Mathematics 351 – Abstract Algebra I
Discussion 1 – More Examples of Rings
September 3, 2007

Background

Last time, we introduced the general definition of a *ring*, an nonempty set R with two binary operations $+, \cdot$ satisfying the following 8 axioms:

1. (closure under $+$) For all $a, b \in R$, $a + b \in R$.
2. (associativity of $+$) For all $a, b, c \in R$, $(a + b) + c = a + (b + c)$.
3. (commutativity of $+$) For all $a, b \in R$, $a + b = b + a$.
4. (additive identity) There is an element $0 \in R$ such that $a + 0 = a$ for all $a \in R$.
5. (additive inverses) For each $a \in R$, there is an element $x = -a \in R$ such that $a+x = 0$.
6. (closure under $\cdot$) For all $a, b \in R$, $a \cdot b \in R$.
7. (associativity of $\cdot$) For all $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
8. (distributive laws) For all $a, b, c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ *and* $(a+b) \cdot c = a \cdot c + b \cdot c$.

If, in addition there is a multiplicative identity 1 in R, we call R a ring *with identity*. If the multiplication operation $\cdot$ is also commutative, we say R is a *commutative* ring.

In a ring R, a *zero divisor* is an element $a \neq 0 \in R$ such that there exists some $b \neq 0$ with $a \cdot b = 0$ or $b \cdot a = 0$. An *integral domain* is a commutative ring with identity that has no zero divisors (i.e. where $a \neq 0$ and $b \neq 0$ implies $a \cdot b \neq 0$).

Finally, a commutative ring with identity $1 \neq 0$, in which every nonzero element has a multiplicative inverse (that is, for each $a \neq 0$, the equation $a \cdot x = 1$ has a solution $x = a^{-1}$ in the ring) is called a *field*.

Discussion Questions

I. Let $\mathbf{Q}$ denote the field of rational numbers, and let p be a (fixed) prime number in $\mathbf{Z} \subset \mathbf{Q}$. Let

$$R_p = \left\{ \frac{a}{b} \in \mathbf{Q} : a, b \in \mathbf{Z} \text{ and } b = p^i \text{ for some } i \geq 0 \right\}.$$

A) Show carefully that R_p is a commutative ring with identity, using the usual sum and product operations from $\mathbf{Q}$. **Note**: Many of the ring properties in R_p will follow from corresponding statements in $\mathbf{Q}$, because R_p is a subset of the ring $\mathbf{Q}$, so if you are thinking about it the right way, this should not be as tedious as it might appear at first(!)

B) Is R_p an integral domain? Is R_p a field? Prove your assertions.

C) The ring R_p here is an example of what is called a *subring* of the ring $\mathbf{Q}$. What should it mean to say a ring S is a subring of a ring R in general?

II. Let $L = \{a \in \mathbf{R} : a > 0\}$. We can define new "sum" and "product" operations on L as follows:

$$a \oplus b = ab \qquad \text{and} \qquad a \otimes b = a^{\ln(b)}.$$

Is L is a ring under these operations? Why or why not? Is it a *subring* of the ring $(\mathbf{R}, +, \cdot)$ ($+, \cdot =$ the usual sum and product of reals)?

III. In class last time, we saw that

$$C^1(\mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} : f' \text{ exists and is continuous at all } x \in R\}$$

is a commutative ring with identity.

A) Let

$$f(x) = \begin{cases} 3x^2 - 2x^3 & \text{if } 0 \le x \le 1 \\ 2x^3 - 9x^2 + 12x - 4 & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Sketch the graph of f and show that f is an element of $C^1(\mathbf{R})$.

B) Is $C^1(\mathbf{R})$ an integral domain? (**Hint:** You can make lots of other functions in $C^1(\mathbf{R})$ by "doing things to" the function f from part A.)

IV. Let $(R, +_R, \cdot_R)$, $(S, +_S, \cdot_S)$ be any two rings. We can define sum and product operations $+, \cdot$ "component-wise" on the Cartesian product:

$$R \times S = \{(a, b) : a \in R, b \in S\}$$

by the following rules:

$$(a, b) + (a', b') = (a +_R a', b +_S b')$$
$$(a, b) \cdot (a', b') = (a \cdot_R a', b \cdot_S b').$$

A) Show that the above operations always make $R \times S$ into a ring.

B) Let $R = \mathbf{Z}_2$ with addition and multiplication mod 2, and let $S = \mathbf{Z}_3$ with addition and multiplication mod 3. Make addition and multiplication tables for the operations $+, \times$ on $\mathbf{Z}_2 \times \mathbf{Z}_3$ defined as above.

C) Is $\mathbf{Z}_2 \times \mathbf{Z}_3$ "the same" as some other ring we know? In what sense?

Assignment

Group writeups due on Monday, September 10.