue, & de quelque autre quantité auſſy connue

Comment ils ſe reſoluent.

Et lors cete racine, ou ligne inconnue ſe trouue ayſement. Car ſi i'ay par exemple

$$z^2 \,\infty\, az + bb$$



ie fais le triangle rectangle N L M, dont le coſté L M eſt eſgal à b racine quarrée de la quantité connue bb, & l'autre L N eſt $\frac{1}{2}a$, la moitié de l'autre quantité connue, qui eſtoit multipliée par z que ie ſuppoſe eſtre la ligne inconnue. puis prolongeant M N la baze de ce triangle,

I shall therefore content myself with the statement that if the student, in solving these equations, does not fail to make use of division wherever possible, he will surely reach the simplest terms to which the problem can be reduced.

And if it can be solved by ordinary geometry, that is, by the use of straight lines and circles traced on a plane surface,[19] when the last equation shall have been entirely solved there will remain at most only the square of an unknown quantity, equal to the product of its root by some known quantity, increased or diminished by some other quantity also known.[20] Then this root or unknown line can easily be found. For example, if I have $z^2 = az + b^2$,[21] I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to $\frac{1}{2}a$, that is, to half the other known quantity which was multiplied by z, which I supposed to be the unknown line. Then prolonging MN, the hypotenuse[22] of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way:[23]

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}.$$

But if I have $y^2 = -ay + b^2$, where y is the quantity whose value is desired, I construct the same right triangle NLM, and on the hypote-

[19] For a discussion of the possibility of constructions by the compasses and straight edge, see Jacob Steiner, *Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises*, Berlin, 1833. For briefer treatments, consult Enriques, *Fragen der Elementar-Geometrie*, Leipzig, 1907; Klein, *Problems in Elementary Geometry, trans. by Beman and Smith*, Boston, 1897; Weber und Wellstein, *Encyklopädie der Elementaren Geometrie*, Leipzig, 1907. The work by Mascheroni, *La geometria del compasso*, Pavia, 1797, is interesting and well known.

[20] That is, an expression of the form $z^2 = az \pm b$. "Esgal a ce qui se produit de l'Addition, ou soustraction de sa racine multiplée par quelque quantité connue, & de quelque autre quantité aussy connue," as it appears in line 14, opposite page.

[21] Descartes proposes to show how a quadratic may be solved geometrically.

[22] Descartes says "prolongeant MN la baze de ce triangle," because the hypotenuse was commonly taken as the base in earlier times.

[23] From the figure $OM.PM = LM^2$. If $OM = z$, $PM = z - a$, and since $LM = b$, we have $z(z-a) = b^2$ or $z^2 = az + b^2$. Again, $MN = \sqrt{\frac{1}{4}a^2 + b^2}$, whence $OM = z = ON + MN = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$. Descartes ignores the second root, which is negative.

nuse MN lay off NP equal to NL, and the remainder PM is y, the desired root. Thus I have

$$y = -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}.$$

In the same way, if I had

$$x^4 = -ax^2 + b^2,$$

PM would be x^2 and I should have

$$x = \sqrt{-\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}},$$

and so for other cases.

Finally, if I have $z^2 = az - b^2$, I make NL equal to $\frac{1}{2}a$ and LM equal to b as before; then, instead of joining the points M and N, I draw MQR parallel to LN, and with N as a center describe a circle through L cutting MQR in the points Q and R; then z, the line sought, is either MQ or MR, for in this case it can be expressed in two ways, namely:[34]

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2},$$

and

$$z = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}.$$

[34] Since $MR.MQ = \overline{LM}^2$, then if $R = z$, we have $MQ = a - z$, and so

$$z(a-z) = b^2 \quad \text{or} \quad z^2 = az - b^2.$$

If, instead of this, $MQ = z$, then $MR = a - z$, and again, $z^2 = az - b^2$. Furthermore, letting O be the mid-point of QR,

$$MQ = OM - OQ = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2},$$

and

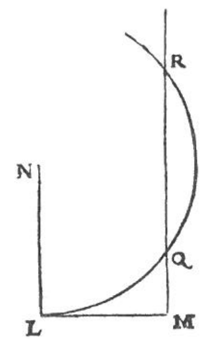
$$MR = MO + OR = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}.$$

Descartes here gives both roots, since both are positive. If MR is tangent to the circle, that is, if $b = \frac{1}{2}a$, the roots will be equal; while if $b > \frac{1}{2}a$, the line MR will not meet the circle and both roots will be imaginary. Also, since $RM.QM = \overline{LM}^2$, $z_1 z_2 = b^2$, and $RM + QM = z_1 + z_2 = a$.

angle, iuſques a O, en ſorte qu'N O ſoit eſgale a N L, la toute O M eſt z la ligne cherchée. Et elle s'exprime en cete ſorte

$$z \propto \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}.$$

Que ſi iay $yy \propto -ay + bb$, & qu'y ſoit la quantité qu'il faut trouuer, ie fais le meſme triangle rectangle N L M, & de ſa baze M N i'oſte N P eſgale a N L, & le reſte P M eſt y la racine cherchée. De façon que iay $y \propto -\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}$. Et tout de meſme ſi i'auois $x^4 \propto -ax^2 + b^2$. P M ſeroit x^2. & i'aurois $x \propto \sqrt{-\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}}$: & ainſi des autres.



Enfin ſi i'ay

$$z^2 \propto az - bb:$$

ie fais N L eſgale à $\tfrac{1}{2}a$, & L M eſgale à b cõme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée z eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \propto \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$, & $z \propto \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne couppe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Au

Au reſte ces meſmes racines ſe peuuent trouuer par vne infinité d'autres moyens, & i'ay ſeulement veulu mettre ceux cy, comme fort ſimples, affin de faire voir qu'on peut conſtruire tous les Probleſmes de la Geometrie ordinaire, ſans faire autre choſe que le peu qui eſt compris dans les quatre figures que i'ay expliquées. Ce que ie ne croy pas que les anciens ayent remarqué. car autrement ils n'euſſent pas pris la peine d'en eſcrire tant de gros liures, ou le ſeul ordre de leurs propoſitions nous fait connoiſtre qu'ils n'ont point eu la vraye methode pour les trouuer toutes, mais qu'ils ont ſeulement ramaſſé celles qu'ils ont rencontrées.

Exemple tiré de Pappus.

Et on le peut voir auſſy fort clairement de ce que Pappus a mis au commencement de ſon ſeptieſme liure, ou aprés s'eſtre areſté quelque tems a denombrer tout ce qui auoit eſté eſcrit en Geometrie par ceux qui l'auoient precedé, il parle enfin d vne queſtion, qu'il dit que ny Euclide, ny Apollonius, ny aucun autre n'auoient ſceu entierement reſoudre. & voycy ſes mots.

Ie cite plutoſt la verſion latine que le texte grec affin que chaſcun l'entende plus ayſement.

Quem autem dicit (Apollonius) in tertio libro locum ad tres, & quatuor lineas ab Euclide perfectum non eſſe, neque ipſe perficere poterat, neque aliquis alius: ſed neque paululum quid addere iis, quæ Euclides ſcripſit, per ea tantum conica, quæ uſque ad Euclidis tempora præmonſtrata ſunt, &c.

Et vn peu aprés il explique ainſi qu'elle eſt cete queſtion.

At locus ad tres, & quatuor lineas, in quo (Apollonius) magnifice ſe iactat, & oſtentat, nulla habita gratia ei, qui prius ſcripſerat, eſt hujuſmodi. Si poſitione datis tribus rectis

And if the circle described about N and passing through L neither cuts nor touches the line MQR, the equation has no root, so that we may say that the construction of the problem is impossible.

These same roots can be found by many other methods,[25] I have given these very simple ones to show that it is possible to construct all the problems of ordinary geometry by doing no more than the little covered in the four figures that I have explained.[26] This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labor into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding all,[27] but rather gathered together those propositions on which they had happened by accident.

This is also evident from what Pappus has done in the beginning of his seventh book,[28] where, after devoting considerable space to an enumeration of the books on geometry written by his predecessors,[29] he finally refers to a question which he says that neither Euclid nor Apollonius nor any one else had been able to solve completely;[30] and these are his words:

"Quem autem dicit (Apollonius) in tertio libro locum ad tres, & quatuor lineas ab Euclide perfectum non esse, neque ipse perficere poterat, neque aliquis alius; sed neque paululum quid addere iis, quæ

[25] For interesting contraction, see Rabuel, p. 23, et seq.

[26] It will be seen that Descartes considers only three types of the quadratic equation in z, namely, $z^2+az-b^2=0$, $z^2-az-b^2=0$, and $z^2-az+b^2=0$. It thus appears that he has not been able to free himself from the old traditions to the extent of generalizing the meaning of the coefficients, — as negative and fractional as well as positive. He does not consider the type $z^2+az+b^2=0$, because it has no positive roots.

[27] "Qu'ils n'ont point eu la vraye methode pour les trouuer toutes."

[28] See Note [9].

[29] See Pappus, Vol. II, p. 637. Pappus here gives a list of books that treat of analysis, in the following words: "Illorum librorum, quibus de loco, 'αναλυόμενος sive resoluto agitur, ordo hic est. Euclidis datorum liber unus, Apollonii de proportionis sectione libri duo, de spatii sectione duo, de sectione determinata duo, de tactionibus duo, Euclidis porismatum libri tres, Apollonii inclinationum libri duo, eiusdem locorum planorum duo, conicorum octo, Aristaei locorum solidorum libri duo." See also the Commandinus edition of Pappus, 1660 edition, pp. 240-252.

[30] For the history of this problem, see Zeuthen: *Die Lehre von den Kegelschnitten im Alterthum*, Copenhagen, 1886. Also, Adam and Tannery, *Oeuvres de Descartes*, vol. 6, p. 723.

Euclides scripsit, per ea tantum conica, quæ usque ad Euclidis tempora præmonstrata sunt, &c." [31]

A little farther on, he states the question as follows:

"At locus ad tres, & quatuor lineas, in quo (Apollonius) magnifice se jactat, & ostentat, nulla habita gratia ei, qui prius scripserat, est hujusmodi.[32] *Si positione datis tribus rectis lineis ab uno & eodem puncto, ad tres lineas in datis angulis rectæ lineæ ducantur, & data sit proportio rectanguli contenti duabus ductis ad quadratum reliquæ: punctum contingit positione datum solidum locum, hoc est unam ex tribus conicis sectionibus. Et si ad quatuor rectas lineas positione datas in datis angulis lineæ ducantur; & rectanguli duabus ductis contenti ad contentum duabus reliquis proportio data sit; similiter punctum datum coni sectionem positione continget. Si quidem igitur ad duas tantum locus planus ostensus est. Quod si ad plures quam quatuor, punctum continget locos non adhuc cognitos, sed lineas tantum dictas; quales autem sint, vel quam habeant proprietatem, non constat: earum unam, neque primam, & quæ manifestissima videtur, composuerunt ostendentes utilem esse. Propositiones autem ipsarum hæ sunt.*

"Si ab aliquo puncto ad positione datas rectas lineas quinque ducantur rectæ lineæ in datis angulis, & data sit proportio solidi parallelepipedi rectanguli, quod tribus ductis lineis continetur ad solidum parallelepipedum rectangulum, quod continetur reliquis duabus, & data quapiam linea, punctum positione datam lineam continget. Si autem ad sex, & data sit proportio solidi tribus lineis contenti ad solidum, quod tribus reliquis continetur; rursus punctum continget positione datam lineam. Quod si ad plures quam sex, non adhuc habent dicere, an data sit proportio cujuspiam contenti quatuor lineis ad id quod reliquis continetur,

[31] Pappus, Vol. II, pp. 677, et seq., Commandinus edition of 1660, p. 251. Literally, "Moreover, he (Apollonius) says that the problem of the locus related to three or four lines was not entirely solved by Euclid, and that neither he himself, nor any one else has been able to solve it completely, nor were they able to add anything at all to those things which Euclid had written, by means of the conic sections only which had been demonstrated before Euclid." Descartes arrived at the solution of this problem four years before the publication of his geometry, after spending five or six weeks on it. See his letters, Cousin, Vol. VI, p. 294, and Vol. VI, p. 224.

[32] Given as follows in the edition of Pappus by Hultsch, previously quoted: "Sed hic ad tres et quatuor lineas locus quo magnopere gloriatur simul addens ei qui conscripserit gratiam habendam esse, sic se habet."

rectis lineis ab uno & eodem puncto, ad tres lineas in datis angulis rectæ lineæ ducantur, & data ſit proportio rectanguli contenti duabus ductis ad quadratum reliquæ: punctum contingit poſitione datum ſolidum locum, hoc eſt unam ex tribus conicis ſectionibus. Et ſi ad quatuor rectas lineas poſitione datas in datis angulis lineæ ducantur; & rectanguli duabus ductis contenti ad contentum duabus reliquis proportio data ſit: ſimiliter punctum datum coni ſectionem poſitione continget. Si quidem igitur ad duas tantum locus planus oſtenſus eſt. Quod ſi ad plures quam quatuor, punctum continget locos non adhuc cognitos, ſed lineas tuntum dictas; quales autem ſint, vel quam habeant proprietatem, non conſtat: earum unam, neque primam, & quæ manifeſtiſſima videtur, compoſuerunt oſtendentes utilem eſſe. propoſitiones autem ipſarum hæ ſunt.

Si ab aliquo puncto ad poſitione datas rectas lineas quinque ducantur rectæ lineæ in datis angulis, & data ſit proportio ſolidi parallelepipedi rectanguli, quod tribus ductis lineis continetur ad ſolidum parallelepipedum rectangulum, quod continetur reliquis duabus, & data quapiam linea, punctum poſitione datam lineam continget. Si autem ad ſex, & data ſit proportio ſolidi tribus lineis contenti ad ſolidum, quod tribus reliquis continetur; rurſus punctum continget poſitione datam lineam. Quod ſi ad plures quam ſex, non adhuc habent dicere, an data ſit proportio cuiuſpiã contenti quatuor lineis ad id quod reliquis continetur, quoniam non eſt aliquid contentum pluribus quam tribus dimenſionibus.

Ou ie vous prie de remarquer en paſſant, que le ſcrupule, que faiſoient les anciens d'vſer des termes de l'Arithmetique en la Geometrie, qui ne pouuoit proceder, que de ce qu'ils ne voyoient pas assés clairement leur rapport, causoit beaucoup d'obscurité, & d'embaras, en la façon dont ils s'expliquoient. car Pappus poursuit en cete sorte.

Acquiescunt autem his, qui paulo ante talia interpretati sunt. neque unum aliquo pacto comprehensibile significantes quod his continetur. Licebit autẽ per coniunctas proportiones hæc, & dicere, & demonstrare universe in dictis proportionibus, atque his in hunc modum. Si ab aliquo puncto ad positione datas rectas lineas ducantur rectæ lineæ in datis angulis, & data sit proportio coniuncta ex ea, quam habet una ductarum ad unam, & altera ad alteram, & alia ad aliam, & reliqua ad datam lineam, si sint septem; si vero octo, & reliqua ad reliquam: punctum continget positione datas lineas. Et similiter quotcumque sint impares vel pares multitudine, cum hæc, ut dixi, loco ad quatuor lineas respondeant, nullum igitur posuerant ita ut linea nota sit, &c.

La question donc qui auoit esté commencée a resoudre par Euclide, & poursuiuie par Apollonius, sans auoir esté acheuée par personne, estoit telle. Ayant trois ou quatre ou plus grand nombre de lignes droites données par position; premierement on demande vn point, duquel on puisse tirer autant d'autres lignes droites, vne sur chascune des données, qui façent auec elles des angles donnés, & que le rectangle contenu en deux de celles, qui seront ainsi tirées d'vn mesme point, ait la proportion donnée auec le quarré de la troisiesme, s'il n'y en a que trois; oubien auec le rectangle des deux autres, s'il y en a quatre; oubien, s'il y en a cinq, que le parallelepipede composé de trois ait la proportion donnée auec le parallelepipede

quoniam non est aliquid contentum pluribus quam tribus dimensionibus." [33]

Here I beg you to observe in passing that the considerations that forced ancient writers to use arithmetical terms in geometry, thus making it impossible for them to proceed beyond a point where they could see clearly the relation between the two subjects, caused much obscurity and embarrassment, in their attempts at explanation.

Pappus proceeds as follows:

"Acquiescunt autem his, qui paulo ante talia interpretati sunt; neque unum aliquo pacto comprehensibile significantes quod his continetur. Licebit autem per conjunctas proportiones hæc, & dicere & demonstrare universe in dictis proportionibus, atque his in hunc modum. Si ab aliquo puncto ad positione datas rectas lineas ducantur rectæ lineæ in datis angulis, & data sit proportio conjuncta ex ea, quam habet una ductarum ad unam, & altera ad alteram, & alia ad aliam, & reliqua ad datam lineam, si sint septem; si vero octo, & reliqua ad reliquam: punctum continget positione datas lineas. Et similiter quotcumque sint

[33] This may be somewhat freely translated as follows: "The problem of the locus related to three or four lines, about which he (Apollonius) boasts so proudly, giving no credit to the writer who has preceded him, is of this nature: If three straight lines are given in position, and if straight lines be drawn from one and the same point, making given angles with the three given lines; and if there be given the ratio of the rectangle contained by two of the lines so drawn to the square of the other, the point lies on a solid locus given in position, namely, one of the three conic sections.

"Again, if lines be drawn making given angles with four straight lines given in position, and if the rectangle of two of the lines so drawn bears a given ratio to the rectangle of the other two; then, in like manner, the point lies on a conic section given in position. It has been shown that to only two lines there corresponds a plane locus. But if there be given more than four lines, the point generates loci not known up to the present time (that is, impossible to determine by common methods), but merely called 'lines'. It is not clear what they are, or what their properties. One of them, not the first but the most manifest, has been examined, and this has proved to be helpful. (Paul Tannery, in the *Oeuvres de Descartes*, differs with Descartes in his translation of Pappus. He translates as follows: Et on n'a fait la synthèse d' aucune de ces lignes, ni montré qu'elle servit pour ces lieux, pas même pour celle qui semblerait la première et la plus indiquée.) These, however, are the propositions concerning them.

"If from any point straight lines be drawn making given angles with five straight lines given in position, and if the solid rectangular parallelepiped contained by three of the lines so drawn bears a given ratio to the solid rectangular parallelepiped contained by the other two and any given line whatever, the point lies on a 'line' given in position. Again, if there be six lines, and if the solid contained by three of the lines bears a given ratio to the solid contained by the other three lines, the point also lies on a 'line' given in position. But if there be more than six lines, we cannot say whether a ratio of something contained by four lines is given to that which is contained by the rest, since there is no figure of more than three dimensions."

impares vel pares multitudine, cum hæc, ut dixi, loco ad quatuor lineas respondeant, nullum igitur posuerunt ita ut linea nota sit, &c.[34]

The question, then, the solution of which was begun by Euclid and carried farther by Apollonius, but was completed by no one, is this:

Having three, four or more lines given in position, it is first required to find a point from which as many other lines may be drawn, each making a given angle with one of the given lines, so that the rectangle of two of the lines so drawn shall bear a given ratio to the square of the third (if there be only three); or to the rectangle of the other two (if there be four), or again, that the parallelepiped[35] constructed upon three shall bear a given ratio to that upon the other two and any given line (if there be five), or to the parallelepiped upon the other three (if there be six); or (if there be seven) that the product obtained by multiplying four of them together shall bear a given ratio to the product of the other three, or (if there be eight) that the product of four of them shall bear a given ratio to the product of the other four. Thus the question admits of extension to any number of lines.

Then, since there is always an infinite number of different points satisfying these requirements, it is also required to discover and trace the curve containing all such points.[36] Pappus says that when there are only three or four lines given, this line is one of the three conic sections, but he does not undertake to determine, describe, or explain the nature of the line required[37] when the question involves a greater number of lines. He only adds that the ancients recognized one of them which they had shown to be useful, and which seemed the sim-

[34] This rather obscure passage may be translated as follows: "For in this are agreed those who formerly interpreted these things (that the dimensions of a figure cannot exceed three) in that they maintain that a figure that is contained by these lines is not comprehensible in any way. This is permissible, however, both to say and to demonstrate generally by this kind of proportion, and in this manner: If from any point straight lines be drawn making given angles with straight lines given in position; and if there be given a ratio compounded of them, that is the ratio that one of the lines drawn has to one, the second has to a second, the third to a third, and so on to the given line if there be seven lines, or, if there be eight lines, of the last to a last, the point lies on the lines that are given in position. And similarly, whatever may be the odd or even number, since these, as I have said, correspond in position to the four lines; therefore they have not set forth any method so that a line may be known." The meaning of the passage appears from that which follows in the text.

[35] That is, continued product.

[36] It is here that the essential feature of the work of Descartes may be said to begin.

[37] See line 19 on the opposite page.

lelepipede compofé des deux qui reftent, & d'vne autre ligne donnée. Ou s'il y en a fix, que le parallelepipede cõpofé de trois ait la proportion donnée auec le parallelepipede des trois autres. Ou s'il y en a fept, que ce qui fe produift lorfqu'on en multiplie quatre l'vne par l'autre, ait la raifon donnée auec ce qui fe produift par la multiplication des trois autres, & encore d'vne autre ligne donnée; Ou s'il y en a huit, que le produit de la multiplication de quatre ait la proportion donnée auec le produit des quatre autres. Et ainfi cete queftion fe peut eftendre a tout autre nombre de lignes. Puis a caufe qu'il y a toufiours vne infinité de diuers poins qui peuuent fatisfaire a ce qui eft icy demandé, il eft auffy requis de connoiftre, & de tracer la ligne, dans laquelle ils doiuent tous fe trouuer. & Pappus dit que lorfqu'il n'y a que trois ou quatre lignes droites données, c'eft en vne des trois fections coniques. mais il n'entreprend point de la determiner, ny de la defcrire. non plus que d'expliquer celles ou tous ces poins fe doiuent trouuer, lorfque la queftion eft propofée en vn plus grand nombre de lignes. Seulement il aioufte que les anciens en auoient imaginé vne qu'ils monftroient y eftre vtile, mais qui fembloit la plus manifefte, & qui n'eftoit pas toutefois la premiere. Ce qui m'a donné occafion d'effayer fi par la methode dont ie me fers on peut aller auffy loin qu'ils ont efté.

Et premierement i'ay connu que cete queftion n'eftant propofée qu'en trois, ou quatre, ou cinq lignes, on peut toufiours trouuer les poins cherchés par la Geometrie fimple; c'eft a dire en ne fe feruant que de la reigle & du — Refponfe à la queftion de Pappus.

Qq 2 compas,

compas, ny ne faisant autre chose, que ce qui a desia esté dit; excepté seulement lorsqu'il y a cinq lignes données, si elles sont toutes paralleles. Auquel cas, comme aussy lorsque la question est proposée en six, ou 7, ou 8, ou 9 lignes, on peut tousiours trouuer les poins cherchés par la Geometrie des solides; c'est a dire en y employant quelqu'vne des trois sections coniques. Excepté seulement lorsqu'il y a neuf lignes données, si elles sont toutes paralleles. Auquel cas derechef, & encore en 10, 11, 12, ou 13 lignes on peut trouuer les poins cherchés par le moyen d'vne ligne courbe qui soit d'vn degré plus composée que les sections coniques. Excepté en treize si elles sont toutes paralleles, auquel cas, & en quatorze, 15, 16, & 17 il y faudra employer vne ligne courbe encore d'vn degré plus composée que la precedente & ainsi a l'infini.

Puis iay trouué aussy, que lorsqu'il ny a que trois ou quatre lignes données, les poins cherchés se rencontrent tous, non seulement en l'vne des trois sections coniques, mais quelquefois aussy en la circonference d'vn cercle, ou en vne ligne droite. Et que lorsqu'il y en a cinq, ou six, ou sept, ou huit, tous ces poins se rencontrent en quelque vne des lignes, qui sont d'vn degré plus composées que les sections coniques, & il est impossible d'en imaginer aucune qui ne soit vtile a cete question; mais ils peuuent aussy derechef se rencontrer en vne section conique, ou en vn cercle, ou en vne ligne droite. Et s'il y en a neuf, ou 10, ou 11, ou 12, ces poins se rencontrent en vne ligne, qui ne peut estre que d'vn degré plus composée que les precedentes; mais toutes celles

qui

plest, and yet was not the most important.[38] This led me to try to find out whether, by my own method, I could go as far as they had gone.[39]

First, I discovered that if the question be proposed for only three, four, or five lines, the required points can be found by elementary geometry, that is, by the use of the ruler and compasses only, and the application of those principles that I have already explained, except in the case of five parallel lines. In this case, and in the cases where there are six, seven, eight, or nine given lines, the required points can always be found by means of the geometry of solid loci,[40] that is, by using some one of the three conic sections. Here, again, there is an exception in the case of nine parallel lines. For this and the cases of ten, eleven, twelve, or thirteen given lines, the required points may be found by means of a curve of degree next higher than that of the conic sections. Again, the case of thirteen parallel lines must be excluded, for which, as well as for the cases of fourteen, fifteen, sixteen, and seventeen lines, a curve of degree next higher than the preceding must be used; and so on indefinitely.

Next, I have found that when only three or four lines are given, the required points lie not only all on one of the conic sections but sometimes on the circumference of a circle or even on a straight line.[41]

When there are five, six, seven, or eight lines, the required points lie on a curve of degree next higher than the conic sections, and it is impossible to imagine such a curve that may not satisfy the conditions of the problem; but the required points may possibly lie on a conic section, a circle, or a straight line. If there are nine, ten, eleven, or twelve lines, the required curve is only one degree higher than the preceding, but any such curve may meet the requirements, and so on to infinity.

[38] See lines 5-10 from the foot of page 23.

[39] Descartes gives here a brief summary of his solution, which he amplifies later.

[40] This term was commonly applied by mathematicians of the seventeenth century to the three conic sections, while the straight line and circle were called plane loci, and other curves linear loci. See Fermat, *Isagoge ad Locos Planos et Solidos*, Toulouse, 1679.

[41] Degenerate or limiting forms of the conic sections.

Finally, the first and simplest curve after the conic sections is the one generated by the intersection of a parabola with a straight line in a way to be described presently.

I believe that I have in this way completely accomplished what Pappus tells us the ancients sought to do, and I will try to give the demonstration in a few words, for I am already wearied by so much writing.

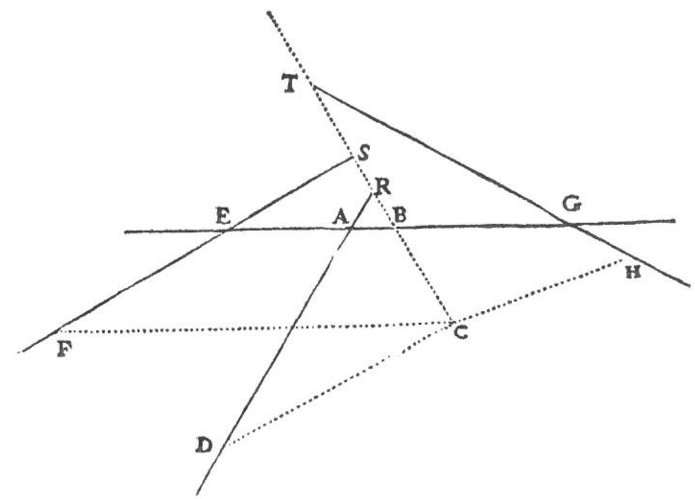
Let AB, AD, EF, GH, ... be any number of straight lines given in position,[42] and let it be required to find a point C, from which straight lines CB, CD, CF, CH, ... can be drawn, making given angles CBA, CDA, CFE, CHG, ... respectively, with the given lines, and

[42] It should be noted that these lines are given in position but not in length. They thus become lines of reference or coördinate axes, and accordingly they play a very important part in the development of analytic geometry. In this connection we may quote as follows: "Among the predecessors of Descartes we reckon, besides Apollonius, especially Vieta, Oresme, Cavalieri, Roberval, and Fermat, the last the most distinguished in this field; but nowhere, even by Fermat, had any attempt been made to refer several curves of different orders simultaneously to one system of coördinates, which at most possessed special significance for one of the curves. It is exactly this thing which Descartes systematically accomplished." Karl Fink, *A Brief History of Mathematics*, trans. by Beman and Smith, Chicago, 1903, p. 229.

Heath calls attention to the fact that "the essential difference between the Greek and the modern method is that the Greeks did not direct their efforts to making the fixed lines of a figure as few as possible, but rather to expressing their equations between areas in as short and simple a form as possible." For further discussion see D. E. Smith, *History of Mathematics*, Boston, 1923-25, Vol. II, pp. 316-331 (hereafter referred to as Smith).

qui ſont d'vn degré plus compoſées y peuuent ſeruir, & ainſi a l'infini.

Au reſte la premiere, & la plus ſimple de toutes aprés les ſections coniques, eſt celle qu'on peut deſcrire par l'interſection d'vne Parabole, & d'vne ligne droite, en la façon qui ſera tantoſt expliquée. En ſorte que ie penſe auoir entierement ſatisfait a ceque Pappus nous dit auoir eſté cherché en cecy par les anciens. & ie taſcheray d'en mettre la demonſtration en peu de mots. car il m'ennuie deſia d'en tant eſcrire.



Soient A B, A D, E F, G H, &c. pluſieurs lignes données par poſition, & qu'il faille trouuer vn point, comme C, duquel ayant tiré d'autres lignes droites ſur les données, comme C B, C D, C F, & C H, en ſorte que les angles C B A, C D A, C F E, C H G, &c. ſoient donnés, &

Q q 3

& que ce qui eſt produit par la multiplication d'vne partie de ces lignes, ſoit eſgal a ce qui eſt produit par la multiplication des autres, oubien qu'ils ayent quelque autre proportion donnée, car cela ne rend point la queſtion plus difficile.

Commēt on doit poſer les termes pour venir à l'Equation en cet exemple.

Premierement ie ſuppoſe la choſe comme deſia faite, & pour me demeller de la cõfuſion de toutes ces lignes, ie conſidere l'vne des données, & l'vne de celles qu'il faut trouuer, par exemple A B, & C B, comme les principales, & auſquelles ie taſche de rapporter ainſi toutes les autres. Que le ſegment de la ligne A B, qui eſt entre les poins A & B, ſoit nommé x. & que B C ſoit nommé y. & que toutes les autres lignes données ſoient prolongées, iuſques a ce qu'elles couppent ces deux, auſſy prolongées s'il eſt beſoin, & ſi elles ne leur ſont point paralleles. comme vous voyes icy qu'elles couppent la ligne A B aux poins A, E, G, & B C aux poins R, S, T. Puis a cauſe que tous les angles du triangle A R B ſont donnés, la proportion, qui eſt entre les coſtés A B, & B R, eſt auſſy donnée, & ie la poſe comme de z à b, de façon qu' A B eſtant x, R B ſera $\frac{bx}{z}$, & la toute C R ſera $y + \frac{bx}{z}$, à cauſe que le point B tombe entre C & R; car ſi R tomboit entre C & B, C R ſeroit $y - \frac{bx}{z}$; & ſi C tomboit entre B & R, C R ſeroit $-y + \frac{bx}{z}$. Tout de meſme les trois angles du triangle D R C ſont donnés, & par conſequent auſſy la proportion qui eſt entre les coſtés C R, & C D, que ie poſe comme de z à c: de façon que C R eſtant $y + \frac{bx}{z}$,

CD

such that the product of certain of them is equal to the product of the rest, or at least such that these two products shall have a given ratio, for this condition does not make the problem any more difficult.

First, I suppose the thing done, and since so many lines are confusing, I may simplify matters by considering one of the given lines and one of those to be drawn (as, for example, AB and BC) as the principal lines, to which I shall try to refer all the others. Call the segment of the line AB between A and B, x, and call BC, y. Produce all the other given lines to meet these two (also produced if necessary) provided none is parallel to either of the principal lines. Thus, in the figure, the given lines cut AB in the points A, E, G, and cut BC in the points R, S, T.

Now, since all the angles of the triangle ARB are known,[43] the ratio between the sides AB and BR is known.[44] If we let $AB : BR = z : b$, since $AB = x$, we have $RB = \frac{bx}{z}$; and since B lies between C and R[45], we have $CR = y + \frac{bx}{z}$. (When R lies between C and B, CR is equal to $y - \frac{bx}{z}$, and when C lies between B and R, CR is equal to $-y + \frac{bx}{z}$.) Again, the three angles of the triangle DRC are known,[46] and therefore the ratio between the sides CR and CD is determined. Calling this ratio $z : c$, since $CR = y + \frac{bx}{z}$, we have $CD = \frac{cy}{z} + \frac{bcx}{z^2}$. Then, since

[43] Since BC cuts AB and AD under given angles.
[44] Since the ratio of the sines of the opposite angles is known.
[45] In this particular figure, of course.
[46] Since CB and CD cut AD under given angles.

the lines AB, AD, and EF are given in position, the distance from A to E is known. If we call this distance k, then $EB = k + x$; although $EB = k - x$ when B lies between E and A, and $E = -k + x$ when E lies between A and B. Now the angles of the triangle ESB being given, the ratio of BE to BS is known. We may call this ratio $z:d$. Then $BS = \frac{dk + dx}{z}$ and $CS = \frac{zy + dk + dx}{z}$.[47] When S lies between B and C we have $CS = \frac{zy - dk - dx}{z}$, and when C lies between B and S we have $CS = \frac{-zy + dk + dx}{z}$. The angles of the triangle FSC are known, and hence, also the ratio of CS to CF, or $z:e$. Therefore, $CF = \frac{ezy + dek + dex}{z^2}$. Likewise, AG or l is given, and $BG = l - x$. Also, in triangle BGT, the ratio of BG to BT, or $z:f$, is known. Therefore, $BT = \frac{fl - fx}{z}$ and $CT = \frac{zy + fl - fx}{z}$. In triangle TCH, the ratio of TC to CH, or $z:g$, is known,[48] whence $CH = \frac{gzy + fgl - fgx}{z^2}$.

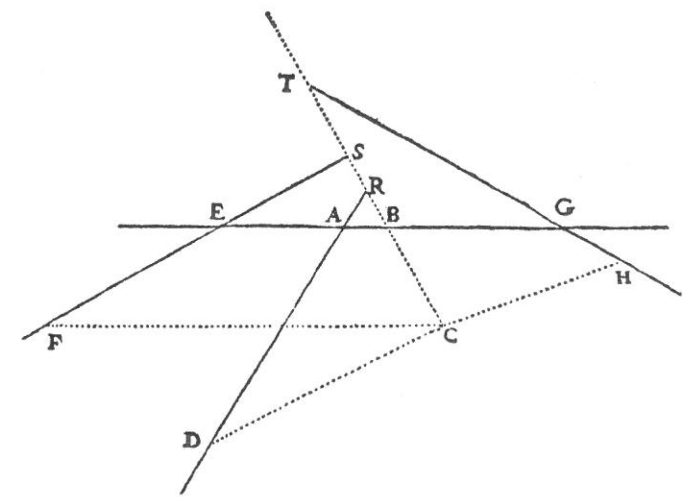
[47] We have

$$CS = y + BS = y + \frac{dk + dx}{z} = \frac{zy + dk + dx}{z},$$

and similarly for the other cases considered below.

The translation covers the first eight lines on the original page 312 (page 32 of this edition.

[48] It should be noted that each ratio assumed has z as antecedent.



C D sera $\frac{cy}{z} + \frac{bcx}{zz}$. Aprés cela pourceque les lignes A B, A D, & E F sont données par position, la distance qui est entre les poins A & E est aussy donnée, & si on la nomme K, on aura E B esgal a $k + x$; mais ce seroit $k - x$, si le point B tomboit entre E & A; & $-k + x$, si E tomboit entre A & B. Et pourceque les angles du triangle E S B sont tous donnés, la proportion de B E a B S est aussy donnée, & ie la pose comme z à d, sibienque B S est $\frac{dk + dx}{z}$, & la toute C S est $\frac{zy + dk + dx}{z}$; mais ce seroit $\frac{zy - dk - dx}{z}$, si le point S tomboit entre B & C; & ce seroit $\frac{-zy + dk + dx}{z}$, si C tomboit entre B & S. De plus les trois angles du triangle F S C sont donnés, & en suite la pro-

proportion de C S à C F, qui ſoit comme de z à e, & la toute C F ſera $\frac{ezy + dek + dex}{zz}$. En meſme façon A G que ie nomme l eſt donnée, & B G eſt $l - x$, & a cauſe du triangle B G T la proportion de B G a B T eſt auſſy donnée, qui ſoit comme de z à f. & B T ſera $\frac{fl - fx}{z}$, & C T ∞ $\frac{zy + fl - fx}{z}$. Puis derechef la proportion de T C a C H eſt donnée, a cauſe du triangle T C H, & la poſant comme de z à g, on aura C H ∞ $\frac{+gzy + fgl - fgx}{zz}$.

Et ainſi vous voyés, qu'en tel nombre de lignes données par poſition qu'on puiſſe auoir, toutes les lignes tirées deſſus du point C a angles donnés ſuiuant la teneur de la queſtion, ſe peuuent touſiours exprimer chaſcune par trois termes; dont l'vn eſt composé de la quantité inconnue y, multipliée, ou diuiſée par quelque autre connue; & l'autre de la quantité inconnue x, auſſy multipliée ou diuiſée par quelque autre connuë, & le troiſieſme d'vne quantité toute connuë. Excepté ſeulement ſi elles ſont paralleles; oubien a la ligne A B, auquel cas le terme composé de la quantité x ſera nul; oubien a la ligne C B, auquel cas celuy qui eſt composé de la quantité y ſera nul; ainſi qu'il eſt trop manifeſte pour que ie m'areſte a l'expliquer. Et pour les ſignes +, & --, qui ſe ioignent à ces termes, ils peuuent eſtre changés en toutes les façons imaginables.

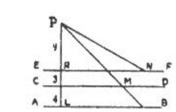
Puis vous voyés auſſy, que multipliant pluſieurs de ces lignes l'vne par l'autre, les quantités x & y, qui ſe trouuent dans le produit, n'y peuuent auoir que chaſcune autant de dimenſions, qu'il y a eu de lignes, a l'explication

And thus you see that, no matter how many lines are given in position, the length of any such line through C making given angles with these lines can always be expressed by three terms, one of which consists of the unknown quantity y multiplied or divided by some known quantity; another consisting of the unknown quantity x multiplied or divided by some other known quantity; and the third consisting of a known quantity.[49] An exception must be made in the case where the given lines are parallel either to AB (when the term containing x vanishes), or to CB (when the term containing y vanishes). This case is too simple to require further explanation.[50] The signs of the terms may be either + or — in every conceivable combination.[51]

You also see that in the product of any number of these lines the degree of any term containing x or y will not be greater than the number of lines (expressed by means of x and y) whose product is found. Thus, no term will be of degree higher than the second if two lines be multiplied together, nor of degree higher than the third, if there be three lines, and so on to infinity.

[49] That is, an expression of the form $ax+by+c$, where a, b, c, are any real positive or negative quantities, integral or fractional (not zero, since this exception is considered later).

[50] The following problem will serve as a very simple illustration: Given three parallel lines AB, CD, EF, so placed that AB is distant 4 units from CD, and CD is distant 3 units from EF; required to find a point P such that if PL, PM, PN



be drawn through P, making angles of 90°, 45°, 30°, respectively, with the parallels. Then $\overline{PM}^2 = PL.PN$.

Let $PR = y$, then $PN = 2y$, $PM = \sqrt{2}(y+3)$, $PL = y+7$. If $\overline{PM}^2 = PN.PL$, we have $\left[\sqrt{2}(y+3)\right]^2 = 2y(y+7)$, whence $y = 9$. Therefore, the point P lies on the line XY parallel to EF and at a distance of 9 units from it. Cf. Rabuel, p. 79.

[51] Depending, of course, upon the relative positions of the given lines.

Furthermore, to determine the point C, but one condition is needed, namely, that the product of a certain number of lines shall be equal to, or (what is quite as simple), shall bear a given ratio to the product of certain other lines. Since this condition can be expressed by a single equation in two unknown quantities,[52] we may give any value we please to either x or y and find the value of the other from this equation. It is obvious that when not more than five lines are given, the quantity x, which is not used to express the first of the lines can never be of degree higher than the second.[53]

Assigning a value to y, we have $x^2 = \pm ax \pm b^2$, and therefore x can be found with ruler and compasses, by a method already explained.[54] If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn.

This method can be used when the problem concerns six or more lines, if some of them are parallel to either AB or BC, in which case

[52] That is, an indeterminate equation. "De plus, à cause que pour determiner le point C, il n'y a qu'une seule condition qui soit requise, à sçavoir que ce qui est produit par la multiplication d'un certain nombre de ces lignes soit égal, ou (ce qui n'est de rien plus mal-aisé) ait la proportion donnee, à ce qui est produit par la multiplication des autres; on peut prendre à discretion l'une des deux quantitez inconnuës x ou y, & chercher l'autre par cette Equation." Such variations in the texts of different editions are of no moment, but are occasionally introduced as matters of interest.

[53] Since the product of three lines bears a given ratio to the product of two others and a given line, no term can be of higher degree than the third, and therefore, than the second in x.

[54] See pages 13, et seq.

cation desquelles elles seruent, qui ont esté ainsi multipliées: ensorte qu'elles n'auront iamais plus de deux dimensions, en ce qui ne sera produit que par la multiplication de deux lignes; ny plus de trois, en ce qui ne sera produit que par la multiplication de trois, & ainsi a l'infini.

De plus, a cause que pour determiner le point C, il n'y a qu'vne seule condition qui soit requise, à sçauoir que ce qui est produit par la multiplication d'vn certain nombre de ces lignes soit esgal, ou (cequi n'est de rien plus malaysé) ait la proportion donnée, à ce qui est produit par la multiplication des autres; on peut prendre a discretion l'vne des deux quantités inconnues x ou y, & chercher l'autre par cete Equation. en laquelle il est euident que lorsque la question n'est point proposée en plus de cinq lignes, la quantité x qui ne sert point a l'expression de la premiere peut tousiours n'y auoir que deux dimensions. de façon que prenant vne quantité connuë pour y, il ne restera que $xx \propto +$ ou $-$ $ax +$ ou $-$ bb. & ainsi on pourra trouuer la quantité x auec la reigle & le compas, en la facon tantost expliquée. Mesme prenant successiuement infinies diuerses grandeurs pour la ligne y, on en trounera aussy infinies pour la ligne x, & ainsi on aura vne infinité de diuers poins, tels que celuy qui est marqué C, par le moyen desquels on descrira la ligne courbe demandée.

Commēt on trouue que ce problesme est plan, lorsqu'il n'est point proposé en plus de 5 lignes.

Il se peut faire aussy, la question estant proposée en six, ou plus grand nombre de lignes; s'il y en a entre les données, qui soient paralleles a BA, ou BC, que l'vne des deux quantités x ou y n'ait que deux dimensions en

Rr l'Equal'Equation, & ainsi qu'on puisse trouuuer le point C auec la reigle & le compas. Mais au contraire si elles sont toutes paralleles, encore que la question ne soit proposée qu'en cinq lignes, ce point C ne pourra ainsi estre trouué, a cause que la quantité x ne se trouuant point en toute l'Equation, il ne sera plus permis de prendre vne quantité connuë pour celle qui est nommée y, mais ce sera elle qu'il faudra chercher. Et pource quelle aura trois dimensions, on ne la pourra trouuer qu'en tirant la racine d'vne Equation cubique. cequi ne se peut generalement faire sans qu'on y employe pour le moins vne section conique. Et encore qu'il y ait iusques a neuf lignes données, pourvû qu'elles ne soient point toutes paralleles, on peut tousiours faire que l'Equation ne monte que iusques au quarré de quarré. au moyen dequoy on la peut aussy tousiours resoudre par les sections coniques, en la façon que i'expliqueray cy aprés. Et encore qu'il y en ait iusques a treize, on peut tousiours faire qu'elle ne monte que iusques au quarré de cube. en suite de quoy on la peut resoudre par le moyen d'vne ligne, qui n'est que d'vn degré plus composée que les sections coniques, en la façon que i'expliqueray aussy cy aprés. Et cecy est la premiere partie de ceque i'auois icy a demonstrer; mais auant que ie passe a la seconde il est besoin que ie die quelque chose en general de la nature des lignes courbes.

LA

either x or y will be of only the second degree in the equation, so that the point C can be found with ruler and compasses.

On the other hand, if the given lines are all parallel even though a question should be proposed involving only five lines, the point C cannot be found in this way. For, since the quantity x does not occur at all in the equation, it is no longer allowable to give a known value to y. It is then necessary to find the value of y.[55] And since the term in y will now be of the third degree, its value can be found only by finding the root of a cubic equation, which cannot in general be done without the use of one of the conic sections.[56]

And furthermore, if not more than nine lines are given, not all of them being parallel, the equation can always be so expressed as to be of degree not higher than the fourth. Such equations can always be solved by means of the conic sections in a way that I shall presently explain.[57]

Again, if there are not more than thirteen lines, an equation of degree not higher than the sixth can be employed, which admits of solution by means of a curve just one degree higher than the conic sections by a method to be explained presently.[58]

This completes the first part of what I have to demonstrate here, but it is necessary, before passing to the second part, to make some general statements concerning the nature of curved lines.

[55] That is, to solve the equation for y.

[56] See page 84.

[57] See page 107.

[58] This line of reasoning may be extended indefinitely. Briefly, it means that for every two lines introduced the equation becomes one degree higher and the curve becomes correspondingly more complex.