Mathematics 351 – Abstract Algebra 1
Discussion 2 – Examples of Groups
October 19, 2007

Background

We will now turn to a class of algebraic structures with just one operation, the *groups* you began the study of in the Algebraic Structures class. We want to recall the definition and study some properties and examples of groups of various kinds.

The definition

Recall that a group $(G, *)$ is a set with a binary operation $*$ satisfying:

1) For all $a, b \in G$, $a * b \in G$.
2) The operation $*$ is associative: for all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
3) There is an identity element $e \in G$ for $*$, satisfying $a * e = a = e * a$ for all $a \in G$.
4) For each $a \in G$, there is an inverse of a in G for the $*$ operation, satisfying $a * a^{-1} = e = a^{-1} * a$.

Discussion Questions

A) Let R be a ring with addition and multiplication operations $+$, $\cdot$. Is $(R, +)$ a group? Is $(R, \cdot)$ always a group?

B) Show that if R is a ring with identity and U is the set of units in R, then $(U, \cdot)$ is a group.

C) If the underlying set G of a group is *finite*, then it is possible to describe the operation $*$ completely by giving a *group operation table* with rows labeled by the elements $g \in G$, and columns also labeled by the elements $h \in G$, and the entry in row g and column h equal to $g * h$. Show using the definition that each row and each column of the group table of a finite group consists of all the elements of G in some order. (That is, all the elements of G appear, and each appears only once.)

D) The 2×2 matrix A given by

$$A = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

is an invertible matrix (why?). What is the *smallest* group G of invertible 2×2 matrices containing A? Explain, and give the group operation table of G.

E) Let H be the set of six 3×3 matrices given below:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is H a group under the operation of *matrix multiplication*? If so, give the operation table. If not, say why not.

E) Let K be the set of six functions:

$$K = \{x, \frac{1}{x}, 1-x, \frac{1}{1-x}, \frac{x-1}{x}, \frac{x}{x-1}\}.$$

Is K a group under the operation of *function composition*? If so, give the operation table. If not, say why not.