Mathematics 351 – Abstract Algebra 1
Solutions for Problem Set 4
September 28, 2007

Section 4.5

1 e. By the Rational Root Test, the possibilities for rational roots of $f(x) = 2x^4+7x^3+5x^2+7x+3 = 0$ are

$$x = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}.$$

Since all of the coefficients in $f(x)$ are positive, note that only the negative numbers here are actually candidates as roots. Testing them one by one, we find $f(-1) = -4$, $f(-3) = 0$, $f\left(\frac{-1}{2}\right) = 0$ and $f\left(\frac{-3}{2}\right) = -\frac{39}{4}$. Hence

$$f(x) = (2x+1)(x+3)(x^2+1),$$

(The x^2+1 is found by dividing $(2x+1)(x+3) = 2x^2+7x+3$ into $f(x)$ and taking the quotient. The remainder must be zero by the Factor Theorem.) Each factor is clearly irreducible in $\mathbb{Q}[x]$.

3. By the Rational Roots Test, if $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in \mathbb{Z}[x]$, then any rational root of f must have the form $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $p|a_0$ and $q|1$. This means $q = \pm 1$, so $x \in \mathbb{Z}$.

4 a. First, by the Rational Roots Test, the only possible roots of the polynomial $f(x) = x^4 + 2x^3 + x + 1$ in $\mathbb{Q}$ are $x = \pm 1$. But neither is a root. This means that no $ax+b \in \mathbb{Z}[x]$ divides $f(x)$. We will use Theorem 4.22 now to say that it is enough to consider factorizations of $f(x)$ into polynomials with *integer coefficients*. However, it is still possible that we have $f(x) = (x^2+ax+b)(x^2+cx+d)$ for some $a, b, c, d \in \mathbb{Z}$. However, this would say $x^4 + 2x^3 + x + 1 =$

$$(x^2+ax+b)(x^2+cx+d) = x^4 + (a+c)x^3 + (b+ac+d)x^2 + (ad+bc) + bd,$$

so $a+c = 2$, $b+ac+d = 0$, $ad+bc = 1$ and $bd = 1$. The last equation says $b = d = 1$ or $b = d = -1$ since $b, d \in \mathbb{Z}$. In the first case the other equations reduce to $a+c = 2, ac+2 = 0$, and $a+c = 1$ This is a contradiction since $a+c$ cannot be 1 and 2(!) The other case is similar. We get $a+c = 2$, $ac-2 = 0$, and $-a-c = 1$. This is also impossible. Hence $f(x)$ must be irreducible.

5. a. Eisenstein applies using $p = 2$ ($2 \not| 1$, $2|4$ and $2|22$, but $4 \not| 22$). So the polynomial is irreducible in $\mathbb{Q}[x]$. b. Eisenstein applies using $p = 5$ so the polynomial is irreducible in $\mathbb{Q}[x]$. c. Eisenstein applies using either $p = 2$ or $p = 3$ so the polynomial is irreducible in $\mathbb{Q}[x]$. *Comment:* Note that part c shows it is possible for several different primes to satisfy the coditions in Eisenstein's Criterion.

6. By the Eisenstein Criterion, $x^9 + 12x^5 - 21x + k$ is irreducible whenever $p = 3$ divides k, but $9 \not| k$. There are infinitely many k that satisfy these requirements: we can take $k = 3\ell$ for any $\ell \in \mathbb{Z}$ satisfying $\ell \equiv 1$ or $2 \bmod 3$.

12. Following the hint to prove the contrapositive form, assume $f(x)$ is reducible in $F[x]$. Then we can factor it as $f(x) = g(x)h(x)$ with $\deg(g), \deg(h) \geq 1$. From this equation, it follows that

for all $c \in F$, $f(x+c) = g(x+c)h(x+c)$. If $g(x) = a_r x^r + \cdots + a_1 x + a_0$, then $g(x+c) = a_r(x+c)^r + \cdots + a_1(x+c) + a_0 = a_r x^r + \cdots$. So $\deg(g(x)) = \deg(g(x+c))$. Similarly for $h(x)$. Hence the equation $f(x+c) = g(x+c)h(x+c)$ shows $f(x+c)$ is also reducible.

13. Let $f(x) = x^4 + 4x + 1$. Then $f(x+1) = x^4 + 4x^3 + 6x^2 + 8x + 6$ (using the binomial theorem to expand $(x+1)^4$). Hence by the Eisenstein Criterion with $p = 2$, $f(x+1)$ is irreducible in $\mathbb{Q}[x]$. By Exercise 12, $f(x)$ is also irreducible.

14. Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Then

$$\begin{aligned} f(x+1) &= (x^4 + 4x^3 + 6x^2 + 4x + 1) + (x^3 + 3x^2 + 3x + 1) + (x^2 + 2x + 1) + x + 1 + 1 \\ &= x^4 + 5x^3 + 10x^2 + 10x + 5 \end{aligned}$$

The Eisenstein Criterion with $p = 5$ shows that this polynomial is irreducible in $\mathbb{Q}[x]$. Hence $f(x)$ is irreducible in $\mathbb{Q}[x]$ as well by Exercise 12. *Comment:* You can also derive the formula above for $f(x+1)$ by noticing that $f(x) = \frac{x^5-1}{x-1}$. Hence

$$f(x+1) = \frac{(x+1)^5 - 1}{x} = x^4 + 5x^3 + 10x^2 + 10x + 5.$$

which explains why we are seeing the binomial coefficients $\binom{5}{\ell}$ in the expansion of $f(x+1)$.